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# THE HAUSDORFF MEANS OF DOUBLE FOURIER SERIES AND THE PRINCIPLE OF LOCALIZATION

FRED USTINA

**In the two dimensional case, as in the one dimensional case, the Hausdorff summability method is generated by a Hausdorff weight function. In this paper, we investigate the conditions which must be imposed on this weight function in order that the resulting means of a double Fourier series will display the principle of localization.**

In this article we examine the conditions under which the Hausdorff means of double Fourier series exhibit the principle of localization. As is well known, these means are a generalization of a number of other well known means, including those of Cesáro and Euler. Our results are summarized in Theorems 1 to 4, together with the appropriate corollaries.

Let $[c, d; a, b]$ denote a rectangle with vertices at $(a, b)$, $(a, d)$, $(c, b)$ and $(c, d)$, $a \leqq c$, $b \leqq d$. For $0 < \delta < \pi$, let $R(\delta) = [\delta, \delta; -\delta, -\delta]$, $N(\delta) = [\pi, \delta; -\pi, -\delta] \cup [\delta, \pi; -\delta, -\pi]$, $C(\delta) = [\pi, \pi; -\pi, -\pi] \sim N(\delta)$, and $E(\delta) = N(\delta) \sim R(\delta)$. For $0 < \tau \leqq 1/2$, let $\Delta(\tau) = [1-\tau, 1-\tau; \tau, \tau]$, and let $\theta(\tau) = [1, 1; 0, 0] \sim \Delta(\tau)$. Then $N(\delta)$ is a cross-neighborhood of the origin, $E(\delta)$ is the deleted cross-neighborhood, and $\theta(\tau)$ is the $\tau$-neighborhood of the boundary of the unit square $[1, 1; 0, 0]$.

Let $f(x, y)$ be a $2\pi$-periodic function, Lebesgue integrable in the period square, and let $\{s_{mn}(x, y)\}$ be the corresponding sequence of partial sums of the Fourier series of $f(x, y)$. In the sequel we relate all such calculations to the origin, so that we will be examining the sequence $\{s_{mn}(0, 0)\}$, which we denote simply by $\{s_{mn}\}$. As is easily shown,

$$\begin{aligned} s_{mn} &= \frac{1}{4\pi^2}\int_{-\pi,-\pi}^{\pi,\pi} f(s, t)\,\frac{\sin(m+1/2)s}{\sin s/2}\,\frac{\sin(n+1/2)t}{\sin t/2}\,ds\,dt \\ &= \frac{1}{4\pi^2}\left\{\int_{R(\delta)} + \int_{E(\delta)} + \int_{C(\delta)}\right\} \qquad (1) \\ &= r_{mn} + e_{mn} + c_{mn}\,. \end{aligned}$$

Now suppose that a regular linear summability method $H$ ([6], Vol. 1, p. 74) is applied to the sequence $\{s_{mn}\}$ and let $\{h_{mn}\}$ denote the corresponding sequence of transforms under the method $H$. Then

$$\begin{aligned} h_{mn} &= H\{s_{mn}\} \\ &= H\{r_{mn}\} + H\{e_{mn}\} + H\{c_{mn}\} \qquad (2) \\ &= \alpha_{mn} + \beta_{mn} + \gamma_{mn}\,. \end{aligned}$$

We will say that the principle of localization holds for the sequence $\{s_{mn}\}$ if for arbitrary, fixed $\delta$, $s_{mn} = r_{mn} + o(1)$, $m, n \to \infty$, and that it holds restrictedly if $s_{mn} = r_{mn} + o(1)$, $m, n \to \infty$, under certain specified restrictions on the subscripts $m$ and $n$. Similarly, we will say that the principle of localization holds for the sequence $\{h_{mn}\}$ if $h_{mn} = \alpha_{mn} + o(1)$, $m, n \to \infty$, and that it holds restrictedly if

$$h_{mn} = \alpha_{mn} + o(1),\ m, n \to \infty\ ,$$

under certain specified restrictions on the subscripts $m$ and $n$.

It follows that a necessary and sufficient condition for the principle of localization to hold in the first case is that

$$e_{mn} + c_{mn} = o(1),\ m, n \to \infty\ ,$$

and in the second case the necessary and sufficient condition is that $\beta_{mn} + \gamma_{mn} = o(1)$, $m, n \to \infty$. Since $f(x, y)$ is assumed to be Lebesgue integrable, the sequence $\{c_{mn}\}$ is a null sequence. Since the method $H$ is assumed to be regular, it follows that the sequence $\{\gamma_{mn}\}$ is also a null sequence. Thus to prove that the principle of localization holds for an arbitrary, Lebesgue integrable function, it is necessary and sufficient to prove that, for an arbitrary fixed $\delta$, $\delta > 0$, $e_{mn} = o(1)$ and $\beta_{mn} = o(1)$, $m, n \to \infty$, in the respective cases.

Now let $H$ be a regular Hausdorff summability method. This method has been investigated in detail by Hallenbach [2]. From the foregoing, it follows that to prove that the principle of localization holds for this method, it is necessary and sufficient to prove that the sequence $\{\beta_{mn}\}$ is a null sequence, where

$$(3)\qquad \beta_{mn} = \sum_{0,0}^{m,n} \binom{m}{k}\binom{n}{l} e_{kl} \int_{0,0}^{1,1} u^k(1-u)^{m-k} v^l (1-v)^{n-l} dg(u, v)\ ,$$

where the Hausdorff weight function $g(u, v)$ is of bounded variation in the unit square $[1, 1; 0, 0]$, $g(u, 0) = g(u, 0^+) = g(0, v) = g(0^+, v) = 0$, and $g(1, 1) = 1$.

We will say that $g(u, v)$ satisfies a Lipschitz condition of order one and Lipschitz constant $M$ on a region $R \subset [1, 1; 0, 0]$ if $g(u, v)$ is continuous on $R$ and if

$$|g(u'', v'') - g(u', v'') - g(u'', v') + g(u', v')| \leqq M|(u'' - u')(v'' - v')|$$

whenever the rectangle $[u'', v''; u', v']$ contains only points of $R$. It is easily seen that in such a case, the absolute value of the measure $dg(u, v)$ is majorized by $Mdudv$ on $R$.

**Summary of the main results.** We assume that $g(u, v)$ is a regular Hausdorff weight function, so that the corresponding Hausdorff method is regular. From (3), we have

$$(4)\quad \beta_{mn} = \sum_{0,0}^{m,n}\binom{m}{k}\binom{n}{l}e_{kl}\left\{\int_{\Delta(\tau)} + \int_{\theta(\tau)}\right\} u^k(1-u)^{m-k}v^l(1-v)^{n-l}dg(u, v)$$
$$= \delta_{mn} + \theta_{mn}\,.$$

THEOREM 1. *If a $2\pi$-periodic function is Lebesgue integrable in the period square, then $\delta_{mn} = o(1)$, $m, n \to \infty$, provided that for some fixed but otherwise arbitrary $\varepsilon$, $0 < \varepsilon \leqq 1$, and $\kappa = \tau(1-\tau)\delta^2/12$, we have that $m \leqq \varepsilon^{-1}e^{\kappa n}$* and *$n \leqq \varepsilon^{-1}e^{\kappa m}$, a fortiori provided that*

$$\varepsilon m^{-k} \leqq m/n \leqq \varepsilon^{-1}n^k$$

*for an arbitrary, fixed positive integer $k$, as $m, n \to \infty$.*

COROLLARY 1. *If $\{h_{mn}\}$ is a sequence of regular Hausdorff transforms of the sequence $\{s_{mn}\}$, corresponding to a $2\pi$-periodic function which is Lebesgue integrable on the period square, then the question of whether the sequence $\{h_{mn}: \varepsilon m^{-k} \leqq m/n \leqq \varepsilon^{-1}n^k\}$ exhibits the principle of localization depends entirely on the behavior of the Hausdorff weight function in an arbitrarily small $\tau$-neighborhood of the boundary of the unit square* [1, 1; 0, 0].

COROLLARY 2. *If, in Corollary* 1, *the sequence $\{h_{mn}\}$ corresponds to a Hausdorff weight function $g(u, v)$ such that for some $\tau > 0$, the measure $dg(u, v)$ is identically zero in a $\tau$-neighborhood of the boundary of the unit square, then the sequence $\{h_{mn}: \varepsilon m^{-k} \leqq m/n \leqq \varepsilon^{-1}n^k\}$ exhibits the principle of localization. In particular, if $0 < \alpha < 1$ and $0 < \beta < 1$, and if $\{\varepsilon_{mn}\}$ is the sequence of the Euler $(\varepsilon; \alpha, \beta)$ transforms of the sequence $\{s_{mn}\}$, corresponding to a $2\pi$-periodic function Lebesgue integrable on the period square, then the sequence*

$$\{\varepsilon_{mn}: \varepsilon m^{-k} \leqq m/n \leqq \varepsilon^{-1}n^k\}$$

*exhibits the principle of localization.*

Corollary 1 is an obvious consequence of Theorem 1. This is also the case with the first part of Corollary 2. The second part of Corollary 2 then follows by observing that the Euler $(\varepsilon; \alpha, \beta)$ means are Hausdorff means corresponding to a function $g(u, v)$ such that the measure $dg(u, v)$ is identically zero except at the point $(\alpha, \beta)$, where $dg(u, v) = 1$.

THEOREM 2. *Let $\sigma$ be any subset of $\theta(\tau)$ such that on every simply connected subset of $\sigma$, the Hausdorff weight function $g(u, v)$ satisfies a Lipschitz condition of order one with Lipschitz constant $M$. Let*

$$(5) \qquad \sigma_{mn} = \sum_{0,0}^{m,n} \binom{m}{k}\binom{n}{l} e_{kl} \int_\sigma u^k(1-u)^{m-k} v^l (1-v)^{n-l} dg(u, v) .$$

*Then we have*

(a) *if* $f(x, y) \in L_p(N(\delta))$, $1 < p < \infty$, *then* $\sigma_{mn} = o(1)$, $m, n \to \infty$, $\varepsilon m^{-\lambda} \leqq m/n \leqq \varepsilon^{-1} n^\lambda$, *where* $\varepsilon$ *and* $\lambda$ *are fixed but arbitrary,* $0 < \varepsilon \leqq 1$, *and* $0 < \lambda < p - 1$.

(b) *if* $f(x, y) \in B(N(\delta))$, *then* $\sigma_{mn} = o(1), m, n \to \infty$, $m \leqq \varepsilon^{-1} e^{n^\lambda}$, $n \leqq \varepsilon^{-1} e^{m^\lambda}$, *where now* $\varepsilon$ *and* $\lambda$ *are fixed,* $0 < \varepsilon \leqq 1$ *and* $0 < \lambda < 1$.

*In particular, the contribution to the mnth term of the sequence* $\{\beta_{mn}\}$, *due to integration with respect to* $g(u, v)$ *over any subset of the unit square on which* $g(u, v)$ *satisfies a Lipschitz condition of order one with Lipschitz constant* $M$, *is* $o(1)$, $m, n \to \infty$, $\varepsilon m^{-\lambda} \leqq m/n \leqq \varepsilon^{-1} n^\lambda$, $0 < \lambda < p - 1$, *if* $f(x, y) \in L_p(N(\delta))$; *and this contribution is*

$$o(1),\ m, n \to \infty,\ m \leqq \varepsilon^{-1} e^{n^\lambda},\ n \leqq \varepsilon^{-1} e^{m^\lambda}, 0 < \lambda < 1 ,$$

*if* $f(x, y) \in B(N(\delta))$.

COROLLARY. *Let* $\{c_{mn}\}$ *denote the sequence of the Cesàro* $(C; \alpha, \beta)$ *means of* $\{s_{mn}\}$, *corresponding to a* $2\pi$-*periodic function* $f(x, y)$. *If* $1 \leqq \alpha, \beta$, *and if* $f(x, y) \in B(N(\delta))$, *then for arbitrary fixed* $\varepsilon, \lambda, 0 < \varepsilon \leqq 1$, $0 < \lambda < 1$, *the sequence* $\{c_{mn}: m \leqq \varepsilon^{-1} e^{n^\lambda}, n \leqq \varepsilon^{-1} e^{m^\lambda}\}$ *exhibits the principle of localization. If* $1 \leqq \alpha, \beta$, *and if* $f(x, y) \in L_p(N(\delta))$, $1 < p < \infty$, *then for arbitrary, fixed* $\varepsilon, \lambda$, $0 < \varepsilon \leqq 1$, $0 < \lambda < p - 1$, *the sequence*

$$\{c_{mn}; \varepsilon m^{-\lambda} \leqq m/n \leqq \varepsilon^{-1} n^\lambda\}$$

*exhibits the principle of localization.*

The corollary follows from the theorem by taking $\sigma = [1, 1; 0, 0]$ and observing that the $(C; \alpha, \beta)$ means are Hausdorff means corresponding to the function $g(u, v) = \{1 - (1-u)^\alpha\}\{1 - (1-v)^\beta\}$. Then $|dg(u, v)| = |\alpha\beta(1-u)^{\alpha-1}(1-v)^{\beta-1}| dudv \leqq M dudv$ for some constant $M \geqq \alpha\beta$ if $1 \leqq \alpha, \beta$.

Since it is already known ([6], Vol. II, p. 304) that for bounded $f(x, y)$, the sequence $\{c_{mn}\}$ exhibits the principle of localization without the restrictions imposed on the subscripts $m, n$ in the above corollary, it would seem that our estimates in Theorem 2 can be improved or a new proof devised to give a better result. Attempts to achieve this have been unsuccessful so far.

THEOREM 3. *Suppose that the Hausdorff weight function* $g(u, v)$ *is discontinuous along finitely many lines*

$$\omega = \{u = u_i, i = 1, 2, \cdots, k; v = v_j, j = 1, 2, \cdots, l\} ,$$

*such that for some* $\tau > 0$, $\tau \leqq u_i \leqq 1 - \tau$, $\tau \leqq v_j \leqq 1 - \tau$ *for all* $i, j$. *Let* $g_i(v) = g(u_i^+, v) - g(u_i^-, v)$ *and* $g_j(u) = g(u, v_j^+) - g(u, v_j^-)$, *and suppose that the functions* $\{g_i(v), g_j(u)\}$ *all satisfy a Lipschitz condition of order one and Lipschitz constant* $M$. *Let*

$$(6) \qquad \omega_{mn} = \sum_{0,0}^{m,n} \binom{m}{k}\binom{n}{l} e_{kl} \int_\omega u^k(1-u)^{m-k} v^l (1-v)^{n-l} dg(u, v)$$

*be the contribution to* $\beta_{mn}$ *due to integration with respect to* $g(u, v)$ *along these lines. Then we have*

(a) *if* $f(x, y) \in L[\pi, \pi; -\pi, -\pi]$, *then* $\omega_{mn} = o(1)$, $m, n \to \infty$, *provided* $\varepsilon \leqq m/n \leqq \varepsilon^{-1}$, *where* $0 < \varepsilon \leqq 1$ *is fixed but otherwise arbitrary.*

(b) *if* $f(x, y) \in L_p(N(\delta))$, *then*

$$\omega_{mn} = o(1),\ m, n \to \infty,\ \varepsilon m^{-\lambda} \leqq m/n \leqq \varepsilon^{-1} n^{\lambda}\ ,$$

*where* $0 < \varepsilon \leqq 1$, *and* $0 < \lambda < p - 1$ *are fixed but otherwise arbitrary.*

(c) *if* $f(x, y) \in B(N(\delta))$, *then* $\omega_{mn} = o(1)$, $m, n \to \infty$, $m \leqq \varepsilon^{-1} e^{n^\lambda}$, $n \leqq \varepsilon^{-1} e^{m^\lambda}$, *where* $\varepsilon$ *is as before, and* $\lambda$ *is again fixed,* $0 < \lambda < 1$.

From Theorems 1, 2 and 3, a number of general statements pertaining to the principle of localization for the sequence $\{h_{mn}\}$ of regular Hausdorff transforms of the sequence $\{s_{mn}\}$ can be made immediately. We give one example. From Theorems 1 and 3, we get

THEOREM 4. *Suppose that* $f(x, y) \in L[\pi, \pi; -\pi, -\pi]$, *that* $f(x, y)$ *is periodic of period* $2\pi$, *and that* $\{s_{mn}\}$ *is the sequence of partial sums of the Fourier series of* $f(x, y)$. *If the Hausdorff weight function* $g(u, v)$ *is such that for some* $\tau > 0$ *the measure* $dg(u, v)$ *is identically zero in the* $\tau$*-neighborhood of the boundary of the unit square, except along finitely many lines* $u = u_i$, $v = v_j$, $\tau \leqq u_i \leqq 1 - \tau$, $\tau \leqq v_j \leqq 1 - \tau$ *for all* $i, j$, *along which the difference functions* $g_i(v)$, $g_j(u)$, *as defined in Theorem 3, all satisfy a Lipschitz condition of order one and Lipschitz constant* $M$, *then the corresponding sequence of Hausdorff transforms* $\{h_{mn}\}$ *exhibits the principle of localization restrictedly, with the restriction that for arbitrary fixed* $\varepsilon$, $0 < \varepsilon \leqq 1$, *we have*

$$\varepsilon \leqq m/n \leqq \varepsilon^{-1}\ .$$

**Preliminary Lemmas.** We collect here a few lemmas to facilitate the proofs of our results.

LEMMA 1. *Let*

$$\rho_1 \sin\alpha = u \sin s,\ \rho_1 \cos\alpha = 1 - u + u\cos s$$
$$\rho_2 \sin\beta = v \sin t,\ \rho_2 \cos\beta = 1 - v + v\cos t\ .$$

*Then*

$$(7)\quad \sum_{0,0}^{m,n}\binom{m}{k}\binom{n}{l}\frac{\sin(k+1/2)s}{\sin s/2}\,\frac{\sin(l+1/2)t}{\sin t/2}\,u^k(1-u)^{m-k}v^l(1-v)^{n-l}$$
$$=\rho_1^m\rho_2^n\,\phi(m;\alpha,s)\,\phi(n;\beta,t)$$

*where*

$$\phi(m;\alpha,s)=\sin m\alpha\cot s/2+\cos m\alpha$$
$$\phi(n;\ \beta,t)=\sin n\beta\ \cot t/2+\cos n\beta\ .$$

*Proof.* The proof is essentially due to Szász [3, p. 443].

$$\sum_0^m\binom{m}{k}\sin(k+1/2)s\,u^k(1-u)^{m-k}$$
$$(8)\qquad =Im\{(1-u+ue^{is})^m e^{is/2}\}$$
$$=Im\{\{\rho_1(\cos\alpha+i\sin\alpha)\}^m(\cos s/2+i\sin s/2)\}$$
$$=\rho_1^m\,Im\{(\cos m\alpha+i\sin m\alpha)(\cos s/2+i\sin s/2)\}$$
$$=\rho_1^m\,(\sin m\alpha\cos s/2+\cos m\alpha\sin s/2)\ .$$

Dividing through by $\sin s/2$ and noting that (7) is a product of two sums of the type (8) completes the proof.

LEMMA 2. *For small values of $t$, say $|t|\leqq\delta$, and $0\leqq v\leqq 1$,*

$$(9)\quad \sin n\beta=\sin ntv+2\cos n(tv+\tfrac{1}{2}rv(1-v)t^3)\sin nrv(1-v)t^3/2\ ,$$

*where $|r|=|r(t,v)|\leqq\delta'$ .*

*Proof.* This result is due to Szász [3, p. 449], taking into account that $\sin(-x)=-\sin x$.

LEMMA 3. $|\phi(m;\alpha,s)|\leqq\cot s/2+1,\ 0<s\leqq\pi,$ *so that for* $0<\delta\leqq s\leqq\pi$, $|\phi(m;\alpha,s)|\leqq\gamma<\infty$ *uniformly in $m$, $\alpha$ and $s$.*

*Proof.* The lemma is obvious.

LEMMA 4.

$$\phi(n;\beta,t)=2\frac{\sin n\beta}{t}+\psi'(n;t,v)$$

*where $\psi'(n;t,v)$ is bounded absolutely and uniformly, $0\leqq v\leqq 1$, $0<|t|\leqq\pi$, $n=1,2,\cdots$.*

*Proof.* From Lemma 1, we have

$$\phi(n;\beta,t) = 2\frac{\sin n\beta}{t} + \sin n\beta\{\cot t/2 - 2/t\} + \cos n\beta$$
$$= 2\frac{\sin n\beta}{t} + \psi'(n;t,v)\ .$$

LEMMA 5. *For small values of $t$, say $|t| \leqq \delta$, and $0 \leqq v \leqq 1$,*

$$\frac{\sin n\beta}{t} = \frac{\sin ntv}{t} + \psi''(n;t,v)\{nv(1-v)t^2/12\}\ ,$$

*where $|\psi''(n;t,v)| \leqq 12\delta'$.*

*Proof.* The lemma follows immediately from Lemma 2 on division by $t$ since $|\cos x| \leqq 1$ and $|\sin x| \leqq |x|$. Thus the second term on the right in (9) is bounded absolutely by the absolute value of $nrv(1-v)t^3$. Thus

$$\sin n\beta = \sin ntv + \psi''(n;t,v)\{nv(1-v)t^3/12\}\ ,$$

where $|\psi''(n;t,v)| \leqq |12r(t,v)| \leqq 12\delta'$.

LEMMA 6. *For $|s| \leqq \pi$ and $0 \leqq u \leqq 1$,*

$$\rho_1^m \leqq e^{-mu(1-u)s^2/12}\ .$$

*Proof.* From the definition in Lemma 1, we have

$$\begin{aligned}\rho_1^2 &= 1 - 2u(1-u)(1-\cos s)\\ &= 1 - 2u(1-u)\{(s^2/2! - s^4/4!) + (s^6/6! - s^8/8!) + \cdots\}\\ &\leqq 1 - 2u(1-u)s^2(1/2! - s^2/4!)\\ &\leqq 1 - u/6(1-u)s^2\\ &\leqq e^{-u/6(1-u)s^2}\end{aligned}$$

since under the restrictions on $s$ and $u$, $u(1-u)$, and each of the paired terms in $s$, is nonnegative, and $1/2! - s^2/4! > 1/12$.

REMARK. It is clear that under similar restrictions on $t$ and $v$, we have

$$\rho_2^n \leqq e^{-nv(1-v)t^2/12}\ .$$

LEMMA 7. *In the interval $0 \leqq v \leqq 1$ and $|t| \leqq \delta$,*

$$\rho_2^n\,\phi(n;\beta,t) = 2\rho_2^n\frac{\sin ntv}{t} + \psi(n;t,v)$$

*where $\psi(n;t,v)$ is bounded absolutely and uniformly in $n$, $t$ and $v$,* *say* $|\psi(n; t, v)| \leqq \delta''$.

*Proof.* By Lemmas 4 and 5,

$$\phi(n; \beta, t) = 2\frac{\sin ntv}{t} + 2\psi''(n; t, v)nv(1-v)t^2/12 + \psi'(n; t, v) . \tag{10}$$

Then

$$\begin{aligned}\rho_2^n\phi(n; \beta, t) &= 2\rho_2^n\frac{\sin ntv}{t} + 2\psi''(n; t, v)\rho_2^n(n/12)v(1-v)t^2 \\ &\qquad + \rho_2^n\psi'(n; t, v) \\ &= 2\rho_2^n\frac{\sin ntv}{t} + \psi(n; t, v) ,\end{aligned}$$

where $\psi(n; t, v)$ is bounded absolutely and uniformly, since $\psi'(n; t, v)$ and $\psi''(n; t, v)$ are so bounded, and

$$\rho_2^n(n/12)v(1-v)t^2 \leqq (n/12)v(1-v)t^2e^{-nv(1-v)t^2/12} \leqq e^{-1}$$

for all values of $n$, $v$ and $t$, since $ze^{-z} \leqq e^{-1}$ for $z \geqq 0$.

LEMMA 8. *For $n$ and $k$ large enough, $0 \leqq v \leqq 1$ and $|t| \leqq \delta$,*

$$|\rho_2^n\,\phi(n; \beta, t)| \leqq 4n$$

*and*

$$|\phi(n; \beta, t)| \leqq 4kn .$$

*Proof.* The first part follows immediately from Lemma 7 since $\rho_2^n \leqq 1$, $\psi(n; t, v)$ is bounded, and $|\sin ntv/t| \leqq nv \leqq n$. The second part follows from (10) since $\psi'(n; t, v)$ and $\psi''(n; t, v)$ are bounded, and $nv(1-v)t^2/12 < n$ for $0 \leqq t \leqq \pi$.

LEMMA 9. *If $K$ and $\kappa$ are any fixed, positive numbers, however, large or small, then*

$$K\{me^{-\kappa n} + ne^{-\kappa m}) = o(1),\ m, n \to \infty ,$$

*provided that for some fixed $\lambda$, $0 < \lambda < \kappa$, and fixed $\varepsilon$, $0 < \varepsilon \leqq 1$, we have $m \leqq \varepsilon^{-1}e^{\lambda n}$ and $n \leqq \varepsilon^{-1}e^{\lambda m}$ as $m, n \to \infty$, a fortiori, provided that $\varepsilon \leqq m/n \leqq \varepsilon^{-1}$ as $m, n \to \infty$.*

*Proof.* We prove only that under these conditions, $Kme^{-\kappa n} = o(1)$, $m, n \to \infty$, the proof of the other part being the same. But then

$$\begin{aligned}Kme^{-\kappa n} &\leqq K\varepsilon^{-1}e^{-(\kappa-\lambda)n} \\ &\to 0,\ n \to \infty .\end{aligned}$$

LEMMA 10. *For* $0 < \delta \leqq s \leqq \pi$,

$$\int_0^1 \rho_1^m du < cm^{-1} ,$$

*where* $c = 48/\delta^2$.

*Proof.* By Lemma 6,

$$\int_0^1 \rho_1^m \, du \leqq \int_0^1 e^{-mu(1-u)s^2/12} \, du$$
$$\leqq \int_0^1 e^{-mu(1-u)\delta^2/12} \, du .$$

Now since the integrand on the right is symmetric in $u$ and $1 - u$, and $u(1 - u) \geqq u/2$, $0 \leqq u \leqq 1/2$, we get

$$\int_0^1 \rho_1^m \, du \leqq 2\int_0^{1/2} e^{-mu(1-u)\delta^2/12} \, du$$
$$\leqq 2\int_0^{1/2} e^{-mu\delta^2/24} \, du$$
$$< 48/m\delta^2 = cm^{-1} .$$

LEMMA 11. *For* $0 \leqq v \leqq 1$ *and* $0 < \delta \leqq \pi$,

$$\int_0^\delta \left| \frac{\sin ntv}{t} \right| dt < 2 \log n, \; n \geqq 10 .$$

*Proof.* If $v = 0$, the lemma is obvious. If $v \neq 0$, set $ntv = t'$. Then

$$\int_0^\delta \left| \frac{\sin ntv}{t} \right| dt = \int_0^{nv\delta} \left| \frac{\sin t}{t} \right| dt$$
$$\leqq \int_0^{\pi n} \left| \frac{\sin t}{t} \right| dt$$
$$< \int_0^1 \frac{\sin t}{t} \, dt + \int_1^{\pi n} \frac{dt}{t}$$
$$< \{1 + \log \pi + \log n\}$$
$$< 2 \log n, \; n \geqq 10 .$$

LEMMA 12. *For* $0 \leqq v \leqq 1$, $0 < \delta \leqq \pi$ *and* $1/p + 1/q = 1$, $1 < p < \infty$, *we have*

$$\left\{\int_0^\delta \left| \frac{\sin ntv}{t} \right|^q dt\right\}^{1/q} < 2n^{1/p} \log n, \; n \geqq 10 .$$

*Proof.* Again, if $v = 0$, the result is obvious. Otherwise set

$ntv = t'$. Then

$$\left\{\int_0^\delta \left|\frac{\sin ntv}{t}\right|^q dt\right\}^{1/q} = \left\{\int_0^{nv\delta} \left|nv\frac{\sin t}{t}\right|^q \frac{dt}{nv}\right\}^{1/q}$$
$$\leqq \left\{(nv)^{q-1}\int_0^{\pi n} \left|\frac{\sin t}{t}\right|^q dt\right\}^{1/q}$$
$$\leqq n^{1/p}\left\{\int_0^{\pi n}\left|\frac{\sin t}{t}\right| dt\right\}^{1/q}$$
$$< n^{1/p}\{1 + \log \pi + \log n\}^{1/q}$$
$$< 2n^{1/p} \log n,\ n \geqq 10 .$$

LEMMA 13. *Given an arbitrary fixed number* $K < \infty$, *and fixed* $\varepsilon$ *and* $\lambda$, $0 < \varepsilon \leqq 1$, $0 < \lambda < p - 1$, *where* $1 < p < \infty$, *then*

$$K\left\{\frac{m^{1/p}\log m}{n} + \frac{n^{1/p}\log n}{m}\right\} = o(1),\ m, n \to \infty ,$$

*provided that* $n/m \leqq \varepsilon^{-1}m^\lambda$ *and* $m/n \leqq \varepsilon^{-1}n^\lambda$ *as* $m, n \to \infty$ .

*Proof*. Again we satisfy ourselves by proving that under these restrictions, $K(m^{1/p}\log m)/n = o(1)$, $m, n \to \infty$, the proof of the other part being the same. But then $m < \varepsilon^{-1}n^{1+\lambda}$, and so, taking $\varepsilon = 1$ for convenience,

$$K\frac{m^{1/p}\log m}{n} < K\frac{n^{1/p+\lambda/p}\log n^{1+\lambda}}{n}$$
$$= K(1+\lambda)\frac{\log n}{n^{1-1/p-\lambda/p}}$$
$$= o(1),\ n \to \infty ,$$

provided that $\lambda < p - 1$.

**Proof of the main results.** By the earlier remarks, to establish the conditions under which the sequence $\{h_{mn}\}$ will exhibit the principle of localization, it is necessary and sufficient to establish the conditions under which the sequence $\{\beta_{mn}\}$ is a null sequence. By (1) and (3),

$$\beta_{mn} = \frac{1}{4\pi^2}\sum_{0,0}^{m,n}\binom{m}{k}\binom{n}{l}\int_{E(\delta)} f(s, t)\,\frac{\sin (k + 1/2)s}{\sin s/2}\,\frac{\sin (l + 1/2)t}{\sin t/2}\,dsdt$$
$$\times \int_{0,0}^{1,1} u^k(1-u)^{m-k}v^l(1-v)^{n-l}dg(u, v) .$$

Interchanging the order of summation and integration, which is permissible in this case since the sum is finite, and applying Lemma 1, we get

$$
\begin{aligned}
\beta_{mn} &= \frac{1}{4\pi^2}\int_{E(\delta)} f(s,t)dsdt\int_{0,0}^{1,1}\rho_1^m\rho_2^n\,\phi(m;\alpha,s)\phi(n;\beta,t)dg(u,v)\\
&= \frac{1}{4\pi^2}\left\{\int_{\delta,-\delta}^{\pi,\delta}+\int_{-\pi,-\delta}^{-\delta,\delta}+\int_{-\delta,\delta}^{\delta,\pi}+\int_{-\delta,-\pi}^{\delta,-\delta}\right\}f(s,t)dsdt \qquad (11)\\
&\quad\times\int_{0,0}^{1,1}\rho_1^m\rho_2^n\,\phi(m;\alpha,s)\phi(n;\beta,t)dg(u,v)\\
&= \lambda_{mn}^1+\lambda_{mn}^2+\lambda_{mn}^3+\lambda_{mn}^4\,.
\end{aligned}
$$

We assume that the mass points of $dg(u, v)$ are all bounded uniformly from the boundary of the unit square, and that $\tau$ is small enough so that $\theta(\tau)$ contains no mass points of $dg(u, v)$. We also assume that $g(u, v)$ satisfies a Lipschitz condition of order one and Lipschitz constant $M$ in every simply connected region of $\theta(\tau)$ over which $g(u, v)$ is continuous, and we let $\sigma$ denote the union of all such regions. Finally, we assume that in $\theta(\tau)$, $g(u, v)$ has finitely many lines of discontinuity, $u = u_i$, $i = 1, 2, \cdots, k$, $v = v_j$, $j = 1, 2, \cdots, l$, such that $\tau \leqq u_i, v_j \leqq 1 - \tau$ for all $i, j$, and that the difference functions $\{g_i(v), g_j(u)\}$, where $g_i(v) = g(u_i^+, v) - g(u_i^-, v)$ and

$$g_j(u) = g(u, v_j^+) - g(u, v_j^-)\ ,$$

all satisfy a Lipschitz condition of order one and Lipschitz constant $M$ on $\theta(\tau)$. We denote the set of these lines by $\omega$.

Thus, $\lambda_{mn}^i$, $i = 1, 2, 3, 4$, has three components, namely, due to integration over $\Delta(\tau)$, then due to integration over $\sigma$, and finally due to integration over $\omega$. We denote these components by $\delta_{mn}^i$, $\sigma_{mn}^i$ and $\omega_{mn}^i$, so that

$$\lambda_{mn}^i = \delta_{mn}^i + \sigma_{mn}^i + \omega_{mn}^i\ , \qquad i = 1, 2, 3, 4, \tag{12}$$

and by (4), (5) and (6), we have

$$
\begin{aligned}
\delta_{mn} &= \delta_{mn}^1+\delta_{mn}^2+\delta_{mn}^3+\delta_{mn}^4\\
\sigma_{mn} &= \sigma_{mn}^1+\sigma_{mn}^2+\sigma_{mn}^3+\sigma_{mn}^4 \qquad (13)\\
\omega_{mn} &= \omega_{mn}^1+\omega_{mn}^2+\omega_{mn}^3+\omega_{mn}^4
\end{aligned}
$$

*Proof of Theorem* 1. From (11) and (12), we get

$$
\begin{aligned}
\left|\delta_{mn}^1\right| &= \frac{1}{4\pi^2}\left|\int_{\delta,-\delta}^{\pi,\delta} f(s,t)dsdt\int_{\Delta(\tau)}\rho_1^m\rho_2^n\,\phi(m;\alpha,s)\phi(n;\beta,t)dg(u,v)\right|\\
&\leqq \frac{1}{4\pi^2}\int_{\delta,-\delta}^{\pi,\delta}|f(s,t)|dsdt\int_{\Delta(\tau)}\rho_1^m\rho_2^n|\phi(m;\alpha,s)||\phi(n;\beta,t)||dg(u,v)|\\
&\leqq \frac{kn\gamma}{\pi^2}e^{-m\tau(1-\tau)\delta^2/12}V(g)\int_{\delta,-\delta}^{\pi,\delta}e^{-n\tau(1-\tau)t^2/12}|f(s,t)|dsdt\\
&= K_1\,ne^{-\kappa m}I_1(n)\ ,
\end{aligned}
$$

where we have applied Lemmas 3, 6 and 8, and set $\kappa = \tau(1-\tau)\delta^2/12$, $K_1 = k\gamma V(g)/\pi^2$, where $V(g)$ denotes the total variation of $g(u, v)$ in the unit square, and

$$I_1(n) = \int_{\delta', -\delta}^{\pi, \delta} e^{-n\tau(1-\tau)t^2/12} |f(s, t)| dsdt ,$$

and used the observation that in $\Delta(\tau)$, $\rho_2^n$ is bounded by $e^{-n\tau(1-\tau)t^2/12}$. Now $e^{-n\tau(1-\tau)t^2/12}|f(s, t)| \downarrow 0$, $n \to \infty$, except on a set of measure zero, so that $I_1(n) \downarrow 0$, $n \to \infty$.

That $|\delta_{mn}^2| \leqq K_2 n e^{-\kappa m} I_2(n)$, where $K_2$ and $I_2(n)$ have the corresponding relation to the rectangle $[-\delta, \delta; -\pi, -\delta]$, is proved in the same manner. To show that $|\delta_{mn}^3| \leqq K_3 m e^{-\kappa n} I_3(m)$ and $|\delta_{mn}^4| \leqq K_4 m e^{-\kappa n} I_4(m)$, we interchange the roles of $s$ and $t$, and thus of $m$ and $n$. Here, $\kappa$ has the same meaning throughout, and the constants $K_i$, $i = 1, 2, 3, 4$, are all finite. Thus

$$\begin{aligned} (14) \qquad |\delta_{mn}| &\leqq |\delta_{mn}^1| + |\delta_{mn}^2| + |\delta_{mn}^3| + |\delta_{mn}^4| \\ &\leqq K\{ne^{-\kappa m}(I_1(n) + I_2(n)) + me^{-\kappa n}(I_3(m) + I_4(m))\} \end{aligned}$$

where $K = \max\{K_1, K_2, K_3, K_4\}$. By the same argument as used to prove that $I_1(n) \downarrow 0$, $n \to \infty$, we have that $I_2(n) \downarrow 0$, $n \to \infty$, and $(I_3(m) + I_4(m)) \downarrow 0$, $m \to \infty$. Applying this to the right hand side of (14), we get $\delta_{mn} = o(1)$, $m, n \to \infty$, provided that $m \leqq \varepsilon^{-1}e^{\kappa n}$ and $n \leqq \varepsilon^{-1}e^{\kappa m}$ as $m, n \to \infty$. This completes the proof of the theorem.

*Proof of Theorem* 2. Taking $\tau = 1/2$, we may assume that the Lebesgue measure of $\sigma$ is unity, that is, the measure of $[1, 1; 0, 0] \sim \sigma$ is zero. As before, we have

$$|\sigma_{mn}^1| \leqq \frac{1}{4\pi^2} \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| dsdt \int_{\sigma} \rho_1^m \rho_2^n |\phi(m; \alpha, s)||\phi(n; \beta, t)||dg(u, v)| ,$$

and since on $\sigma$ the measure $|dg(u, v)|$ is majorized by $Mdudv$, we have by Lemmas 3 and 7,

$$\begin{aligned} |\sigma_{mn}^1| &\leqq \frac{M\gamma}{2\pi^2} \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| dsdt \int_{0,0}^{1,1} \rho_1^m \frac{\sin ntv}{t} |dudv \\ (15) \qquad &+ \frac{M\gamma}{4\pi^2} \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| dsdt \int_{0,0}^{1,1} \rho_1^m |\psi(n; t, v)| dudv . \end{aligned}$$

Now

$$\begin{aligned} \int_{0,0}^{1,1} \rho_1^m |\psi(n; t, v)| dudv &= \int_0^1 \rho_1^m du \int_0^1 |\psi(n; t, v)| dv \\ &\leqq \delta'' c m^{-1} , \end{aligned}$$

uniformly in $t$, $m$ and $n$, $0 < \delta \leqq s \leqq \pi$, by Lemmas 7 and 10. Thus the last term on the right in (15) is bounded by $(M\gamma Ic\delta'')/(4\pi^2 m) = o(1)$, $m \to \infty$, where we have set

$$I = \int_{\delta,-\delta}^{\pi,\delta} |f(s, t)|\, dsdt .$$

Next, applying Lemma 10 and changing the order of integration in the first term on the right in (15), we get

$$(16) \qquad \begin{aligned} |\sigma^1_{mn}| &\leqq \frac{M\gamma c}{2\pi^2 m}\int_0^1 \left\{\int_{\delta,-\delta}^{\pi,\delta} |f(s, t)| \left|\frac{\sin ntv}{t}\right| dsdt \right\} dv \\ &\quad + o(1), \ m \to \infty . \end{aligned}$$

Now suppose that $f(s, t) \in B(N(\delta))$, that is, $|f(s, t)| \leqq B < \infty$ almost everywhere in $N(\delta)$. Then

$$\begin{aligned} |\sigma^1_{mn}| &\leqq \frac{BM\gamma c\pi}{2\pi^2 m}\int_0^1 \left\{\int_{-\delta}^{\delta} \left|\frac{\sin ntv}{t}\right| dt\right\} dv + o(1) \\ &= \frac{BM\gamma c}{\pi m}\int_0^1 \left\{\int_0^{\delta} \left|\frac{\sin ntv}{t}\right| dt\right\} dv + o(1) \\ &< 2\,\frac{BM\gamma c}{\pi m} \log n + o(1) \\ &= K\frac{\log n}{m} + o(1),\ m \to \infty,\ n \geqq 10 \end{aligned}$$

by Lemma 11. Thus $\sigma^1_{mn} = o(1)$, $m, n \to \infty$, provided that for some fixed but otherwise arbitrary $\varepsilon$ and $\lambda$, $0 < \varepsilon \leqq 1, 0 < \lambda < 1$, we have $n \leqq \varepsilon^{-1} e^{m^\lambda}$ as $m, n \to \infty$.

That $\sigma^2_{mn} = o(1)$, $m, n \to \infty$; $n \leqq \varepsilon^{-1} e^{m^\lambda}$, is proved in a similar manner. To prove that $\sigma^3_{mn} + \sigma^4_{mn} = o(1)$, $m, n \to \infty$, $m \leqq \varepsilon^{-1} e^{n^\lambda}$, we again reverse the roles of $s$ and $t$, and so of $m$ and $n$. This completes the proof of Theorem 2 for the case where $f(s, t) \in B(N(\delta))$.

If $f(s, t) \in L_p(N(\delta))$, $1 < p < \infty$, then in (16),

$$(17) \qquad \begin{aligned} &\int_{\delta,-\delta}^{\pi,\delta} |\hat{f}(s, t)| \left|\frac{\sin ntv}{t}\right| dsdt \\ &\leqq \left\{\int_{\delta,-\delta}^{\pi,\delta} |f(s, t)|^p dsdt\right\}^{1/p} \left\{\int_{\delta,-\delta}^{\pi,\delta} \left|\frac{\sin ntv}{t}\right|^q dsdt\right\}^{1/q} \\ &< 2In^{1/p} \log n \\ &= K' n^{1/p} \log n,\ n \geqq 10 , \end{aligned}$$

where we equated the first integral to $I$, and applied Lemma 12 to the second integral. Then by (16),

$$|\sigma^1_{mn}| \leqq \frac{K'M\gamma c}{2\pi^2} \cdot \frac{n^{1/p}\log n}{m} + o(1)$$
$$\leqq K_1 \frac{n^{1/p}\log n}{m}, \; m \to \infty, \; n \geqq 10 ,$$

the $o(1)$ term being absorbed in the first term.

Proceeding as in the first part of the proof with respect to $\sigma^i_{mn}$, $i = 2, 3, 4$, and combining the results, we get

$$|\sigma_{mn}| \leqq K\left\{\frac{n^{1/p}\log n}{m} + \frac{m^{1/p}\log m}{n}\right\}, \; m, n \geqq 10$$
$$= o(1), \; m, n \to \infty$$

by Lemma 13, provided that $m/n \leqq \varepsilon^{-1}n^{\lambda}$ and $n/m \leqq \varepsilon^{-1}m^{\lambda}$, or equivalently provided that $\varepsilon m^{-\lambda} \leqq m/n \leqq \varepsilon^{-1}n^{\lambda}$, $0 < \varepsilon \leqq 1$, $0 < \lambda < p - 1$. This completes the proof of Theorem 2.

*Proof of Theorem* 3. Let $\omega^{1,i}_{mn}$ be the contribution to $\omega^1_{mn}$ due to integration along the line $u = u_i$, and let $\omega^{1,j}_{mn}$ be the contribution to $\omega^1_{mn}$ due to integration along the line $v = v_j$. Then

$$\omega^{1,i}_{mn} = \frac{1}{4\pi^2}\int_{\delta,-\delta}^{\pi,\delta} f(s, t)dsdt\int_{u_i^-,0}^{u_i^+,1} \rho_1^m\rho_2^n\phi(m; \alpha, s)\phi(n; \beta, t)dg(u, v) .$$

By Lemma 6, $\rho_1^m \leqq e^{-\kappa m}$, where again $\kappa = \tau(1-\tau)\delta^2/12$. Then by Lemmas 3 and 8,

$$|\omega^{1,i}_{mn}| \leqq \frac{Mn\gamma e^{-\kappa m}}{\pi^2}\int_{\delta,-\delta}^{\pi,\delta} |f(s, t)|\, dsdt$$
$$= \frac{M\gamma I_1}{\pi^2} ne^{-\kappa m}$$
$$= K_1' \, ne^{-\kappa m}$$

where $I_1$ and $K_1'$ have the obvious meanings, $K_1'$ being an absolute constant independent of $i$. Since there are at most $k < \infty$ such lines, the total contribution of these $k$ lines to $\omega^1_{mn}$ does not exceed

$$kK_1' \, ne^{-\kappa m} = K' \, \mathrm{ne}^{-\kappa m} . \tag{18}$$

Next, we have

$$\omega^{1,j}_{mn} = \frac{1}{4\pi^2}\int_{\delta,-\delta}^{\pi,\delta} f(s, t)dsdt\int_{0,v_j^-}^{1,v_j^+} \rho_1^m\rho_2^n\phi(m; \alpha, s)\phi(n; \beta, t)dg(u, v) , \tag{19}$$

and so, by Lemmas 3, 6 and 8,

$$
\begin{aligned}
|\omega_{mn}^{1,j}| &\leqq \frac{kn\gamma M}{\pi^2}\int_{\delta,-\delta}^{\pi,\delta} |f(s,t)|\, e^{-n\tau(1-\tau)t^2/12} ds dt \int_0^1 \rho_1^m\, du \\
&< \frac{k\gamma Mc}{\pi^2}\,\frac{n}{m}\, I_1(n) \\
&= K_1''\frac{n}{m} I_1(n)
\end{aligned}
$$

where $k$ now is the constant of Lemma 8, and $I_1(n)$ is the same as in the proof of Theorem 1. Here again $K_1''$ is an absolute constant independent of $j$, and $I_1(n) \downarrow 0$, $n \to \infty$. Since there are at most $l < \infty$ such lines, the total contribution of these to $\omega_{mn}^1$ is bounded absolutely by

$$
lK_1''\frac{n}{m} I_1(n) = K''\frac{n}{m} I_1(n) \,. \tag{20}
$$

Thus by (18) and (20),

$$
\begin{aligned}
|\omega_{mn}^1| &\leqq K'\, ne^{-\kappa m} + K''\frac{n}{m} I_1(n) \\
&\leqq K_1(ne^{-\kappa m} + \frac{n}{m} I_1(n)) \,.
\end{aligned}
$$

By similar reasoning, we get

$$
\begin{aligned}
|\omega_{mn}^2| &\leqq K_2(ne^{-\kappa m} + \frac{n}{m} I_2(n)) \\
|\omega_{mn}^3| &\leqq K_3(me^{-\kappa n} + \frac{m}{n} I_3(m)) \\
|\omega_{mn}^4| &\leqq K_4(me^{-\kappa n} + \frac{m}{n} I_4(m)) \,.
\end{aligned}
$$

Taking $K = 2.\ \max\{K_1, K_2, K_3, K_4\}$ and combining the results, we have

$$
\begin{aligned}
|\omega_{mn}| &\leqq K\{me^{-\kappa n} + ne^{-\kappa m}\} \\
&\quad + K\Big\{\frac{n}{m}(I_1(n) + I_2(n)) + \frac{m}{n}(I_3(m) + I_4(m))\Big\} \\
&= o(1),\ m, n \to 0,\ \varepsilon \leqq m/n \leqq \varepsilon^{-1} \,,
\end{aligned}
$$

where the first part goes to zero by Lemma 9, and the second part because of the convergence of the integrals $I_1(n)$, $I_2(n)$, $I_3(m)$ and $I_4(m)$ to zero. This completes the proof for $f(s,t) \in L[\pi, \pi; -\pi, -\pi]$.

If $f(s,t) \in L_p(N(\delta))$, $1 < p < \infty$, then

$$|\omega_{mn}^{1,i}| \leqq \frac{M\gamma e^{-\kappa m}}{4\pi^2}\int_{\delta,-\delta}^{\pi,\delta}|f(s,t)|dsdt\int_0^1|\rho_2^n\phi(n;\beta,t)|dv$$
$$\leqq \frac{M\gamma e^{-\kappa m}}{2\pi^2}\int_0^1\left\{\int_{\delta,-\delta}^{\pi,\delta}|f(s,t)|\left|\frac{\sin ntv}{t}\right|dsdt\right\}dv$$
$$+\frac{M\gamma e^{-\kappa m}}{4\pi^2}\int_{\delta,-\delta}^{\pi,\delta}|f(s,t)|dsdt\int_0^1|\psi(n;t,v)|dv$$

by Lemma 7. By (17), the integral in brackets in the first term on the right is bounded by $K_1' n^{1/p}\log n$. Thus the first term on the right is bounded $M\gamma K_1' e^{-\kappa m} n^{1/p}\log n/2\pi^2$. Since $\psi(n;t,v)$ is bounded absolutely and uniformly, the second term is $o(1)$ as $m\to\infty$. Since there are at most $k$ such lines, their total contribution to $\omega_{mn}^1$ is bounded by

$$\text{(21)}\qquad \frac{kM\gamma K_1'}{2\pi^2}e^{-\kappa m}n^{1/p}\log n + k\cdot o(1) = K'e^{-\kappa m}n^{1/p}\log n,\ m\to\infty\ .$$

On the other hand, by (19),

$$|\omega_{mn}^{1,j}| \leqq \frac{\gamma Mc}{2\pi^2 m}\int_{\delta,-\delta}^{\pi,\delta}|f(s,t)|\left|\frac{\sin ntv_j}{t}\right|dsdt$$
$$+\frac{\gamma Mc}{4\pi^2 m}\int_{\delta,-\delta}^{\pi,\delta}|f(s,t)||\psi(n;t,v_j)|dsdt$$
$$=\frac{\gamma McK_1'}{2\pi^2}\frac{n^{1/p}\log n}{m}+o(1),\ m\to\infty$$
$$=K_1''\frac{n^{1/p}\log n}{m}+o(1),\ m\to\infty\ .$$

The contribution of finitely many, say $l$, such lines is then bounded by

$$\text{(22)}\qquad lK_1''\frac{n^{1/p}\log n}{m}+l\cdot o(1) = K''\frac{n^{1/p}\log n}{m},\ m\to\infty\ ,$$

the $o(1)$ term being absorbed in the first term. Combining (21) and (22), and setting $K_1=\max\{K', K''\}$, we have

$$|\omega_{mn}^1| \leqq K_1 n^{1/p}\log n\left\{e^{-\kappa m}+\frac{1}{m}\right\},\ m\to\infty$$
$$= o(1),\ m, n\to\infty,\ n/m \leqq \varepsilon^{-1}m^{\lambda},\ 0<\lambda<p-1$$

by Lemma 13, since for $m$ sufficiently large, $e^{-\kappa m}<m^{-1}$.

Proceeding as before with regard to $\omega_{mn}^i$, $i=2, 3, 4$, and combining the results, we get

$$\omega_{mn} = o(1),\ m, n\to\infty,\ \varepsilon m^{-\lambda} \leqq m/n \leqq \varepsilon^{-1}n^{\lambda}\ ,$$

where $0 < \lambda < p - 1$ is arbitrary but fixed.

If $f(s, t) \in B(N(\delta))$, then a similar calculation yields the result

$$\omega_{mn} = o(1),\ m,\ n \to \infty,\ m \leqq \varepsilon^{-1} e^{n^\lambda},\ n \leqq \varepsilon^{-1} e^{m^\lambda}\ ,$$

where $\varepsilon$ and $\lambda$ are arbitrary but fixed, $0 < \varepsilon \leqq 1$, and now $0 < \lambda < 1$. Since the calculations are rather obvious, we avoid the details. This completes the proof of Theorem 3.

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