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B-SETS AND PLANAR MAPS

SHERMAN K. STEIN

In this paper we examine the relation between B-sets, which are a purely set-theoretic concept, and various concepts associated with planar maps, for instance, four-colorings, five-colorings, Hamiltonian circuits, and Petersen's theorem. Moreover, the introduction of the notion of a B-set into graph theory enables us to ask questions which may be more tractable than the four-color conjecture and shed light on it.

1. Introduction and definitions. Let F be a family of sets. A set that meets every member of F and yet contains none of the members of F is called a B-set for F. Observe that if B is a B-set for F, then so is its complement, $(\bigcup_{E \in F} E) - B$. In fact, F has a E-set if and only if $\bigcup_{E \in F} E$ can be partitioned into two sets, A and B, such that neither A nor B contains a member of F. Observe that if F has a B-set, and if $G \subseteqq F$, then G has a B-set. Also, if $G \subseteqq F$, and every member of F contains some member of G, and if G has a B-set, then F also has a B-set. (The notion of B-set goes back to Bernstein, who used it in 1908 to deal with a topological question.)

We shall be concerned with maps covering the surface of the sphere, S^2. For the most part, we will assume that these maps are 3-regular, that is, each vertex has degree three. Each region of the map will be a topological cell. Two regions are *adjacent* if they share at least one edge.

A sequence of distinct regions $R_1, R_2, \cdots, R_{n+1}$, $n \geqq 1$ such that R_i is adjacent to R_{i+1}, $1 \leqq i \leqq n$ is a *path of regions*. If we have $R_{n+1} = R_1$, and $R_1, R_2, \cdots R_n$ are still distinct, we call the path a *region-cycle* of length n (or *n-region-cycle*). A region-cycle consisting simply of the regions around a vertex we call a *basic cycle*. In a 3-regular map the basic cycles have length three.

If the union of any two regions in a map is simply connected, then the regions bordering any given region form a region-cycle, which we call a *face cycle*. Its length is just the number of edges of the surrounded region.

We shall not be interested in region-cycles of length two, unless their union is not simply connected. A region-cycle is *odd* if its length is odd; otherwise, it is *even*.

2. B-sets, four-coloring, and Hamiltonian Circuits. It is well known that the vertices of a graph can be colored in four colors if and only if the set of vertices can be partitioned into two sets such that the graph that each determines is bipartite. We translate this result and its proof into the language of B-sets in Theorem 2.1, which, as the reader will notice, suggests the introduction of B-sets and the questions and results that follow.

THEOREM 2.1. *Let M be a 3-regular map covering the sphere. Then M can be colored with four colors if and only if the family of all odd region-cycles in M has a B-set.*

Proof. If the map is colored with the colors c, d, e, f, replace c and d by the label A, and e and f by the label B. Neither the set of regions labelled A, nor the set of regions labelled B contains an odd region-cycle. Hence each is a B-set of the family of odd region-cycles.

The converse is established by alternating two colors in a B-set of the odd region-cycles and in the complement of such a B-set.

Theorem 2.1 suggests a way of strengthening or weakening the four-color conjecture. We may demand more, for instance by requiring that the family of all region-cycles of length at least three have a B-set. Or we may demand less, for instance by requiring only that the family of basic cycles, or perhaps the family of face cycles have a B-set. Theorem 2.2 concerns the stronger version. Section 3 concerns weaker versions.

The next theorem contradicts an assertion made in [2: p. 616]. (The alleged counter-example presented there has three regions labelled α surrounding a vertex on the lower base.)

THEOREM 2.2. *Let M be a 3-regular map covering the sphere. Then the one-skeleton of M has a Hamiltonian circuit if and only if the family of all region-cycles of regions in M of length at least three has a B-set.*

Proof. Assume that the family of region-cycles of length at least three has a B-set S. Then the complementary set of regions, T, is also a B-set of the same family. Let $|S|$ and $|T|$ be the unions of these sets. Both $|S|$ and $|T|$ are closed subsets of S^2.

Since $|S|$ contains no region-cycles of length at least three, the first homology group of $|S|$ is trivial. By the Alexander duality theorem, therefore, $|T|$ is connected. By symmetry, $|S|$ is connected. Thus both $|S|$ and $|T|$ are connected and simply-connected. Consequently, each is homeomorphic to a disk.

Their common boundary, h, is a circuit. Since both S and T are B-sets of the family of basic cycles, h passes through each vertex of M. Hence h is a Hamiltonian circuit.

Consider now the converse. Let h be a Hamiltonian circuit. By the Jordan curve theorem, h separates S^2 into two sets. Let S be the set of regions "inside" h and let T be the set of regions "outside" h. Both $|S|$ and $|T|$ are homeomorphic to disks.

First, S contains no region-cycle whose union is homeomorphic to a disk, for it would then contain a basic cycle. This is impossible since h passes through each vertex of M.

Second, S contains no region-cycle whose union is not homeomorphic to a disk, for such a region-cycle would separate h from regions surrounded by such a cycle. This contradicts the fact that h passes through each vertex of M and hence meets each region of M.

Thus any map covering S^2 whose one-skeleton has no Hamiltonian circuit also has the property that the family of its region-cycles of length three or more has no B-set. In any particular such map, since it is finite, the family of region-cycles of restricted length has no B-set. We illustrate this by examining Tutte's example of a 3-regular map with 25 regions but with no Hamiltonian circuit [6].

THEOREM 2.3. *Let M be Tutte's map, which involves three copies of the figure below overlapping at the point Q. Then the family of 3-region-cycles, face 4-cycles and the 5-region-cycles has no B-set.*

Proof. It suffices to show that in any labelling of the regions in the map shown in Figure 1 with the letters A and B in such a way that none of the region-cycles mentioned in the theorem is labelled all A or all B the two regions sharing the edge PQ have different labels.

To do so, we assume that the two regions meeting at PQ are labelled the same, say A. Since no three-cycle is all A, we may label two other regions B, as shown. The remaining six regions we name $x_1, x_2, \cdots, x_6$.

Since there are no 3-basic-cycles nor face 4-cycles either x_1 is labelled A and x_2 is labelled B, or vice versa. We show that the first case is impossible; the second is treated similarly. We would then have x_3 labelled B (no 3-region-cycle is labelled all A), x_6 labelled A (no 3-region-cycle is labelled all B), x_5 labelled B (no 3-region-cycle is labelled all A). But then x_4 must be simultaneously A (to avoid a 3-region-cycle labelled all B) and B (to avoid a 5-region-cycle labelled all A.) Thus the two regions at PQ must have different labels.

When three copies of the figure are joined at vertex Q, the labels of the three regions meeting at Q must alternate, which is impossible. (Tutte's map is completed by joining the six loose ends in pairs. In addition to the 21 regions in the three triangles there are three 10-gons and a 9-gon.)

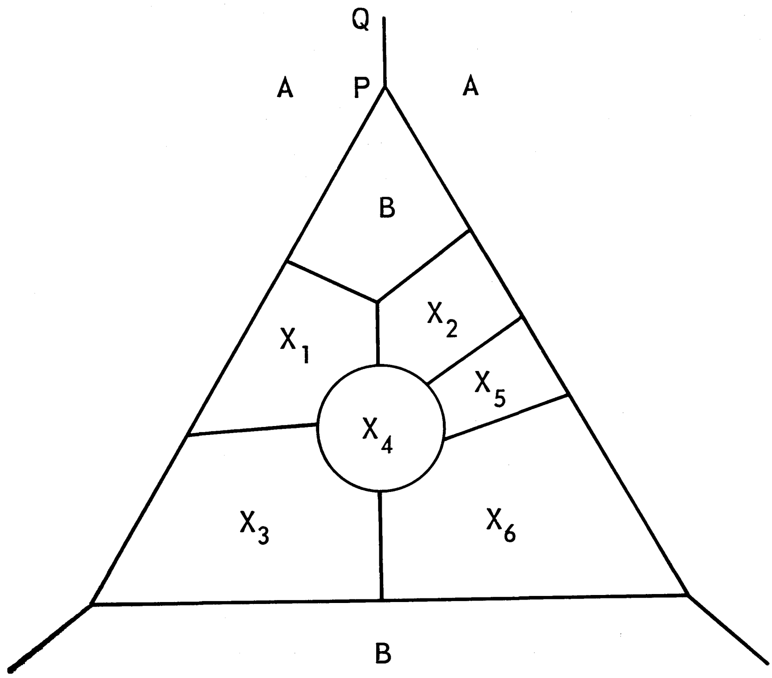

FIG. 1

3. *B*-sets of restricted families of cycles. A weak form of the four--color conjecture is that the family of all 3-region-cycles has a B-set. We shall show that the family of basic 3-region-cycles has a B-set. Indeed this assertion is closely related to Petersen's theorem, as the proof of Theorem 3.1 shows.

THEOREM 3.1. *The family of basic 3-region cycles of a map M covering the sphere has a B-set if and only if the edges of the one-skeleton of M can be labelled r or g in such a way that at each vertex one edge is labelled r and two edges are labelled g.*

Proof. Assume first that the family of basic 3-region-cycles has a B-set. Label an edge r if both regions incident to it are either both in the B-set or both not in the B-set. Label an edge g if exactly one of the regions incident to it is in the B-set. Clearly this defines a labelling of the edges with the desired properties.

Assume, conversely, that the edges have been labelled r or g as in the statement of the theorem. The family of edges labelled g is the disjoint union of circuits, each of which separates the sphere into an "interior" and an "exterior". The set of regions situated in an odd number of the "interiors" is clearly a B-set for the family of basic 3-region-cycles. This proves the theorem.

That such an r, g labelling exists for maps in which the union of two adjacent regions is always simply connected is a consequence of Petersen's theorem ([4]; pp. 186–192). Schonberger [5] strengthened Petersen's theorem by showing that two edges may be preassigned the label g. (D. Barnette has pointed out [private communication] that using Shonberger's theorem, one can give an inductive proof that the set of all 3-region-cycles has a B-set.)

We now consider the family of face cycles. As the map shown in Figure 2, consisting of six regions, shows, this family need not have a B-set.

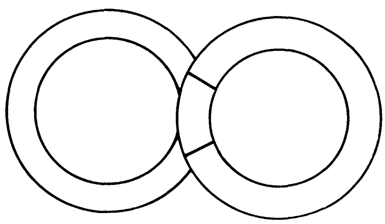

FIG. 2

We have not been able to show that the family of odd face-cycles has a B-set. However, the following theorem suggests that more may be true.

THEOREM 3.2. *Let M be a regular map covering the sphere such that each 3-region-cycle is basic and the union of any two adjacent regions is simply connected. Then, if the number of regions is a multiple of 4, the family of face cycles has a B-set consisting of precisely half the regions.*

Proof. By a theorem of Whitney [7] there is a region-cycle consisting of all the regions of M (that is, a Hamiltonian circuit through the vertices of the dual of M.) Starting at some region, sweep out this region-cycle, labelling the first two regions x, the next two regions y, the next two x, and so on, the letters alternating in pairs. The regions labelled x are a B-set for the family of face-cycles.

Note that the technique used in the proof of Theorem 3.2 also shows that if the number of regions is odd then the family of all face cycles but one (which may be preassigned) has a B-set. If the number of regions is twice an odd number, the family of all but two face-cycles has a B-set.

THEOREM 3.3. *Let M be an infinite map covering the plane, composed of compact regions, and satisfying the same conditions on 2-*

and 3-region-cycles as the maps in the previous theorem. Then the family of all face-cycles of M has a B-set.

Proof. Consider any finite subset, M_1, of the regions of M. We will show that M_1 is contained in a finite set of regions M' such that the regions of M' can be labelled A or B in such a way that no face-cycle situated entirely in M' is all A or all B. By [3], which involves the axiom of choice, or by a direct argument that uses the denumerability of M, the theorem would follow.

To do so, choose first a finite subset, M_2, of regions of M such that $|M_2|$ is homeomorphic to a disk and $M_2 \supseteqq M_1$. Construct a map, M_3, occupying the northern hemisphere of S_2 and isomorphic to M_2. Select a circle in the southern hemisphere parallel to the equator. Partition the band between the equator and the circle into "rectangular" brick-like regions. Finally, adjoin the region below the circle, thus obtaining a regular map, M_4, covering the sphere. Moreover, by putting a suitable number of regions in the band we may assume that this map, M_4, satisfies all the conditions of Theorem 3.2.

By Theorem 3.2 we may label the regions of M_4 A or B in such a way that no face-cycle is all A or all B. By restricting this labelling to M_3 we obtain a labelling of the regions of M_3 with the symbols A and B such that no face-cycle situated entirely in M_3 is all A or all B. From this the theorem follows.

4. A generalization of the five-color theorem. So far our results have been motivated by the four-color conjecture. The five-color theorem suggests another approach closely related to B-sets and which permits a considerable strengthening of the five-color theorem.

Assume that a regular map is colored with the five colors c_1, c_2, c_3, c_4, c_5. Replace c_1 throughout by the label x, c_2 and c_3 by the label y, and c_4 and c_5 by the label z. In this manner we have partitioned the set of regions into three sets: the first consists of isolated regions, while the second and third contain no odd region-cycles. Conversely, from such a partition one may obtain a five-coloring of the map. Note that we may demand, in addition, that no more than one-fifth of the regions be in the set of isolated regions. While the five colors play symmetric roles, the three sets in the partition do not.

In [1] it is proved that the set of regions of a 3-regular map can be partitioned into three sets such that none of these sets contains a region-cycle of length at least three. The proof is like that of the five-color theorem but the result is neither weaker nor stronger than the five-color theorem. Note that the three sets of the partition play symmetric roles in the result.

We strengthen this result to a 'best possible' one, which is stronger than the five-color theorem. In it the three sets of the partition mentioned do not play symmetric roles.

THEOREM 4.1. *It is possible to partition the set of regions of a 3-regular map covering the sphere in such a way that one set consists of isolated regions, while neither of the other two sets contains a region-cycle of length at least three.*

Proof. We only sketch the proof, which is inductive, like that of the five-color theorem, but whose inductive step is divided into many cases because of the asymmetric roles of the sets of the partition. To carry the induction past a 3- or 4-gon, coalesce that region with one of its neighbors, and past a 5-gon, coalesce it with two of its neighbors.

Treatment of the various cases is left to the reader.

Theorem 4.1 is 'best possible' in the sense that there are maps for which the set of isolated regions cannot be empty (namely the maps without Hamiltonian circuits, by Theorem 2.2). However, it is not clear that we may insist that the set of isolated regions contain at most one-fifth of the regions of the map.

5. Edge labellings. Let M be a map, not necessarily regular, covering the sphere, which can be colored in four colors. Consider the dual map, M^*, whose vertices can therefore be labelled A or B such that no odd cycle of vertices is all A or all B. (A vertex cycle in a graph is defined in the obvious ways, two vertices being adjacent if they are the ends of an edge of the graph; we call the cycle odd if there are an odd number of vertices in it.) From the labelling of the vertices of M^* we obtain a labelling of the edges of M^* as follows. Label an edge α if its two ends have the same label it β if its two ends have different labels. This edge-labelling has a property which we express in terms of edge cycles.

By an *edge-cycle* in M^* we shall mean a sequence of directed edges of M^*, $e_1, e_2, \cdots, e_n$, corresponding to distinct undirected edges, such that terminal vertex of e_i coincides with the initial vertex of e_{i+1}, $1 \leqq i \leqq n-1$, and the terminal vertex of e_n is the initial vertex of e_1. If n is odd, we call the cycle itself odd. An *edge path* is defined similarly, without the insistence that the terminal vertex of e_n be the initial vertex of e_1.

THEOREM 5.1. *The family of odd vertex cycles of M^* has a B-set if and only if the edges of M^* can be labelled α or β such that each even edge cycle of M^* has an even number of edges labelled β, and each odd edge cycles has a nonzero even number of edges labelled β.*

Proof. If the family of odd vertex-cycles of M^* has a B-set, then the labelling of the edges already described satisfies the condition of the theorem.

Conversely, from an α, β-labelling of the edges satisfying the condition of the theorem, we may obtain a B-set for the family of odd vertex-cycles, as follows.

Pick a vertex of M^* and label it A. To label any other vertex, V, select an edge path from A to V, say $e_1, e_2, \cdots, e_n$. Label in order the vertices through which the path passes A or B in this manner: having labelled the initial vertex of e_i, label its terminal vertex the same if e_i is labelled α, and different if e_i is labelled β. It is easy to check that this provides a well-defined labelling of the vertices. The vertices labelled A is the desired B-set.

Note that the α, β-labelling of the edges of M^* provide a B-set for the family of odd edge-cycles of M^*. By duality we conclude: If the four-color conjecture is true, then the family of odd edge-cycles in the one-skeleton of a map covering the sphere has a B-set.

In contrast to Theorem 2.1, the existence of such a B-set does not imply a four-coloring.

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UNIVERSITY OF CALIFORNIA, DAVIS