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ON COVERING SPACES AND GALOIS EXTENSIONS

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Let X be a connected compact Hausdorff space, and G a finite abelian group. In this note we obtain a short exact sequence (Theorem 1) which describes the group of isomorphism classes of regular covering spaces of X with group G. The sequence is derived as an immediate translation of a similar sequence involving the group of commutative Galois extensions with group G of $C(X)$, the ring of complex-valued continuous functions on X.

The translation is obtained in part by showing (Theorem 2) that there is an equivalence between the category of finite covering spaces of X and the dual of the category of separable $C(X)$-algebras which are finitely generated projective $C(X)$-modules. This equivalence may be known to students of [8], but I am unaware of any reference for it, so we sketch a proof here.

1. The sequence. Let $X, C(X)$ be as above, and let G be a finite group. A not necessarily connected covering space Y over X is called regular with group G if G acts as a fixed point free group of homeomorphisms of Y which preserve the covering map. Denote by $\mathrm{Cov}(X, G)$ the set of isomorphism classes of regular covering spaces of X with group G (where Y, Z, two covering spaces with group G, are isomorphic if there is a homeomorphism from Y to Z which commutes with the covering maps and the action of G). Denote by $\mathrm{Pic}(X)$ the group (under tensor product of fibers) of isomorphism classes of line bundles on X. Denote by $H_s^2(G, U(C(X)))$ the subgroup of $H^2(G, U(C(X)))$ (group cohomology, with G acting trivially on $U(C(X))$) which is the image of the symmetric 2-cocycles—those cocycles f from $G \times G$ into the units of $C(X)$ which satisfy $f(s, t) = f(t, s)$ for all s, t in G.

THEOREM 1. *Let G be a finite abelian group. Then $\mathrm{Cov}(X, G)$ has an abelian group structure so that the following sequence of abelian groups is exact:*

$$0 \to H_s^2(G, U(C(X))) \to \mathrm{Cov}(X, G) \to \mathrm{Hom}(G, \mathrm{Pic}(X)) \to 0 .$$

Proof. If G is a group of order n and R is a commutative ring with unity, a commutative R-algebra S is a Galois extension of R with group G if G acts as a group of R-algebra automorphisms of S, R is the fixed ring under the action of G, and ([3, 1.3f]) for each maximal ideal m of S and $\sigma \neq 1$ in G, there is an s in S so that $\sigma(s) - s \notin m$. Two Galois extensions S', S'' with group G are isomorphic if there is an R-algebra isomorphism $S' \to S''$ which preserves the action of G. If G is an abelian group, Harrison [9] has shown that the set of isomorphism classes of commutative Galois extensions with group G forms an abelian group, $\mathrm{Comm}(R, G)$. Viewing a Galois extension of R with group G as a rank one projective $R[G]$-module defines a homomorphism from $\mathrm{Comm}(R, G)$ to $\mathrm{Pic}(R[G])$ (the group under tensor product of isomorphism classes of rank one projective $R[G]$-modules), whose kernel consists of the set $NB(R, G)$ of isomorphism classes of commutative Galois extensions with normal basis. If R has no idempotents but 0 and 1 and contains $1/n$ and a primitive nth root of unity, then the image is isomorphic to $\mathrm{Hom}(G, \mathrm{Pic}(R))$ [6, Theorem 9], so that we have the short exact sequence

$$(*) \quad 0 \to NB(R, G) \to \mathrm{Comm}(R, G) \to \mathrm{Hom}(G, \mathrm{Pic}(R)) \to 0 .$$

We set $R = C(X)$ in (*) and translate. $\mathrm{Pic}(C(X)) \cong \mathrm{Pic}(X)$ by Swan [11]; $NB(C(X), G) \cong H^2_s(G, U(X)))$ by Theorems 2.2 and 4.4 of [5]. It suffices to show that $\mathrm{Comm}(C(X), G) \cong \mathrm{Cov}(X, G)$. This will be a corollary of Theorem 2.

2. The equivalence. X, $C(X)$ are as above.

THEOREM 2. *There are category equivalences between the category of finite covering spaces of X, the dual of the category of separable $C(X)$-algebras which are finitely generated projective $C(X)$-modules, and the category of nonramified affine coverings of* $\mathrm{Spec}(C(X))$ [8]. *The first equivalence is induced by: if Y is a covering space, $Y \to C(Y)$; if S is a separable R-algebra, $S \to$* $\mathrm{Max}(S)$. *The functor from the first to the third sends S to* $\mathrm{Spec}(S)$.

Here $\mathrm{Max}(S)$ is the space of maximal ideals of S with the Stone topology (= the topology induced on the geometric points from the Zariski topology on $\mathrm{Spec}(S)$).

COROLLARY. $\mathrm{Cov}(X, G)$ *is an abelian group isomorphic to* $\mathrm{Comm}(C(X), G)$.

Proof of corollary. If $R = C(X)$, $S = C(Y)$, it follows easily from the definition of Galois extension given above that S is a Galois extension of R with group G if and only if Y is a regular covering space of X with group G. Hence there is a bijection between $\mathrm{Cov}(X, G)$ and $\mathrm{Comm}(R, G)$. The group structure on $\mathrm{Cov}(X, G)$ is the one induced from Comm(R, G).

Concerning Theorem 2, we have included the category of non-ramified affine coverings of Spec$(C(X))$ only to make more explicit the relationship with [8]. We shall show only that the correspondences $Y \to C(Y)$, $S \to \mathrm{Max}(S)$ give inverse bijections between the objects of the first two categories; the proof of the rest of the theorem is straightforward and will be omitted.

In what follows, the phrase "S is a finitely generated projective R-algebra" will mean that S is an R-algebra which is finitely generated and projective as an R-module.

Proof of Theorem 2. We recall some facts about a compact Hausdorff space X (see [7]): The topology of X has a basis consisting of the complements of zero sets of continuous functions on X ("cozero sets"). [7, 3.2, p. 38]. If f is a continuous function let $Z(f) =$ the zeros of f and $V(f) = X - Z(f)$. For any closed set F, if $C(X)|_F$ denotes the restriction to F of the continuous functions on X, then $C(X)|_F = C(F)$ [7, 3.11(c), p. 43]. For any open set $V = V(f)$, $C(X)|_V = C(X)_f$, the localization of $C(X)$ with respect to the multiplicative set consisting of the powers of f. Max$(C(X))$, the set of maximal ideals of $C(X)$, is in one-to-one correspondence with the points of X, since any maximal ideal of $C(X)$ is of the form $\{f \text{ in } C(X) \mid f(p) = 0\}$ for some point p of X. If Max$(C(X))$ is given the Stone topology: basic closed sets are of the form $\{x \mid f \text{ is in } x\} = Z(f)$ for f in $C(X)$, then Max$(C(X))$ is homeomorphic to X [7, 4.9, p. 58].

Assume now that X is a compact Hausdorff space and Y a finite covering space, that is, there is a continuous map $p: Y \to X$ and for each x in X a neighborhood U of x (a canonical neighborhood) such that $p^{-1}(U)$ is the disjoint union of a finite number n of open sets of Y each homeomorphic to U.

Fix an x in X, let U be a canonical neighborhood, and let F be a closed subneighborhood. Then $p^{-1}(F)$ is a disjoint union of closed sets of Y each homeomorphic to F, so $C(p^{-1}(F)) \cong C(F)^n$ (the product as rings of n copies of $C(F)$). Let $V = V(f)$ be a cozero set containing x and contained in F. Then $V_Y(f) = V_Y(f \circ p) = p^{-1}(V_X(f)) \subseteq p^{-1}(F)$ is a disjoint union of open sets of Y each homeomorphic to $V_X(f)$. So $C(Y)_f = C(Y)|_{V_Y(f)} = C(p^{-1}(F))|_{\overset{-1}{p}(V_X(f))}$ is the ring of continuous functions on a finite disjoint union of open sets each homeomorphic to $V_X(f)$, whence $C(Y)_f \cong (C(X)_f)^n$. Thus for each maximal ideal x of $C(X)$ there is a f not in x so that $C(Y)_f$ is a finitely generated projective separable $C(X)_f$-algebra. By [4, § 5] and [1] $C(Y)$ is therefore a finitely generated projective separable $C(X)$-algebra.

For the other direction we need a

LEMMA. *Let S be a finitely generated projective separable R-algebra, $R = C(X)$. Then for each x in X there is a neighborhood $V(h)$ of x so that $S_h \cong (R_h)^n$, a product as rings of n copies of R_h.*

Proof of lemma. R_x, the localization of R with respect to the maximal ideal x, is a local ring, so by Theorems 3.1, 2.2 and 2.8 of [10], $S_x = R_x[\theta]$, where θ satisfies a polynomial $f(t) = t^n + a_1t^{n-1} + \cdots + a_n$ with coefficients in R_x and with n distinct roots $s_1, \ldots, s_n$ in $R/x \cong \boldsymbol{C}$. Let $\theta = a/b$ with a in S, b in $R - x$, let k be the product of the denominators of the coefficients of $f(t)$, and let d be the discriminant [10] of $f(t)$. Then on $V(g)$ with $g = bdk$, $R_g[\theta]$ is a separable R_g-algebra contained in S_g, and is a finitely generated projective R_g-module of the same rank as S_g. So S_g is a projective $R_g[\theta]$-module of rank one. But since S_g is a finitely generated projective $R_g[\theta]$-algebra, $R_g[\theta]$ is a $R_g[\theta]$-direct summand of S_g. Thus $S_g = R_g[\theta]$. Since $f(t)$ has coefficients in R_g, $S_g \cong R_g[t]/(f(t))$.

Claim: There exists a subneighborhood $V = V(h)$ of $V(g)$ containing x, and continuous functions $r_1, \ldots, r_n$ in $C(X)$ so that on V, $f(t) = \prod_{i=1}^{n}(t - r_i)$. This follows from the implicit function theorem applied to the function $F(\bar{a}, t) = t^n + a_1t^{n-1} + \cdots + a_n$ at $\bar{a} = \bar{a}(x) = (a_1(x), \ldots, a_n(x)) \in \boldsymbol{C}^n$, $t = s_i$. For since $F(\bar{a}(x), t)$ has n distinct roots, the partial derivative $F_t(\bar{a}(x), s_i) \neq 0$, so there exists a neighborhood U of $\bar{a}(x)$ in $\boldsymbol{C}^n$ and continuous functions $t_i\colon U \to \boldsymbol{C}$ such that $F(\bar{a}, t_i(a)) = 0$ for all $\bar{a}$ in U. Since $t_i(\bar{a}(x)) = s_i \neq s_j = t_j(\bar{a}(x))$ for all $i \neq j$ we can pick U so small that $t_i(U) \cap t_j(U) = \varnothing$ for all $i \neq j$. If we set $\chi\colon V(g) \to \boldsymbol{C}^n$ by $\chi(y) = \bar{a}(y)$, then χ is a continuous function, so $V(g) \cap \chi^{-1}(U)$ is a neighborhood of x on which there exist continuous functions $\tilde{r}_i = t_i \circ \chi$, $i = 1, \ldots, n$, with disjoint images, which are roots of f. Hence there is a basic open subneighborhood $V(h)$ (containing x) of a closed subneighborhood of $V(g) \cap \chi^{-1}(U)$ and n elements $r_1, \ldots, r_n$ of $C(X)$ so that on $V(h)$, $f(t) = \prod_{i=1}^{n}(t - r_i)$. The lemma follows easily.

Suppose now that S is a finitely generated projective separable R-algebra, and set $Y = \operatorname{Max}(S)$. Put the Stone topology on Y. Let $p\colon Y \to X$ by $p(y) = y \cap R$, a maximal ideal since S is integral over R ([2]).

Let x be a point of X. By the lemma there exists an h in $C(X)$ so that $S_h = (R_h)^n$, a product as rings of copies of R_h. Then $V(h)$ is easily seen to be a canonical neighborhood of x, and p is continuous, so that Y is a covering space of X.

It is easy to verify that the topology defined on Y makes Y into a compact Hausdorff space.

If Y is a covering space and a compact Hausdorff space, then $Y = \operatorname{Max}(C(Y))$ as topological spaces by [7, 3.6, p. 40]. On the other hand, given S, a finitely generated projective separable R-algebra, S is clearly contained in $C(\mathrm{Max}(S))$. Replacing $\mathrm{Max}(S)$ by each of its connected components as necessary, we may assume that S and $C(\mathrm{Max}(S))$ have no nontrivial idempotents. Embed $C(\mathrm{Max}(S))$ in a finitely generated projective Galois extension T of R containing no idempotents but 0 and 1 (possible by [10, 1.13]). By the fundamental theorem of Galois theory ([3, Theorem 2.3]) S is the fixed ring of some subgroup of $\mathrm{Aut}_R(T)$. But any group which fixes S fixes $C(\mathrm{Max}(S))$, whence $S = C(\mathrm{Max}(S))$. This completes the proof of Theorem 2.

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