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ATOMIC AND DIFFUSE FUNCTIONALS ON A C^*-ALGEBRA

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It is shown that the notion and basic properties of atomic and diffuse measures have exact analogues in the theory of functionals on operator algebras.

We regard a C^*-algebra A as the non-commutative analogue of an algebra $C_0(T)$ of continuous functions vanishing at infinity on some locally compact space T. It has been shown in [3], [4], [5], [13] and [14] that, at least when A is separable, there is also a reasonable analogue of the Borel functions on T, namely the σ-closure $\mathscr{B}_A$ of A. In this paper we prove that $\mathscr{B}_A$ has an abundance of minimal projections, corresponding to points in T, and thus the notion of atomic and diffuse measures on T can be generalized to the non-commutative situation, since the diffuse measures are characterized as those measures that vanish at all points of T.

Let A be a separable C^*-algebra and denote by P the set of pure states of A. Choose in P a maximal set $\{f_t : t \in T\}$ of pairwise inequivalent pure states of A. If (π_t, H_t) denotes the irreducible representation of A corresponding to f_t, we define the *reduced atomic representation* ρ of A as operators on the Hilbert space $H_a = \Sigma^{\oplus} H_t$ by

$$\rho(x)(\Sigma^{\oplus}\xi_t) = \Sigma^{\oplus}\pi_t(x)\xi_t \ .$$

The reduced atomic representation is faithful and each pure state of A is a vector functional from H_a. Since any other choice of a maximal set in P will give an equivalent representation, the reduced atomic representation is essentially unique. In particular the cardinality of T is uniquely determined as the cardinality of the set $\hat{A}$ of equivalence classes of irreducible representations of A. In what follows we shall identify A with its image $\rho(A)$.

Let $\mathscr{B}_A^R$ denote the monotone σ-closure of the self-adjoint part of A. Then $\mathscr{B}_A (= \mathscr{B}_A^R + i\mathscr{B}_A^R)$ is a C^*-algebra in $B(H_a)$ called the *Baire operators* of A [14, Theorem 1]. Each representation (π, H) of A extends to a σ-normal representation of $\mathscr{B}_A$ ([3, Theorem 3.2]) such that $\pi(\mathscr{B}_A)$ is the monotone σ-closure of $\pi(A)$ in $B(H)$ ([13, Proposition 4.2]). In particular, if (π, H) is irreducible we have $\pi(\mathscr{B}_A) = B(H)$.

THEOREM 1. *There is a bijective correspondence between pure states of A and minimal projections of $\mathscr{B}_A$.*

Proof. Since A is separable, each point f in P is a closed G_δ set. Hence there is a peaking element x in A^+, with $\|x\| = 1$, such that $f(x) = 1$ and $g(x) < 1$ for each state $g \neq f$ ([7, Theorem 9]). We have $x^n \searrow p$, where p is a projection in $\mathscr{B}_A$. If ξ is a unit vector in H_a representing f, then $(p\xi \mid \xi) = 1$. For any unit vector η in H_a which is not a multible of ξ we have $(y\eta \mid \eta) \neq (y\xi \mid \xi)$ for some y in A; hence $(p\eta \mid \eta) < 1$. It follows that p is the one-dimensional projection on the subspace spanned by ξ, and consequently minimal.

If, conversely, p is a minimal projection in $\mathscr{B}_A$, then $p\mathscr{B}_A p$ is a commutative algebra, isomorphic with the complex field. The functional f on $\mathscr{B}_A$, defined by $f(x) = pxp$, is the unique state extension of the identity map on $p\mathscr{B}_A p$ ([11, Theorem 1.2]), which is pure; and therefore f is a σ-normal pure state of $\mathscr{B}_A$. But then $f \in P$. If ξ is a unit vector in pH_a, then $f(x) = (x\xi \mid \xi)$ and it follows from the first part of the proof that p is one-dimensional. Thus the correspondence is bijective, and the theorem follows.

COROLLARY 2. *There is a bijective correspondence between elements in $\hat{A}$ and minimal projections in the center $\mathscr{C}$ of $\mathscr{B}_A$.*

REMARK. Since $\mathscr{B}_A$ has a unit, we can identity $\mathscr{C}$ with a σ-closed algebra of bounded functions on $\hat{A}$. The projections in $\mathscr{C}$ then constitute the sets in a σ-field on $\hat{A}$, called the *Davies-Borel structure* on $\hat{A}$. The above corollary tells us that points in $\hat{A}$ are Davies-Borel sets (cf. [5, Theorem 2.9]).

Let $\mathscr{F}$ denote the smallest monotone closed C^*-subalgebra of $\mathscr{B}_A$, which contains all minimal projections of $\mathscr{B}_A$. Then $\mathscr{F}$ can be indentified with the set of operators x in the direct sum $\Sigma^{\oplus}_{t \in T} B(H_t)$, such that $x_t = 0$ except for countably many t in T. In particular, $\mathscr{F}$ is an ideal of $\mathscr{B}_A$.

DEFINITION. A positive functional f on A is called *atomic* if there is a projection p in $\mathscr{F}$ such that $f(1 - p) = 0$; f is called *diffuse* if it vanishes at all minimal projections of $\mathscr{B}_A$.

PROPOSITION 3. *Each positive functional f on A has a unique decomposition $f = f_a + f_d$ such that f_a is atomic and f_d is diffuse. Moreover, f_a and f_d are centrally orthogonal.*

Proof. Let α be the norm of the functional $f \mid \mathscr{F}$ on $\mathscr{F}$. There is then a sequence $\{p_n\}$ in the unit ball of $\mathscr{F}^+$ such that $f(p_n) \nearrow \alpha$. Replacing p_n with its range projection, we may assume that all p_n are projections. Let p be the central support of $\vee p_n$. Then

$p \in \mathscr{C} \cap \mathscr{F}$ and $f(p) = \alpha$. Put $f_a(x) = f(px)$ and $f_d(x) = f((1-p)x)$. Then $f_a(1-p) = 0$, hence f_a is atomic; and for each x in $\mathscr{F}^+$, with $\|x\| \leqq 1$, we have $f(x(1-p)+p) \leqq \alpha$, hence $f_d(x) = 0$, and thus f_d is diffuse. By construction f_a and f_d are centrally orthogonal.

REMARK. We see from the proof that a bounded functional will be atomic (respectively diffuse) if and only if the restriction to $\mathscr{C}$ induces an atomic (respectively diffuse) measure on the Davies-Borel structure of $\hat{A}$.

PROPOSITION 4. *A positive functional f on A is atomic exactly if it has the form $f = \Sigma \alpha_n f_n$, with f_n in P. Moreover, the summands can be chosen such that $f_n \perp f_m$ for $n \neq m$.*

Proof. If f is atomic and $f(1-p) = 0$ for a projection p in $\mathscr{F}$, then, assuming that $p \in \mathscr{C}$, we have $p = \Sigma p_k$, where each p_k is a minimal projection in $\mathscr{C}$. Thus $f = \Sigma f_k$, where $f_k(x) = f(p_k x)$, and each f_k is a σ-normal functional on $B(H_k)(= p_k \mathscr{B}_A)$. There is then for each k an orthonormal basis $\{\xi_{nk}\}$ for H_k and a sequence $\{\alpha_{nk}\}$ of positive constants such that $f_k(x) = \Sigma \alpha_{nk}(x\xi_{nk} \mid \xi_{nk})$, for all x in $\mathscr{B}_A$. If f_{nk} denotes the pure state of A determined by ξ_{nk}, then $f = \Sigma \alpha_{nk} f_{nk}$, and since the f_{nk}'s are supported by pairwise orthogonal (minimal) projections in $\mathscr{B}_A$, they are themselves orthogonal.

Conversely, if $f = \Sigma \alpha_n f_n$, with all f_n in P, then for each n let p_n be the minimal projection in $\mathscr{B}_A$ such that $f_n(p_n) = 1$. Then $p = \vee p_n \in \mathscr{F}$ and $f(1-p) = 0$. Hence f is atomic, completing the proof.

DEFINITION. An *atom* for a positive functional f on A is a projection p in $\mathscr{B}_A$, such that $f(p) > 0$, but $f(q)f(p-q) = 0$, for any projection q in $\mathscr{B}_A$ smaller than p.

PROPOSITION 5. *A positive functional is diffuse exactly if it has no atoms.*

Proof. Assume that p is an atom for f. Then the state g of A given by $g(x) = f(p)^{-1}f(pxp)$ is multiplicative, hence pure, on $p\mathscr{B}_A p$. Since g is the unique state extension from $p\mathscr{B}_A p$ to $\mathscr{B}_A$ ([11, Theorem 1.2]), we conclude that $g \in P$. There is then a minimal projection q in $\mathscr{B}_A$ such that $g(q) = 1$. Since $pqp \in \mathscr{F}$, f is not diffuse.

Conversely, if $f = f_a + f_d$, with $f_a \neq 0$, then from Proposition 4 there is a minimal projection p in $\mathscr{B}_A$ such that $f_a(p) > 0$. Clearly p is an atom for f, completing the proof.

The following proposition generalizes a well-known theorem from measure theory.

PROPOSITION 6. *If f is a diffuse functional on A then, corresponding to each projection p in $\mathscr{B}_A$ and each positive $\alpha < f(p)$, there is a projection q in $\mathscr{B}_A$, with $q \leqq p$ and $f(q) = \alpha$.*

Proof. Since p is not an atom for f, there is a projection $p_0 < p$ such that $0 < f(p_0) < f(p)$. Then either $f(p_0) \leqq \frac{1}{2} f(p)$ or $f(p - p_0) \leqq \frac{1}{2} f(p)$. Repeating this procedure we see that for any $\varepsilon > 0$ there is a projection $q_0 \leqq p$ such that $0 < f(q_0) < \varepsilon$.

Now let (π_f, H_f) be the representation of A corresponding to f, and let ξ_f be a vector in H_f such that $f(x) = (\pi_f(x)\xi_f | \xi_f)$, for all x in $\mathscr{B}_A$. Let $\{p_i\}$ be a maximal family of nonzero, orthogonal projections in $\pi_f(\mathscr{B}_A)$ such that $\Sigma\, p_i \leqq \pi_f(p)$, and $\Sigma(p_i\xi_f | \xi_f) \leqq \alpha$ for all finite sums. Since $\pi_f(\mathscr{B}_A)$ is a von Neumann algebra ([8, Theorem 2]), $p_0 = \Sigma\, p_i \in \pi_f(\mathscr{B}_A)$. It follows from spectral theory that there is a projection q in $\mathscr{B}_A$, with $q \leqq p$ and $\pi_f(q) = p_0$.

If we had $f(q) < \alpha$, then from the above we could find a projection $q_0 \leqq p - q$, such that $0 < f(q_0) < \alpha - f(q)$. But then $\pi_f(q_0)$ could be adjoined to the family $\{p_i\}$; a contradiction. Therefore $f(q) = \alpha$, completing the proof.

For the sake of convenience we have stated all theorems in terms of bounded functionals. However, it is quite easy to extend the results to a large and important class of unbounded functionals.

Let f be an extended valued, positive, σ-normal functional on $\mathscr{B}_A^+$ which is majorized by an invariant convex functional ρ on $\mathscr{B}_A^+$ (see [13, § 2] for definition). Assume furthermore that there is a sequence $\{e_n\}$ in $\mathscr{B}_A^+$ such that $\Sigma\, e_n = 1$, and $\rho(e_n) < \infty$ for each n. These conditions are satisfied if f is a σ-finite σ-trace on $\mathscr{B}_A$ ([4], [5]) — take $\rho = f$ — or f is a C^*-integral of A ([1, Proposition 4.4] and [13, Theorem 2.5]).

For each x in $\mathscr{B}_A$ and each n, the element $e_n x$ belongs to the set of definition for f, and

$$|f(e_n x)|^2 \leqq f(e_n) f(x^* e_n x) \leqq \rho(e_n)\rho(x^* e_n x) \leqq \|x\|^2 \rho(e_n)^2 .$$

If $f_n(x) = f(e_n x)$, then $\{f_n\}$ is a sequence of ρ-normal bounded functionals on $\mathscr{B}_A$ such that $f(x) = \Sigma\, f_n(x)$ for all x in $\mathscr{B}_A^+$. For each n there is a central projection p_n in $\mathscr{F}$ such that $f_n(p_n \cdot)$ is atomic and $f_n((1 - p_n)\cdot)$ is diffuse. Using $p = \vee p_n$ $(\in \mathscr{C} \cap \mathscr{F})$ we see that Proposition 3 is valid for f.

To show that most of Proposition 4 holds also for an unbounded atomic functional f of the above type, we notice that, as in the proof of Proposition 4, we can write $f = \Sigma f_k$, where each f_k is a σ-normal (unbounded) functional on an algebra $B(H_k)$. If p is a one-dimensional projection in H_k, then $e_n p \neq 0$ for some n and therefore $p = \|pe_n p\|^{-1} pe_n p$. It follows that

$$\rho(p) = \|pe_n p\|^{-1}\rho\left(e_n^{\frac{1}{2}} p e_n^{\frac{1}{2}}\right) \leqq \|pe_n p\|^{-1}\rho(e_n) < \infty \ .$$

This proves that f_k on $B(H_k)$ is majorized by an invariant convex functional, which is finite at each operator in $B(H_k)$ of finite rank. Thus f_k is a C^*-integral on the C^*-algebra of compact operators on H_k, and from [10, Theorem 3.8] there is a b_k in $B(H_k)^+$ such that $f_k(x) = \mathrm{tr}(b_k x)$, for all x in $B(H_k)^+$. Hence f_k (and f) can be expressed as a sum $\Sigma\, \alpha_{nk} f_{nk}$ with f_{nk} in P for all n (and k). If each b_k can be diagonalized in $B(H_k)$ then f can be written as a weighted sum of mutually orthogonal pure states. This is trivially the case if f is a trace, since then each f_k is a multible of tr. In general b_k can not be diagonalized, hence a decomposition in mutually orthogonal bounded functionals is not possible.

Proposition 5 and 6 can be generalized with the same ease. We leave the details to the reader.

Finally we notice that the condition of separability for the C^*-algebra A is used primarily to ensure that $\mathscr{B}_A$ is "large enough" under irreducible representations of A. When A is nonseparable there may be irreducible representations of A on nonseparable Hilbertspaces. The pure states of A will then correspond to minimal projections in the Jordan algebra of *Borel operators* of A defined in [2, § 2.4]. This provides a method for studying atomic functionals on nonseparable C^*-algebras. Another way is to define an atomic functional on A as one which is supported on a sequence of minimal projections from the enveloping von Neumann algebra of A. But in this case the relation to measure theory becomes less clear.

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