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CO-ABSOLUTES OF REMAINDERS OF STONE-CECH COMPACTIFICATIONS

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Let X be a completely regular Hausdorff space. Denote the "absolute" (also called the "projective cover") of X by $E(X)$, the Boolean algebra of regular closed subsets of X by $R(X)$, and the Stone-Cech compactification of X by βX. In this paper it is proved that the canonical map $k: E(\beta X) \to \beta X$ maps $\beta E(X) - E(X)$ irreducibly onto $\beta X - X$ if and only if the map $A \to cl_{\beta X} A - X$ is a Boolean algebra homomorphism from $R(X)$ into $R(\beta X - X)$. This latter condition is shown to hold for a wide class of spaces X. These results are used to calculate absolutes and well-known co-absolutes of $\beta X - X$ under several different sets of hypotheses concerning the topology of X.

Throughout this paper we use without further comment the notation and terminology of the Gillman-Jerison text [6]. In particular, the cardinality of a set S is denoted by $|S|$. The countable discrete space is denoted by N, and the set of nonnegative integers (used as an index set) is denoted by N. The symbol $[CH]$ appearing before the statement of a theorem indicates that the continuum hypothesis ($\aleph_1 = 2^{\aleph_0}$) is used in the proof of the theorem. The cardinal $2^{\aleph_0}$ will be denoted by the letter c. All topological spaces considered in this paper are assumed to be completely regular Hausdorff spaces. This assumption is repeated for emphasis from time to time.

In § 1 we give a brief summary of known results and define some notation and terminology. Some of the results in later sections are generalizations of results appearing in [17]. Background material on Boolean algebras appears in [15].

1. Preliminaries.

The concept of the absolute of a topological space has been considered by several authors, notably Gleason [7], Iliadis [8], Flachsmeyer [5], Ponomarev [12], and Strauss [16]. In the first part of this section we give a brief outline of this theory. Although a theory of absolutes can be developed for a wider class of topological spaces, we shall assume that all spaces considered are completely regular and Hausdorff.

Recall that a subset A of a topological space X is said to be *regular closed* if $A = cl_X(\text{int}_X A)$. Let $R(X)$ denote the family of all regular closed subsets of X. The following theorem is well-known; see, for example, § 1 and § 20 of [15].

THEOREM 1.1. *The family $R(X)$ is a complete Boolean algebra under the following operations:*

(i) $A \leqq B$ *if and only if* $A \subseteqq B$.

(ii) $\bigvee_\alpha A_\alpha = cl_X[\bigcup_\alpha A_\alpha]$

(iii) $\bigwedge_\alpha A_\alpha = cl_X \operatorname{int}_X [\bigcap_\alpha A_\alpha]$

(iv) $A' = cl_X(X - A)$ *(A' denotes the Boolean-algebraic complement of A).*

LEMMA 1.2. *Let X be a dense subspace of a space T. Then the map $A \rightarrow cl_T A$ is a Boolean algebra isomorphism from $R(X)$ onto $R(T)$.*

The proof of 1.2 is straightforward and hence is not included.

The following result is a well-known theorem of Marshall Stone (see 7.1 and 8.2 of [15]).

THEOREM 1.3. *Let U be a Boolean algebra and let $S(U)$ be the set of all ultrafilters on U. For each $x \in U$ put $\lambda(x) = \{\alpha \in S(U): x \in \alpha\}$. If a topology τ is assigned to $S(U)$ by letting $\{\lambda(x): x \in U\}$ be an open base for τ, then $(S(U), \tau)$ is a compact Hausdorff totally disconnected space and the map $x \rightarrow \lambda(x)$ is a Boolean algebra isomorphism from U onto the Boolean algebra of open-and-closed subsets of $S(U)$.*

The set $S(U)$, topologized as above, is called the *Stone space* of U.

Recall that a continuous map k from a space X onto a space Y is said to be *irreducible* if the image under k of each proper closed subset of X is a proper closed subset of Y. The following result, due to Gleason, comprises part of theorem 3.2 of [7].

THEOREM 1.4. *Let Y be a compact Hausdorff space. Then the map $k: S(R(Y)) \rightarrow Y$ defined by*

$$k(\alpha) = \bigcap \{A \in R(Y): \alpha \in \lambda(A)\}$$

is a well-defined irreducible continuous map from $S(R(Y))$ onto Y (λ is as defined in 1.3).

Note that $k[\lambda(A)] = A$ for each $A \in R(Y)$.

PROPOSITION 1.5. *If $f: X \rightarrow Y$ is an irreducible map onto Y, and if S is a dense subset of Y, then $f^{\leftarrow}[S]$ is dense in X.*

Recall that a (completely regular Hausdorff) space X is said to be *extremally disconnected* if the closure of every open subset of X is open. The following properties of extremally disconnected spaces are discussed in [7], [16] and 6M of [6].

THEOREM 1.6. (i) *If Y is any compact Hausdorff space then $S(R(Y))$ is extremally disconnected.*

(ii) *If T is an extremally disconnected space, then every dense subspace of T is extremally disconnected and C^*-embedded in T. Hence if Y is compact and W is dense in $S(R(Y))$, then $S(R(Y)) = \beta W$* (see 6.5 of [6]).

THEOREM 1.7. *Let X be a completely regular Hausdorff space and let $k: S(R(\beta X)) \rightarrow \beta X$ be as in 1.4. Then $k^{\leftarrow}[X]$ is a dense, extremally disconnected subspace of $S(R(\beta X))$, and the restriction of k to $k^{\leftarrow}[X]$ is an irreducible perfect map from $k^{\leftarrow}[X]$ onto X.*

Proof. Since X is dense in βX and k is irreducible, by 1.5 $k^{\leftarrow}[X]$ is dense in $S(R(\beta X))$. Hence $k^{\leftarrow}[X]$ is extremally disconnected by 1.6. Since k is a perfect map, its restriction to a preimage of a subspace of βX will also be perfect. If V is a nonempty open subset of $S(R(\beta X))$, then as k is irreducible it follows that $\beta X - k[S(R(\beta X)) - V] \neq \varnothing$. Hence

$$X - k[k^{\leftarrow}[X] - V] = X - k[S(R(\beta X)) - V] \neq \varnothing$$

since X is dense in βX. Hence $k|k^{\leftarrow}[X]$ is irreducible.

Absolutes and Co-absolutes 1.8.

(i) For each space X, there is a unique (up to homeomorphism) extremally disconnected space that can be mapped irreducibly onto X by a perfect map (see [16]). This space is called the *absolute* of X, and is denoted by $E(X)$. We may identify $E(X)$ with the space $k^{\leftarrow}[X]$ described in 1.7.

(ii) Note that if X is compact then $E(X) = S(R(X))$.

(iii) For any space X, $E(\beta X) = \beta E(X)$; this follows from 1.6(ii) and the fact that $E(X)$ is dense in the extremally disconnected space $S(R(\beta X)) = E(\beta X)$

(iv) Two spaces X and Y are said to be *co-absolute* if $E(X)$ and $E(Y)$ are homeomorphic.

PROPOSITION 1.9. *If there is a perfect irreducible map from X onto Y, then X and Y are co-absolute.*

Proof. Since the composition of two perfect irreducible maps is a perfect irreducible map, there is a perfect irreducible map from $E(X)$ onto Y. The above-mentioned uniqueness of the absolute implies that $E(X)$ and $E(Y)$ are homeomorphic.

The converse to 1.9 is untrue; for example, let αN be the one-point compactification of N. Then βN and αN are co-absolute by

1.9 (the extension to βN of the embedding $i: N \to \alpha N$ is the required map), but as $|\alpha N| < |\beta N|$, there is no irreducible perfect map from αN onto βN.

We conclude this section with some miscellaneous known results.

PROPOSITION 1.10. *Let $W(X)$ denote the set of points at which the space X is locally compact. Then $W(X) = \beta X - cl_{\beta X}(\beta X - X)$.*

Proof. Let $p \in W(X)$ and let A be a compact subset of X such that $p \in \mathrm{int}_X A$. By 3.15(b) of [6], $p \in \mathrm{int}_{\beta X} A$ thus $p \notin cl_{\beta X}(\beta X - X)$. Conversely, if $p \in X - W(X)$ and V is a βX-neighborhood of p, then there is a compact subset K of V such that $p \in \mathrm{int}_{\beta X} K$. Since $p \notin W(X)$, $K - X \neq \emptyset$; thus $p \in cl_{\beta X}(\beta X - X)$.

The following theorem, and its corollary, are due to Parovičenko [10] and Rudin [14].

THEOREM 1.11. [*CH*]. *Let Y be a compact totally disconnected Hausdorff space without isolated points. If:*

(i) *Every zero-set of Y is regular closed*

(ii) *Y is an F-space*

(iii) *Y has c open-and-closed subsets,*

then Y is homeomorphic to $\beta N - N$.

COROLLARY 1.12. [*CH*]. *Let X be a locally compact, σ-compact, noncompact space with a base of c open-and-closed sets. Then $\beta X - X$ is homeomorphic to $\beta N - N$.*

Proof. Since X is locally compact and σ-compact, by 14.27 of [6] $\beta X - X$ is a compact F-space. Since X is σ-compact, it is Lindelöf and hence realcompact; it therefore follows from 3.1 of [3] that condition (i) of 1.11 is satisfied by $\beta X - X$. As X is realcompact, $\beta X - X$ has no isolated points. Since X is Lindelöf and has a basis of open-and-closed subsets, by 16.17 of [6] $\beta X - X$ is totally disconnected and has a base of c open-and-closed subsets. Hence by 1.11 $\beta X - X$ and $\beta N - N$ are homeomorphic.

Finally, the following result is due to Comfort and Negrepontis [1].

THEOREM 1.13. [*CH*]. *Let D be the discrete space of cardinality $\aleph_1$ and let Ω be the subspace of $\beta D - D$ consisting of all the points in the βD-closure of some countable subset of D. Then:*

(i) *Ω can be expressed in the form $\Omega = \bigcup_{\alpha<\omega_1} \Omega(\alpha)$, where each $\Omega(\alpha)$ is a homeomorph of $\beta N - N$, each $\Omega(\alpha)$ is open-and-closed in Ω,* *and* $\Omega(\alpha) \supsetneqq \bigcup_{\gamma<\alpha} \Omega(\gamma)$ (α *ranges through the set of ordinals less than the first uncountable ordinal* ω_1).

(ii) *Up to homeomorphism* Ω *is the only space satisfying the conditions in* (i).

(iii) *The one-point compactification of* Ω *is homeomorphic to* $\beta N - N$.

2. $\mathscr{B}$-pleasant spaces. If A is a closed subset of X, let A^* denote the set $cl_{\beta X}A - X$ (in particular, $\beta X - X = X^*$). We are interested in knowing when the map $A \rightarrow A^*$ is a Boolean algebra homomorphism from $R(X)$ into $R(X^*)$. It turns out that this is so precisely when $[cl_X(X - A)]^* = cl_{X^*}(X^* - A^*)$ for all $A \in R(X)$. We are accordingly motivated to make the following definition.

DEFINITION 2.1. Let $\mathscr{B}$ be a family of closed subsets of X. We shall call X a *$\mathscr{B}$-pleasant space* if $[cl_X(X - B)]^* = cl_{X^*}[X^* - B^*]$ for all $B \in \mathscr{B}$.

Before considering $R(X)$-pleasant spaces, we make several observations and notational conventions.

REMARKS 2.2. (i) We shall let $K(X)$, $L(X)$, and $\mathscr{Z}(X)$ denote respectively the families of all compact subsets, closed subsets, and zero-sets of the space X.

(ii) For any space X and any $A \in L(X)$, it is evident that $X^* = A^* \cup [cl_X(X - A)]^*$, and hence that $cl_{X^*}(X^* - A^*) \subseteq [cl_X(X - A)]^*$.

(iii) In [9], Mandelker defines a space to be "μ-compact" if the intersection of all the free maximal ideals of $C(X)$ is precisely those functions in $C(X)$ with compact support. It is an easy consequence of theorem 4.2 of [9] that X is μ-compact if and only if it is $\mathscr{Z}(X)$-pleasant. Thus the concept of a $\mathscr{B}$-pleasant space is a generalization of Mandelker's concept of a μ-pleasant space.

PROPOSITION 2.3. *The map* $A \rightarrow A^*$ *is a Boolean algebra homomorphism from* $R(X)$ *into* $R(X^*)$ *if and only if* X *is* $R(X)$*-pleasant.*

Proof. Assume that X is $R(X)$-pleasant and that $A \in R(X)$. Then

$$\begin{aligned} cl_{X^*}(\mathrm{int}_{X^*} A^*) &= cl_{X^*}[X^* - cl_{X^*}(X^* - A^*)] \\ &= cl_{X^*}[X^* - [cl_X(X - A)]^*] \\ &= [cl_X[X - cl_X(X - A)]]^* \\ &= [cl_X(\mathrm{int}_X A)]^* \\ &= A^* . \end{aligned}$$

Thus $A^* \in R(X^*)$ and the map $A \rightarrow A^*$ maps $R(X)$ into $R(X^*)$. If $A, B \in R(X)$, then

$$(A \vee B)^* = (A \cup B)^* = A^* \cup B^* = A^* \vee B^*$$

and

$$(A')^* = [cl_X(X - A)]^* = cl_{X^*}(X^* - A^*) = (A^*)' .$$

Thus our map preserves complements and finite joins, and hence is a Boolean algebra homomorphism.

Conversely, if $A \rightarrow A^*$ is a Boolean algebra homomorphism, then $(A^*)' = (A')^*$ for each $A \in R(X)$, and this implies that X is $R(X)$-pleasant.

We wish to show that the class of $L(X)$-pleasant spaces includes several familiar classes of spaces. We need some preliminary results. The topological boundary of a subset S of a space X will be denoted by $bd_X S$.

LEMMA 2.4. *Let X be any space. If $A \in R(X)$, $B \in L(X)$, and $A^* \subseteqq B^*$, then $cl_X(A - B)$ is pseudocompact.*

Proof. Put $S = cl_X(A - B)$ and $V = (\text{int}_X A) - B$. As $A \in R(X)$, it follows that $S = cl_X V$. Suppose that S is not pseudocompact, and choose $h \in C(S) - C^*(S)$. Then h is unbounded on V. It follows from 1.20 of [6] that V contains a countable set $D = (d_n)_{n \in N}$, C-embedded in S, such that h is unbounded on D. As D is countable it is realcompact and hence it follows from 8A.1 of [6] that D is closed in S and hence in X. As h is unbounded on D, D is not compact and so $D^* \neq \phi$

As S is completely regular, for each $n \in N$ we may choose $f_n \in C(S)$ such that $f_n(d_n) = 1$ and $f_n[bd_X S] = \{0\}$. Let $Z = \bigcap \{Z(f_n): n \in N\}$. By 1.14(a) of [6], Z is a zero-set of S that contains $bd_X S$ and is disjoint from D. As D is C-embedded in S, by 1.18 of [6] D is completely separated from Z in S. Hence there exists $f \in C(S)$ such that $f[D] = \{1\}$ and $f[Z] = \{0\}$. Define a real-valued function g on X by $g[X - S] = \{0\}$, $g | S = f$. As $f[bd_X S] = \{0\}$, it is evident that g belongs to $C(X)$ and completely separates D and B. Thus by 6.5 of [6] it follows that $D^* \cap B^* = \phi$. But $D \subseteqq A$ and so $\phi \neq D^* \subseteqq A^*$. This contradicts our assumption that $A^* \subseteqq B^*$. Hence $cl_X(A - B)$ is pseudocompact.

COROLLARY 2.5. *If $B \in L(X)$ and $X^* = B^*$, then $cl_X(X - B)$ is pseudocompact.*

The following proposition is a generalization of a portion of Theorem 4.2 of [9]. The proof that (iii) implies (iv) appears, in essence, both in [9] and in [11].

LEMMA 2.6. *Let* $\mathscr{B}$ *be a family of closed subsets of* X. *Assume that* $\mathscr{B}$ *is closed under finite unions and that* $\{B^*: B \in \mathscr{B}\}$ *is a base for the closed subsets of* X^*. *The following are then equivalent*:

(i) X *is* $\mathscr{B}$*-pleasant.*

(ii) *For any* $B \in \mathscr{B}$, $X^* = B^*$ *implies that* $cl_X(X - B)$ *is compact.*

(iii) *For any* $B \in \mathscr{B}$, $X^* \subseteq cl_{\beta X}B$ *implies* $X^* \subseteq \mathrm{int}_{\beta X}\, cl_{\beta X}B$.

(iv) *For any* $B \in \mathscr{B}$, $\mathrm{int}_{X^*} B^* = (\mathrm{int}_{\beta X}\, cl_{\beta X}B) - X$.

Proof. (i) *implies* (ii): If X is $\mathscr{B}$-pleasant and $X^* = B^*$ for $B \in \mathscr{B}$, then $[cl_X(X - B)]^* = cl_{X^*}(X^* - B^*) = \phi$ and so $cl_X(X - B)$ is compact.

(ii) *implies* (iii): If $X^* \subseteq cl_{\beta X}B$, by (ii) $cl_{\beta X}(X - B) \subseteq X$. Thus $X^* \subseteq \beta X - cl_{\beta X}(X - B) \subseteq cl_{\beta X}B$ and (iii) holds.

(iii) *implies* (iv): It is always true that $(\mathrm{int}_{\beta X}\, cl_{\beta X}B) - X \subseteq \mathrm{int}_{X^*}B^*$. Let $p \in \mathrm{int}_{X^*} B^*$. Since $\{B^*: B \in \mathscr{B}\}$ is a base for the closed subsets of X^*, there exists $A \in \mathscr{B}$ such that $p \in X^* - A^* \subseteq B^*$. Thus $X^* = (A \cup B)^*$ and as $\mathscr{B}$ is closed under finite unions, $A \cup B \in \mathscr{B}$. Hence by hypothesis $X^* \subseteq \mathrm{int}_{\beta X}\, cl_{\beta X}(A \cup B)$. Thus

$$p \in \mathrm{int}_{\beta X}\, cl_{\beta X}(A \cup B) \cap (\beta X - cl_{\beta X}A) \subseteq cl_{\beta X}B \ ,$$

so $p \in \mathrm{int}_{\beta X}\, cl_{\beta X}B$. Thus (iv) holds.

(iv) *implies* (i): If $B \in \mathscr{B}$, then

$$\begin{aligned} cl_{X^*}(X^* - B^*) &= X^* - \mathrm{int}_{X^*} B^* \\ &= (\beta X - \mathrm{int}_{\beta X}\, cl_{\beta X}B) - X \qquad \text{(by (iv))} \\ &= cl_{\beta X}(\beta X - cl_{\beta X}B) - X \\ &= cl_{\beta X}(X - B) - X \\ &= [cl_X(X - B)]^* \ , \end{aligned}$$

and the lemma is proved.

The conditions imposed on $\mathscr{B}$ in 2.6 are obviously satisfied if $\mathscr{B} = L(X)$ or $\mathscr{B} = \mathscr{Z}(X)$. It is easy to show that $\{[cl_X(\mathrm{int}_X Z)]^*: Z \in \mathscr{Z}(X)\}$ is always a base for the closed subsets of X^* (see [17], 2.10); hence $R(X)$ also satisfies the hypotheses imposed on $\mathscr{B}$ in 2.6.

THEOREM 2.7. *The class of all* $L(X)$*-pleasant spaces includes the class of all realcompact spaces, the class of all metric spaces, and the class of all nowhere locally compact spaces.*

Proof. Let $B \in L(X)$ and suppose $X^* = B^*$. By 2.5 $cl_X(X - B)$ is pseudocompact. If X is realcompact, its closed subspace $cl_X(X - B)$ is both realcompact and pseudocompact (8.10 of [6]), and hence is compact (5H.2 of [6]). If X is metric, then $cl_X(X - B)$ is a pseudocompact metric space and hence is compact (by 3D.2 of [6], every pseudocompact normal space is countably compact). In either case 2.6 implies that X is $L(X)$-pleasant.

If X is nowhere locally compact, choose $B \in L(X)$. If $X^* \subseteq cl_{\beta X}B$, by 1.10 $cl_{\beta X}B = \beta X$ and so $B = X$. Thus X is $L(X)$-pleasant by 2.6.

REMARKS 2.8. (i) Theorem 8.19 of [6], theorem 4.2 of [9], and 2.6 together imply that every realcompact space is $\mathscr{Z}(X)$-pleasant. Theorem 2.7 can be viewed as an extension of this result.

(ii) If X is metric then $\mathscr{Z}(X) = L(X)$. It is proved in [9] that every metric space is $\mathscr{Z}(X)$-pleasant, and hence $L(X)$-pleasant.

The following result is an immediate consequence of 2.3 and 2.7.

THEOREM 2.9. *If X is either realcompact, or metric, or nowhere locally compact, then the map $A \to A^*$ is a Boolean algebra homomorphism from $R(X)$ into $R(X^*)$.*

Since every locally compact σ-compact space is realcompact, 2.9 is a generalization of theorem 2.8 of [17].

TWO EXAMPLES 2.10. In this section we give an example of a space that is $\mathscr{Z}(X)$-pleasant but not $R(X)$-pleasant, and an example of a space that is $R(X)$-pleasant but not $\mathscr{Z}(X)$-pleasant.

(i) Let W denote the space of all countable ordinal numbers. Then $\beta W = W \cup \{\omega_1\}$, where ω_1 is the first uncountable ordinal. By 8.19 of [6] and 4.2 of [9], W is μ-compact and thus $\mathscr{Z}(W)$-pleasant (see 2.2 (iii) and 2.6). If $\alpha \in W$ let α^+ denote the smallest ordinal greater than α. Put $U = \{\alpha^+ : \alpha$ is a limit ordinal in $W\}$ and $V = \{\alpha^+ : \alpha \in U\}$. Then $cl_W U$ and $cl_W V$ are in $R(W)$. Evidently $U \cap V = \phi$, and so $cl_W U \wedge cl_W V = \phi$. As U and V are cofinal subsets of W, evidently $(cl_W U)^* \wedge (cl_W V)^* = \{\omega_1\} \wedge \{\omega_1\} = \{\omega_1\}$. Hence the map $A \to A^*$ is not a Boolean algebra homomorphism from $R(W)$ into $R(W^*)$, so by 2.3 W is not $R(W)$-pleasant.

(ii) Let F be a finite subset of $\beta N - N$, and put $X = \beta N - F$; then $\beta X = \beta N$. By 1.6 X is extremally disconnected, and so every regular closed subset of X is open-and-closed in X. Hence if $A \in R(X)$, $[cl_X(X - A)]^* = X^* - A^* = cl_{X^*}(X^* - A^*)$ (since $X^* = F$). Thus X is $R(X)$-pleasant.

As βN is an infinite compact space, by 4K.1 and 4L.1 of [6] there exists $Z \in \mathscr{Z}(\beta N)$ such that $Z - \text{int}_{\beta N} Z \neq \phi$. Choose $p \in Z - \text{int}_{\beta N} Z$; evidently $p \in \beta N - N$, so without loss of generality assume that $Z \cap F = \{p\}$. Put $H = Z \cap X$; then $H \in \mathscr{Z}(X)$. If $p \notin cl_{\beta N}H$, there exists $f \in C(\beta N)$ such that $f(p) = 0$ and $f[cl_{\beta N}H] = \{1\}$. Thus $\{p\} = Z \cap Z(f)$, which contradicts 9.6 of [6]. Hence $\{p\} = cl_{\beta X}H - X = H^*$. Thus

$cl_{X^*}(X^* - H^*) = F - \{p\}$. But $cl_{\beta X}(X - H) = cl_{\beta N}(\beta N - Z)$, which contains p since $p \notin \mathrm{int}_{\beta N} Z$. Thus $p \in [cl_X(X - H)]^*$ and so X is not $\mathscr{Z}(X)$-pleasant.

Recall that $bd_X A$ denotes the topological boundary in X of a subset A of X.

PROPOSITION 2.11. *Let X be an $L(X)$-pleasant space, and let $A \in L(X)$. Then $(bd_X A)^* \subseteqq bd_{X^*} A^*$. If X is normal, then $(bd_X A)^* = bd_{X^*} A^*$.*

Proof. Since A is closed in X we have

$$\begin{aligned}(bd_X A)^* &= [A \cap cl_X(X - A)]^* \\ &\subseteqq A^* \cap [cl_X(X - A)]^* \\ &= A^* \cap cl_{X^*}(X^* - A^*) \\ &= bd_{X^*} A^* \ .\end{aligned}$$

If X is normal, a modification of the argument used in 6.4 of [6] shows that the above inclusion is in fact an equality.

3. **Co-absolutes of βX-X.** Let X be any completely regular Hausdorff space. It is evident that the family $K(X) \cap R(X)$ is an ideal of the Boolean algebra $R(X)$. Let us denote the factor algebra $R(X)/K(X) \cap R(X)$ by $\mathscr{A}(X)$. If X is $R(X)$-pleasant, then obviously $R(X) \cap K(X)$ is the kernel of the homomorphism defined in 2.3, and hence $\{A^*: A \in R(X)\}$ is isomorphic to $\mathscr{A}(X)$. For each $A \in R(X)$ this isomorphism takes the subset A^* of $\beta X - X$ to the equivalence class $[A]$ of $\mathscr{A}(X)$.

It is an immediate consequence of 3.15(b) of [6] that $K(X) \cap R(X) = K(X) \cap R(\beta X)$, and this equality will be used repeatedly. Throughout this section k will denote the map from $E(\beta X)$ onto βX defined in 1.4 and 1.7, and $\beta X - X$ will be denoted by X^*.

THEOREM 3.1. *Let X be any completely regular Hausdorff space.*

(i) *$S(\mathscr{A}(X))$ and $cl_{\beta E(X)}[\beta E(X) - E(X)]$ are homeomorphic.*

(ii) *The space X is $R(X)$-pleasant if and only if the restriction of k to $\beta E(X) - E(X)$ is a perfect irreducible map from $\beta E(X) - E(X)$ onto X^*.*

Proof. (i) Let Δ be an ideal of the Boolean algebra U, let λ be the canonical isomorphism defined in 1.3, and put

$$H = S(U) - \bigcup \{\lambda(u): u \in \Delta\} \ .$$

In § 10 of [15] it is shown that the map g defined by $g([u]) = \lambda(u) \cap H$ is a Boolean algebra isomorphism from the factor algebra U/Δ to the open-and-closed subsets of H. Since H is closed in $S(U)$ and hence compact and totally disconnected, the well-known duality between Boolean algebras and compact totally disconnected spaces implies that H and $S(U/\Delta)$ are homeomorphic.

Now let $U = R(\beta X)$ and $\Delta = R(\beta X) \cap K(X)$. The isomorphism defined in 1.2 fixes Δ elementwise, and so U/Δ is isomorphic to $\mathscr{A}(X)$. Hence $S(U/\Delta)$ and $S(\mathscr{A}(X))$ are homeomorphic, and so by the above remarks it suffices to show that in this case $H = cl_{\beta E(X)}[\beta E(X) - E(X)]$.

Evidently $S(U) = S(R(\beta X)) = E(\beta X) = \beta E(X)$ (see 1.8), and $H = \beta E(X) - \bigcup\{\lambda(A): A \in R(\beta X) \cap K(X)\}$. Suppose that $p \in \beta E(X) - E(X)$. Thus $k(p) \in X^*$. If $A \in R(\beta X)$ and $p \in \lambda(A)$, then $k(p) \in k[\lambda(A)] = A$ and so $A - X \neq \phi$. Thus $A \notin R(\beta X) \cap K(X)$ and so $p \in H$. Hence $\beta E(X) - E(X) \subseteqq H$, and as H is closed in $\beta E(X)$ we have

$$cl_{\beta E(X)}[\beta E(X) - E(X)] \subseteqq H .$$

Conversely, if $p \notin cl_{\beta E(X)}[\beta E(X) - E(X)]$, by 1.10 $p \in W(E(X))$. Hence there is a compact $E(X)$-neighborhood A of p, and as $\beta(E(X))$ is totally disconnected we may assume that A is open-and-closed in $\beta(E(X))$. By 1.3 $A = \lambda(F)$ for some $F \in R(\beta X)$. As $A \subseteqq k^{\leftarrow}[X]$, it follows that $k[A] = F \subseteqq X$; thus $F \in R(\beta X) \cap K(X)$. Hence $p \notin H$ and so $H = cl_{\beta E(X)}[\beta E(X) - E(X)]$. Hence (i) is true.

(ii) Since k is perfect and $\beta E(X) - E(X) = k^{\leftarrow}[X^*]$, evidently $k \mid \beta E(X) - E(X)$ is perfect. The only question is whether this restriction of k is irreducible.

Suppose that X is $R(X)$-pleasant, and let B be a proper closed subset of $\beta E(X) - E(X)$. Then we can find an open-and-closed subset F of $\beta E(X)$ such that $B \subseteqq F - E(X)$ and $[\beta E(X) - E(X)] - F \neq \phi$. By 1.2 and 1.3 there exists $A \in R(X)$ such that $F = \lambda(cl_{\beta X}A)$. Thus $k[B] \subseteqq k[F - E(X)] = cl_{\beta X}A - X = A^*$. Suppose that $A^* = X^*$. Since X is $R(X)$-pleasant, by 2.6 $cl_X(X - A)$ is compact, and hence a member of $R(\beta X)$; i.e. $cl_{\beta X}(\beta X - cl_{\beta X}A) \in R(\beta X) \cap K(X)$. As λ preserves Boolean-algebraic complements, it follows that

$$\beta E(X) - F = \lambda(cl_{\beta X}(\beta X - cl_{\beta X}A)) \subseteqq \beta E(X) - H .$$

Hence $\beta E(X) - F \subseteqq E(X)$ which contradicts our choice of F. Hence $A^* \neq X^*$, $k[B]$ is a proper closed subset of X^*, and $k \mid \beta E(X) - E(X)$ is irreducible.

Conversely, assume that $k \mid \beta E(X) - E(X)$ is irreducible. We shall prove that the contrapositive of 2.6 (ii) holds. Let $A \in R(X)$ and suppose that $cl_X(X - A)$ is not compact. Then $cl_{\beta X}(\beta X - cl_{\beta X}A) - X \neq \phi$, and so $\lambda(cl_{\beta X}(\beta X - cl_{\beta X}A)) - E(X) \neq \phi$. Thus

$$[\beta E(X) - E(X)] - \lambda(cl_{\beta X}(\beta X - cl_{\beta X}A))$$

is a proper closed subset of $\beta E(X) - E(X)$; in other words, $\lambda(cl_{\beta X}A) - E(X)$ is a proper closed subset of $\beta E(X) - E(X)$. By hypothesis this implies that $k[\lambda(cl_{\beta X}A) - E(X)]$, i.e. A^*, is a proper closed subset of X^*. This establishes the contrapositive of 2.6(ii) and hence by 2.6 X is $R(X)$-pleasant.

COROLLARY 3.2. *If X is a metric space, or nowhere locally compact, or realcompact, then there is an irreducible perfect mapping from $\beta E(X) - E(X)$ onto $\beta X - X$, and these two spaces are co-absolute.*

Proof. This follows immediately from 3.1, 2.7, and 1.9.

We now consider co-absolutes of specific classes of spaces. Our first result is obtained by elementary means and does not require the machinery developed in § 2.

THEOREM 3.3. *Let X be nowhere locally compact. Then:*
(i) *$\beta E(X) - E(X)$ is extremally disconnected.*
(ii) *If X is extremally disconnected, so is X^**
(iii) *$E(X^*) = \beta E(X) - E(X)$ (up to homeomorphism).*

Proof. (i) As X is nowhere locally compact, by 1.10 X^* is dense in βX. Hence by 1.5 $k^{\leftarrow}[X^*]$, which is $\beta E(X) - E(X)$, is dense in $\beta E(X)$. Thus by 1.6 $\beta E(X) - E(X)$ is extremally disconnected.
(ii) This follows immediately from (i).
(iii) Either using 3.2 or by direct calculation, we see that the restriction of k to $k^{\leftarrow}[X^*]$ is a perfect irreducible map from $k^{\leftarrow}[X^*]$ onto X^*, and so by 1.8(i) it follows that $E(X^*) = \beta E(X) - E(X)$.

COROLLARY 3.4. [CH]. *Let $\boldsymbol{Q}$ and $\boldsymbol{I}$ denote respectively the spaces of rational and irrational numbers. Then $E(\boldsymbol{Q}^*)$ can be partitioned into two disjoint subspaces, one homeomorphic to $E(\boldsymbol{I})$ and the other homeomorphic to $\boldsymbol{N}^*$. The preceding statement is also valid when "$\boldsymbol{Q}$" and "$\boldsymbol{I}$" are interchanged.*

Proof. Since $\boldsymbol{Q}$ is a dense subspace of the space $\boldsymbol{R}$ of real numbers, by 1.5 the space $k^{\leftarrow}[\boldsymbol{Q}]$ is a dense subspace of $E(\beta\boldsymbol{R})$, and hence is extremally disconnected (see 1.6(ii)). (In this case k is the canonical irreducible map from $E(\beta\boldsymbol{R})$ onto $\beta\boldsymbol{R}$). Hence by 1.8(i) $k^{\leftarrow}[\boldsymbol{Q}]$ may be identified with $E(\boldsymbol{Q})$, and by 1.6(ii) we may identify $E(\beta\boldsymbol{R})$ with $\beta E(\boldsymbol{Q})$. Similarly $k^{\leftarrow}[\boldsymbol{I}]$ may be identified with $E(\boldsymbol{I})$. Thus

$$\begin{aligned}\beta E(\boldsymbol{Q}) - E(\boldsymbol{Q}) &= k^{\leftarrow}[\beta\boldsymbol{R} - \boldsymbol{Q}] \\ &= k^{\leftarrow}[\boldsymbol{R}^* \cup \boldsymbol{I}] \\ &= [\beta E(\boldsymbol{R}) - E(\boldsymbol{R})] \cup E(\boldsymbol{I}) \,.\end{aligned}$$

Since $|R(\boldsymbol{R})| = c$, and since $k|E(\boldsymbol{R})$ is a perfect irreducible map from $E(\boldsymbol{R})$ onto $\boldsymbol{R}$, it follows that $E(\boldsymbol{R})$ is locally compact, σ-compact, and noncompact, and has a basis of c open-and-closed sets. Hence by 1.12 $\beta E(\boldsymbol{R}) - E(\boldsymbol{R})$ is homeomorphic to N^*, and so $E(\boldsymbol{Q}^*) = E(\boldsymbol{I}) \cup N^*$. As "$\boldsymbol{Q}$" and "$\boldsymbol{I}$" can be interchanged in the above argument, the corollary follows.

In Theorem 2.19 of [17], we have proved [CH] that if X is a locally compact, σ-compact, noncompact Hausdorff space and if $|\mathscr{X}(X)| = c$, then there is an irreducible map from N^* onto X^*. The following result is a slightly modified version of this. Note that its proof is considerably more efficient than that employed in 2.19 of [17].

THEOREM 3.5. [CH]. *If X is locally compact, σ-compact, and noncompact, and if $|R(X)| = c$, then there is an irreducible map from N^* onto X^*.*

Proof. Since $k|E(X)$ is a perfect map from $E(X)$ onto X, our assumptions imply that $E(X)$ is locally compact, σ-compact, and noncompact. Since $|R(X)| = c$, $E(X)$ has a basis of c open-and-closed subsets, so by 1.12 $\beta E(X) - E(X)$ is homeomorphic to N^*. But X is σ-compact and hence realcompact (see 8.2 of [6]). Hence X is $R(X)$-pleasant, and our theorem follows from 3.1(ii).

REMARK 3.6. Let X be locally compact, realcompact and noncompact. As $k|E(X)$ is a perfect map, it follows that $E(X)$ is locally compact and noncompact. As X is realcompact, so is $E(X)$ by 8.13 of [6]. It follows from 3.1 of [3] that

$$\mathscr{X}(\beta E(X) - E(X)) \subsetneqq R(\beta E(X) - E(X)) .$$

In an extremally disconnected space every regular closed set is open-and-closed; hence if $\beta E(X) - E(X)$ were extremally disconnected, every zero-set of it would be open-and-closed. It follows from 4J and 9.12 of [6] that $\beta E(X) - E(X)$ would be an infinite compact P-space, which by 4K of [6] is impossible. Hence $\beta E(X) - E(X)$ is not extremally disconnected, and although there is a perfect irreducible map from $\beta E(X) - E(X)$ onto X^* (see 3.1), nonetheless $\beta E(X) - E(X) \neq E(X^*)$.

We now identify some co-absolutes of X^* when X is a locally compact metric space.

THEOREM 3.7. *Let X be a locally compact, noncompact metric* *space without isolated points, and let δX denote the smallest cardinal number m such that X has a dense subset of cardinality m. Then:*

(i) *There is an irreducible perfect map from Y^* onto X^*, where Y is the free union of δX copies of E ([0, 1]).*

(ii) [CH] *If either $\delta X = \aleph_0$ or $\delta X = \aleph_1$, then X^* and N^* are co-absolute.*

Proof. (i) A theorem of A. H. Stone (see 9.5.3 of [2], for example) states that every metric space is paracompact; it is also known ([2], 11.7.3) that every locally compact paracompact Hausdorff space is a free union of locally compact σ-compact Hausdorff spaces. Since a σ-compact metric space is separable, it follows easily that either $\delta X = \aleph_0$ and X is σ-compact, or else $\delta X > \aleph_0$ and X is the free union of δX locally compact, σ-compact, noncompact metric spaces.

Suppose first that $\delta X = \aleph_0$. As $k|E(X)$ is a perfect map from $E(X)$ onto X, it follows that $E(X)$ is locally compact, σ-compact and noncompact. As X has no isolated points and $k|E(X)$ is irreducible, it follows that $E(X)$ has no isolated points. By 11.7.2 of [2], since $E(X)$ is locally compact and σ-compact it can be written in the form $\bigcup_{n \in N} cl_{E(X)} V(n)$, where for each $n \in N$, $V(n)$ is open, $cl_{E(X)} V(n)$ is compact, and $cl_{E(X)} V(n) \subseteqq V(n+1)$. As $E(X)$ is noncompact, the last inclusion may be assumed to be proper. Put $B(0) = cl_{E(X)} V(0)$ and $B(n) = cl_{E(X)} V(n) - cl_{E(X)} V(n-1)$ if $n \geqq 1$. As $E(X)$ is extremally disconnected, its regular closed sets are all open-and-closed; hence $\{B(n): n \in N\}$ is a family of compact, pairwise disjoint subspaces of $E(X)$ whose union is $E(X)$. As each $B(n)$ is open-and-closed in $E(X)$ it is extremally disconnected (see 1H of [6]). As $E(X)$ has no isolated points, neither have any of the $B(n)$. The restriction of $k|E(X)$ to each $B(n)$ is easily seen to be an irreducible map from $B(n)$ onto $k[B(n)]$; hence $k[B(n)]$ is a compact metric space without isolated points whose absolute is $B(n)$. But any two compact metric spaces without isolated points have homeomorphic absolutes (see § 9C of [15]); hence each $B(n)$ is homeomorphic to $E([0, 1])$. Thus $E(X)$ is expressible as a free union of $\aleph_0$ copies of $E([0, 1])$.

If X were not σ-compact, then as noted above, $X = \dot{\bigcup}_{\alpha \in \Sigma} X(\alpha)$, where each $X(\alpha)$ is locally compact, σ-compact, and noncompact, and $|\Sigma| = \delta X$. Thus $E(X) = \dot{\bigcup}_{\alpha \in \Sigma} k^{\leftarrow}[X(\alpha)]$. As each $k^{\leftarrow}[X(\alpha)]$ is open in $E(X)$ and thus is extremally disconnected, the argument of the previous paragraph shows that $E(X)$ is a free union of $\delta X \cdot \aleph_0 = \delta X$ copies of $E([0, 1])$.

In either case, since X is metric by 2.9 and 3.1 there is an irreducible map from $\beta E(X) - E(X)$ onto X^*. Hence (i) is true.

(ii) If $\delta X = \aleph_0$, then $|R(X)| = c$ and the proof of 3.5 immediately shows that $\beta E(X) - E(X)$ is homeomorphic to N^*. Thus by 1.9 N^* and X^* are co-absolute.

If $\delta X = \aleph_1$, well-order the $\aleph_1$ copies of $E([0, 1])$ whose free union is $E(X)$ and write $E(X) = \dot{\bigcup}_{\alpha<\omega_1} F(\alpha)$, where each $F(\alpha)$ is a copy of $E([0, 1])$. (ω_1 is the first uncountable ordinal.) Put $Y = E(X)$. Let $(\lambda(\alpha))\ \alpha < \omega_1$ be a well-ordering of the countable limit ordinals and for each $\alpha < \omega_1$, put $G(\alpha) = \bigcup\{F(\gamma): \gamma < \lambda(\alpha)\}$ and $\Omega(\alpha) = cl_{\beta Y} G(\alpha) - Y$. Finally, put $J = \bigcup_{\alpha<\omega_1} \Omega(\alpha)$. By 1.12 each $\Omega(\alpha)$ is homeomorphic to N^*, and since $\gamma < \alpha$ implies $G(\alpha) - G(\gamma)$ is not compact, for each α we have $\Omega(\alpha) \supsetneqq \bigcup_{\gamma<\alpha} \Omega(\gamma)$. Since each $G(\alpha)$ is open-and-closed in Y^*, each $\Omega(\alpha)$ is open-and-closed in J. Hence by 1.13 (ii) J is homeomorphic to Ω. But J is evidently dense in Y^*, so Y^* is a compactification of J. By 1.13 (iii) N^* is homeomorphic to the one-point compactification of J, so there is an irreducible map from Y^* onto N^*. Thus by 1.9 Y^* and N^* are co-absolute. But by part (i) of this theorem, Y^* and X^* are co-absolute; hence X^* and N^* are co-absolute.

4. Absolutes and remote points.

The main result in this section is 4.5, which identifies the absolute of a compact metric space without isolated points with the Stone-Cech compactification of a certain set of remote points (under assumption of the continuum hypothesis).

A point $p \in \beta X$ is called a *remote point* of βX if p is not in the βX-closure of any discrete subspace of X. We shall denote the set of remote points of βX by $T(\beta X)$. Remote points have been studied by several authors (see [4], [11], and [13]). One of the better characterizations of $T(\beta X)$ has been given by Plank in Theorems 5.3 and 5.5 of [11]. The following is a statement of these results of Plank.

THEOREM 4.1. *Let X be a metric space without isolated points. Then*

$$T(\beta X) = \bigcap \{X^* - bd_{X^*} Z^*: Z \in \mathscr{Z}(X)\} .$$

If in addition X is separable and noncompact, and if the continuum hypothesis is assumed, then $|T(\beta X)| = 2^c$ and $T(\beta X)$ is dense in X^*.

As before, let k denote the canonical irreducible map from $E(\beta X)$ onto βX and let λ denote the Boolean algebra isomorphism from $R(\beta X)$ onto the open-and-closed subsets of $E(\beta X)$.

LEMMA 4.2. *Let X be a metric space without isolated points. If $p \in T(\beta X)$ then $|k^{\leftarrow}(p)| = 1$.*

Proof. Let $p \in X^*$. Suppose that x and y are distinct points of $\beta E(X)$ and that $k(x) = k(y) = p$. By 1.2 there exists $A \in R(X)$ such that $x \in \lambda(cl_{\beta X}A)$ and $y \in \beta E(X) - \lambda(cl_{\beta X}A)$. This latter set is

$$\lambda(cl_{\beta X}(\beta X - cl_{\beta X}A)) ,$$

since λ is an isomorphism. Thus $p \in [cl_{\beta X}(\beta X - cl_{\beta X}A) \cap cl_{\beta X}A] - X = [cl_X(X - A)]^* \cap A^*$. As X is $L(X)$-pleasant, it follows that

$$p \in cl_{X^*}(X^* - A^*) \cap A^* = bd_{X^*}A^* .$$

As $R(X) \subseteq \mathscr{Z}(X)$ since X is metric, we have $A \in \mathscr{Z}(X)$ and so by 4.1 $p \notin T(\beta X)$. The lemma now follows.

THEOREM 4.3. *Let X be a metric space without isolated points. Then $T(\beta X)$ and $k^{\leftarrow}[T(\beta X)]$ are homeomorphic.*

Proof. By 4.2 $k|k^{\leftarrow}[T(\beta X)]$ is a one-to-one continuous mapping from $k^{\leftarrow}[T(\beta X)]$ onto $T(\beta X)$. As k is a closed mapping from $\beta E(X)$ onto βX, evidently $k|k^{\leftarrow}[T(\beta X)]$ is a closed mapping from $k^{\leftarrow}[T(\beta X)]$ onto $T(\beta X)$, and hence is a homeomorphism.

COROLLARY 4.4. *Let X be a metric space without isolated points. If $T(\beta X)$ is dense in X^*, then $\beta E(X) - E(X)$ contains a dense homeomorphic copy of $T(\beta X)$.*

Proof. Since X is metric and hence $R(X)$-pleasant, by 3.1 $k|\beta E(X) - E(X)$ is an irreducible map from $\beta E(X) - E(X)$ onto X^*. Hence by 1.5 $k^{\leftarrow}[T(\beta X)]$ is dense in $\beta E(X) - E(X)$, and the corollary follows from 4.3.

THEOREM 4.5. [CH]. *Let X be a separable, nowhere locally compact metric space without isolated points. Then $T(\beta X)$ is extremally disconnected and $E(\beta X) = \beta[T(\beta X)]$.*

Proof. By 4.1 $T(\beta X)$ is dense in X^*, which in turn is dense in βX by 1.10. Hence by 1.5 $k^{\leftarrow}[T(\beta X)]$ is dense in the extremally disconnected space $\beta E(X)$. By 1.6 and 4.3 it follows that $T(\beta X)$ is extremally disconnected and that $\beta E(X) = \beta[T(\beta X)]$.

As remarked in the proof of 3.7, all compact separable metric spaces without isolated points are co-absolute: since every separable metric space without isolated points has a metric compactification, all compactifications of separable metric spaces without isolated points are co-absolute (see 1.2). Hence, for example, it follows from 4.5 that $E([0, 1]) = \beta[T(\beta \boldsymbol{Q})]$, where $\boldsymbol{Q}$ denotes the rationals.

REMARK 4.6. [CH]. The assumption in 4.5 that X is nowhere locally compact cannot be dropped. To see this, assume that X is a locally compact, noncompact separable metric space without isolated points. An easy generalization of the proof of 6.2 of [11] shows that there exists $p \in T(\beta X)$ that is not a P-point of X^*. Hence there exists $Z \in \mathscr{Z}(X^*)$ such that $p \in bd_{X^*}Z$. Since X is locally compact and realcompact, by 3.1 of [3] $Z = cl_{X^*}(\text{int}_{X^*} Z)$. Now consider $T(\beta X) \cap \text{int}_{X^*} Z$ and $T(\beta X) - Z$. The former is an open subset of $T(\beta X)$, the latter is a cozero-set of $T(\beta X)$, and they are disjoint. As $T(\beta X)$ is dense in X^*, p belongs to the $T(\beta X)$-closure of both of these sets. Hence $T(\beta X)$ cannot even be basically disconnected (see 1H of [6]), let alone extremally disconnected.

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