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SARIO POTENTIALS ON RIEMANNIAN SPACES

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In potential theory on Riemann surfaces three kernels are considered: the Green's kernel on hyperbolic Riemann surfaces; the Evans kernel on parabolic Riemann surfaces; and the Sario kernel on arbitrary Riemann surfaces. Since the Sario kernel has no restriction on the domain surface, in contrast with the two other kernels, its potential theory enjoys the advantage of full generality. From the point of view of Riemannian spaces potential theory on Riemann surfaces is included in that on Riemannian spaces.

The object of this note is to construct the Sario kernel and to develop the corresponding theory of Sario kernel on Riemannian spaces of dimension $n \geqq 3$. The Sario kernel, which is positive, symmetric and jointly continuous, posseses the property of Riez type decomposition (Theorem 1). The continuity principle, unicity principle, Frostman's maximum principle, energy principle and capacity principle are valid for potentials with respect to the Sario kernel. It is also shown that a set of capacity zero with respect to the Sario kernel is, considered locally, of Newtonian capacity zero (Theorem 7), and so the relation of capacity function and the equilibrium Newtonian potential in Euclidean n-space is obtained.

Historically the Sario kernel on Riemann surface is constructed by Sario ([8], [9], [10]) as a generalization of the elliptic kernel on the Riemann sphere, and the potential theory corresponding to the Sario kernel has been systematically investigated by Nakai ([3], [4], [5], [6]). Our main tools are similar to those of Nakai.

First we shall construct a Sario kernel which is positive and symmetric and demonstrate its joint continuity (Theorem 1). These properties will enable us to prove the continuity principle, unicity principle, Frostman's maximum principle, energy principle and capacity principle for the Sario kernel, i.e., for Potentials with respect to the Sario kernel. It will also be shown that a set of capacity zero with respect to the Sario kernel is, considered locally, of Newtonian capacity zero (Theorem 7). In view of this result we obtain a solution of problem (10) in the monograph of Rodin-Sario [8, p. 257] and Sario [12], i.e., the relation between the capacity function and the equilibrium Newtonian potential in Euclidean space.

Let R^n be a Riemannian space of dimension n, that is, a connected countable oriented C^∞-manifold of dimension $n \geqq 3^2$ with C^∞-metric tensor $g_{ij}(1 \leqq i, j \leqq n)$. Throughout our presentation we denote by V_a a parametric ball with center at a in R^n, by ∂V_a the boundary sphere of V_a and by $g_{V_a}(\zeta, a)$ the Green's function of V_a with pole at a. For the Green's function $g_{v_a}(\zeta, a)$, we always take the normalization $\int_{\partial V_a} {}^*dgV_a(\zeta, a) = 1$ and take $^*d$ to be the exterior normal. The distance between two points $\zeta = (\zeta^1, \cdots, \zeta^n)$ and $a = (a^1, \cdots, a^n)$ in a parametric ball V will be denoted $|\zeta - a| = (\sum_{i=1}^n (\zeta^i - a^i)^2)^{\frac{1}{2}}$.

1. Construction of the Sario kernel. We shall construct a Sario kernel on an arbitrary Riemannian space R^n. On R^n take arbitrary but then fixed points $\zeta_j (j = 0, 1)$ and parametric balls $V_j (j = 0, 1)$ about the ζ_j with disjoint closures in R^n. Let $t_0(\zeta) = t(\zeta, \zeta_0, \zeta_1)$ be a harmonic function on $R^n - \{\zeta_0, \zeta_1\}$ with the following properties $(1°) \sim (5°)$:

(1°) $$t_0(\zeta) - 2g_{V_0}(\zeta, \zeta_0) \in H(\bar{V}_0) ,$$

(2°) $$t_0(\zeta) + 2g_{V_1}(\zeta, \zeta_1) \in H(\bar{V}_1) ,$$

(3°) $$t_0 = (I)L_1 t_0$$

in a neighborhood A of the ideal boundary β of R^n, where $(I)L_1$ is the normal operator with respect to the identity boundary partition. By $\bar{V}_j (j = 0, 1)$ we mean the closure of V_j.

(4°) $$t_0 | A = 0(1), t_0 - \sigma = O(1) ,$$

with singularity function σ for the operator L_1 defined by

$$\sigma(\zeta) = \begin{cases} 2g_{V_0}(\zeta, \zeta_0) & \text{in } \bar{V}_0 \\ -2g_{V_1}(\zeta, \zeta_1) & \text{in } \bar{V}_1 \\ 0 & \text{in } A . \end{cases}$$

Since the function $t_0(\zeta)$ satisfying $(1°) \sim (4°)$ is uniquely determimed up to an additive constant, we impose the normalization condition:

(5°) $$t_0(\zeta) - 2g_{V_0}(\zeta, \zeta_0) \to 0, \text{ as } \zeta \to \zeta_0 \text{ in } V_0 .$$

For the construction of $t_0(\zeta)$ we refer to Rodin-Sario [8] or Sario-Schiffer-Glasner [13].

Next we define the function $s_0(\zeta)$ by

$$s_0(\zeta) = \log(1 + e^{t_0(\zeta)}) .$$

Since $t_0 | V_0 = 2g_{V_0}(\zeta, \zeta_0) + O(1)$, $s_0 | V_0 = 2g_{V_0}(\zeta, \zeta_0) + O(1)$. Also by (4°) $t_0 | R^n - V_0 - V_1 = O(1)$. Thus we obtain:

LEMMA 1. *The function $s_0(\zeta)$ is nonnegative on R^n, finitely* *continuous on* $R^n - \{\zeta_0\}$, *and*

$$s_0 | V_0 = 2gV_0(\zeta, \zeta_0) + O(1) .$$

For an arbitrary point a in $R^n - \{\zeta_0\}$, we construct $t(\zeta, a) = t(\zeta, a, \zeta_0)$ in the same manner as $t(\zeta, \zeta_0, \zeta_1)$. In this case we denote by V_0' and V_2' the closure disjoint parametric balls with centers at ζ_0 and a respectively, and choose the normalization condition:

$$t(\zeta, a) + 2g_{V_0'}(\zeta, \zeta_0) \rightarrow s_0(a), \text{ as } \zeta \rightarrow \zeta_0 \text{ in } \bar{V}_0' .$$

Let $s_1(\zeta, a) = s_0(\zeta) + t(\zeta, a)$ and $s_1(\zeta, \zeta_0) = s_0(\zeta_0)$, i.e., $t(\zeta, \zeta_0) = 0$. The functions $s_1(\zeta, a)$ and $s_1(\zeta, a) + 2g_{V_0'}(\zeta, \zeta_0)$ are finitely continuous on $R^n - \{a\}$ and $\bar{V}_0'$, respectively. Hence by Lemma 1 $s_1 | V_0'' = O(1)$ for the smaller parametric ball V_0'' of V_0 and V_0'. Also by the property of $t(\zeta, a)$, $t | R^n - V_0'' > O(1)$. Thus $s_1(\zeta, a) > O(1)$ and we obtain:

LEMMA 2. *$s_1(\zeta, a)$ is bounded from below.*

For later use we list three properties of $t(\zeta, a)$ which are easily seen from the definition:

(a) $\zeta \rightarrow t(\zeta, a)$ is harmonic on U for fixed $a \in V$,
(b) $a \rightarrow t(\zeta, a)$ is finitely continuous on V for fixed $\zeta \in U$,
(c) $(\zeta, a) \rightarrow t(\zeta, a)$ is bounded from below on $U \times V$,

where U and V are closure disjoint parametric balls about ζ and a respectively, and $\zeta_0 \notin U$.

We finally define $s(\zeta, a) = s_1(\zeta, a) + C$, where the constant C is so chosen that

$$(1) \qquad s(\zeta, a) > 0$$

for all $(\zeta, a) \in R^n \times R^n$. Then $s(\zeta, a)$ is symmetric:

LEMMA 3. *For any* $(\zeta, a) \in R^n \times R^n$

$$(2) \qquad s(\zeta, a) = s(a, \zeta) .$$

Proof. It suffices to prove that $s_1(a, b) = s_1(b, a)$ for any $(a, b) \in R^n \times R^n$. Let $\{\Omega\}$ be a regular exhaustion of R^n such that every Ω properly contains the disjoint parametric balls V_0, V_1 and V_2 about the points ζ_0, a and b, $\bar{V}_j (j = 0, 1, 2)$ properly. Let $t_\Omega(\zeta, a)$, $t_0^\Omega(\zeta)$, and $s_0^\Omega(\zeta)$ be the functions constructed in Ω corresponding to $t(\zeta, a)$, $t_0(\zeta)$ and $s_0(\zeta)$ respectively. Take level spheres α_0, α_1 and α_2 of $g_{V_0}(\zeta, \zeta_0)$,

$g_{V_1}(\zeta, a)$ and $g_{V_2}(\zeta, b)$ in V_0, V_1 and V_2 respectively, and orient α_0, α_1, and α_2 so that $\partial\Omega - \alpha_0 - \alpha_1 - \alpha_2$, with $\partial\Omega$ the boundary surface of Ω, is positively oriented. Then by Green's formula we obtain

$$(3) \qquad \int_{\partial\Omega-\alpha_0-\alpha_1-\alpha_2} t_\Omega(\zeta, a) * dt_\Omega(\zeta, b) - t_\Omega(\zeta, b) * dt(\zeta, a) = 0 .$$

By the L_1-behavior of $t_\Omega(\zeta, a)$ and $t_\Omega(\zeta, b)$ on $\partial\Omega$, the integral of (3) on $\partial\Omega$ vanishes. The integral of (3) on $\alpha_1 + \alpha_2$ yields $-2t_\Omega(a, b) + 2t_\Omega(b, a)$. Also the calculation on α_0 yields $2s_0^\Omega(b) - 2s_0^\Omega(a)$ and we obtain from (3)

$$s_0^\Omega(a) + t_\Omega(a, b) = s_0^\Omega(b) + t_\Omega(b, a) ,$$

on letting $r \to 0$ with r the radius of the parametric balls. Since the convergences $t_\Omega(\zeta, a) \to t(\zeta, a)$ and $t_0^\Omega(\zeta) \to t_0(\zeta)$ as $\Omega \to R^n$ are uniform on compacta (Rodin-Sario [8]; p. 246) the same is true of $s_0^\Omega(\zeta)$. Hence on letting $\Omega \to R^n$, we obtain the equation $s_1(b, a) = s_1(a, b)$.

From (1) and Lemma 3, (ζ, a) is a positive symmetric function. We shall refer to it as the Sario kernel on R^n.

To obtain the subharmonicity of $s_0(\zeta)$, we show:

LEMMA 4. *On* $R^n - \{\zeta_0, \zeta_1\}$

$$\Delta_\zeta s_0(\zeta) = e^{t_0(\zeta)}(1 + e^{t_0(\zeta)})^{-2} |\operatorname{grad}\ t_0(\zeta)|^2 ,$$

holds, and hence $\Delta_\zeta s_0(\zeta)$ *is nonnegative there.*

In terms of a local parameter $\zeta = (\zeta^1, \zeta^2, \cdots, \zeta^n)$, the Laplacian and gradient are

$$\Delta_\zeta \cdot = \sum_{i,j=1}^n g^{ij} \frac{\partial^2}{\partial\zeta^i \partial\zeta^i} \cdot + \sum_{i,j=1}^n \left(\frac{\partial}{\partial\zeta^i} g^{ij} + g^{ij} \frac{\partial}{\partial\zeta^i} \log\sqrt{G} \right) \frac{\partial}{\partial\zeta^i} \cdot ,$$

and

$$|\operatorname{grad} \cdot|^2 = \sum_{i,j=1}^n g^{ij} \frac{\partial}{\partial\zeta^i} \cdot \frac{\partial}{\partial\zeta^i} \cdot ,$$

where the g^{ij}'s are elements of the inverse of the matrix (g_{ij}) and $G = \det(g_{ij})$. For any point $\zeta = (\zeta^1, \zeta^2, \cdots, \zeta^n)$ in $R^n - \{\zeta_0, \zeta_1\}$,

$$\frac{\partial}{\partial\zeta^j} s_0(\zeta) = e^{t_0(\zeta)}(1 + e^{t_0(\zeta)})^{-1} \frac{\partial}{\partial\zeta^j} t_0(\zeta), \ \sum_{i,j=1}^n \left(\frac{\partial}{\partial\zeta^i} g^{ij} + g^{ij} \frac{\partial}{\partial\zeta^i} \log\sqrt{G} \right) \frac{\partial}{\partial\zeta^j} s_0(\zeta)$$
$$= e^{t_0(\zeta)}(1 + e^{t_0(\zeta)})^{-1} \sum_{i,j=1}^n \left(\frac{\partial}{\partial\zeta^i} g^{ij} + g^{ij} \frac{\partial}{\partial\zeta^i} \log\sqrt{G} \right) \frac{\partial}{\partial\zeta^j} t_0(\zeta) ,$$

and

$$\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2}{\partial\zeta^i\partial\zeta^j} s_0(\zeta) = e^{t_0(\zeta)}(1 + e^{t_0(\zeta)})^{-2} |\text{grad } t_0(\zeta)|^2$$
$$+ e^{t_0(\zeta)}(1 + e^{t_0(\zeta)})^{-1} \sum_{i,j=1}^{n} g^{ij} \frac{\partial^2}{\partial\zeta^i\partial\zeta^j} t_0(\zeta) .$$

Since $\Delta_\zeta t_0(\zeta) = 0$, we obtain, on $R^n - \{\zeta_0, \zeta_1\}$,

$$\Delta_\zeta s_0(\zeta) = e^{t_0(\zeta)}(1 + e^{t_0(\zeta)})^{-2} |\text{grad } t_0(\zeta)|^2 .$$

Now consider an n-form $\lambda(\zeta)dV_\zeta$ on $R^n - \{\zeta_0, \zeta_1\}$ defined by

$$\lambda^2(\zeta) = \Delta_\zeta s_0(\zeta) = e^{t_0(\zeta)}(1 + e^{t_0(\zeta)})^{-2} |\text{grad } t_0(\zeta)|^2 ,$$

with $\lambda(\zeta) \geqq 0$ and dV_ζ the local Euclidean volume element on R^n, i.e., locally, $dV_\zeta = n\omega_n r^{n-1}dr$ with $\omega_n r^n$ the volume of a ball of radius r. Hence $\omega_n = \pi^{n/2}/\Gamma(n/2 + 1)$. Since $\lambda^2(\zeta)dV_\zeta = O(1)$ at ζ_0 and ζ_1, $\lambda^2(\zeta)dV_\zeta$ can be continued to a nonnegative finitely continuous n-form on R^n. Also since $s(\zeta, a) = s_0(\zeta) + t(\zeta, a)$, on $R^n - \{a, \zeta_0, \zeta_1\}$ we have

$$\Delta_\zeta s(\zeta, a) = \Delta s_0(3) = \lambda^2(\zeta) . \tag{4}$$

The two points ζ_0 and ζ_1 satisfy the removable condition of subharmonicity so that $s(\zeta, a)$ is a finitely continuous subharmonic function of ζ on $R^n - \{a\}$.

We shall prove the joint continuity and Riesz type decomposition of $s(\zeta, a)$.

THEOREM 1. *The Sario kernel $s(\zeta, a)$ is continuous on $R^n \times R^n$ and finitely continuous on $R^n \times R^n$ outside the diagonal set. Moreover, for every regular region Ω of R^n the decomposition*

$$s(\zeta, a) = 2g_\Omega(\zeta, a) + v_\Omega(\zeta, a) \tag{5}$$

is valid, where g_Ω is the Green's kernel on Ω and v_Ω is a finitely continuous function on $\Omega \times \Omega$.

The proof of the first part is deduced from (a), (b), (c) and Harnack's inequality. If we prove the following two lemmas, the second part is clear. To this end we define the following functions on Ω and $\Omega \times \Omega$:

$$G_\Omega(\zeta) = \int_\Omega \lambda^2(b) g_\Omega(b, \zeta) dV_b ,$$

$$H_\Omega(\zeta, a) = \int_{\partial\Omega} v_\Omega(b, a) * dg_\Omega(b, \zeta) ,$$

where

$$v_\Omega(b, a) = s(\zeta, a) - 2g_\Omega(\zeta, a) .$$

LEMMA 5. *The function* $G_\Omega(\zeta)$ *is continuous on* Ω *and* $H_\Omega(\zeta, a)$ *is finitely continuous on* $\Omega \times \Omega$.

Proof. Let $\zeta' \in \Omega$ and U be a parametric ball with center at ζ' and radius 1 such that $\bar{U} \subset \Omega$. Denote by U_r the parametric ball $|b - \zeta'| < r$ in U with $0 < r < 1$, and by $g_U(\cdot, \zeta')$ the Green's kernel on U with pole at ζ'. Then

$$g_U(b, \zeta) = O(|b - \zeta|^{2-n}) \leqq c\,|b - \zeta|^{2-n}$$

for a suitable constant c. Since $g_\Omega(b, \zeta) - g_U(b, \zeta) > 0$ is finitely continuous on $U \times U$, $\sup\,\{g_\Omega(b, \zeta) - g_U(b, \zeta)\}$ in $U_{\frac{1}{2}} \times U_{\frac{1}{2}}$ is a finite number M. Also for $0 < \varepsilon < 1/6$ and $\zeta \in U_\varepsilon$, $g_\Omega(\zeta, b) \leqq g_U(\zeta, b) + M \leqq c\,|b - \zeta|^{2-n} + M$ in $\{|b - \zeta| < 2\varepsilon\}$. On setting $m = \sup\{\lambda^2(b)\,|\, b \in U\} < \infty$ we obtain

$$\int_{\{|b-\zeta|<\varepsilon\}} \lambda^2(b) g_\Omega(b, \zeta) dV_b \leqq m \cdot n \cdot \omega_n \left\{ c \int_0^{2\varepsilon} r dr + M \int_0^{2\varepsilon} r^{n-1}\, dr \right\} = O(\varepsilon) .$$

Also for any $\zeta'' \in U_\varepsilon$.

$$|G_\Omega(\zeta') - G_\Omega(\zeta'')| \leqq \int_{\Omega - U_\varepsilon} \lambda^2(b)\,|g_\Omega(b, \zeta') - g_\Omega(b, \zeta'')|\, dV_b$$
$$+ \sum_{\zeta=\zeta', \zeta''} \int_{\{|b-\zeta|<2\varepsilon\}} \lambda^2(b) g_\Omega(b, \zeta'') dV_b .$$

Since $g_\Omega(b, \zeta'') \to g_\Omega(b, \zeta')$ uniformly on $\Omega - U_\varepsilon$ as $\zeta'' \to \zeta'$, we obtain

$$\limsup_{\zeta'' \to \zeta'} |G_\Omega(\zeta'') - G_\Omega(\zeta')| < O(\varepsilon) .$$

Thus

$$\lim_{\zeta'' \to \zeta'} G_\Omega(\zeta'') = G_\Omega(\zeta') .$$

LEMMA 6. *The function* $v_\Omega(\zeta, a)$ *has the representation*

$$v_\Omega(\zeta, a) = -G_\Omega(\zeta) - H_\Omega(\zeta, a)$$

on $\Omega \times \Omega$. *Consequently* $v_\Omega(\zeta)$ *is finitely continuous there.*

Since this lemma may be proved in a manner similar to the case $n = 2$ (Nakai [3] or Rodin-Sario [8; p. 309]), we omit the proof.

2. Sario potential. Having obtained a positive symmetric kernel $s(\zeta, a)$ on $R^n \times R^n$, we shall now construct the potential with kernel function $s(\zeta, a)$ and investigate its potential-theoretic properties.

By a regular Borel measure μ on R^n with compact support S_μ in R^n, we mean a measure μ such that for every parametric ball V with $V \cup S_\mu \neq \emptyset$ and every local parameter $x = \varphi(\zeta)$ of V, $\mu(\varphi(\zeta)) \mid V \cap S$ is a regular Borel measure in n-dimensional Euclidean space E^n, where $\mu(\varphi(\zeta)) \mid V \cap S_\mu$ is the restriction of $\mu(\varphi(\zeta))$ to $V \cap S_\mu$. Unless specified otherwise we consider only nonnegative regular Borel measures μ with compact support S_μ in R^n. We define

$$s_\mu(\zeta) = \int s(\zeta, a) d\mu(a)$$

and, as in the case $n = 2$, call $s_\mu(\zeta)$ the (n-dimensional) Sario potential with respect to the measure μ. By (1) it is nonnegative, and positive unless $\mu \equiv 0$. As a consequence of Theorem 1, s_μ is lower semicontinuous on R^n and finitely continuous on $R^n - S_\mu$. By (4) it is subharmonic on $R^n - S_\mu$.

For convenience we list below several lemmas without proof. In these lemmas we always suppose that R^n is hyperbolic, i.e., a Green's kernel $g(\zeta, a)$ exists on R^n. Thus one can consider potentials $g_\mu(\zeta) = \int g(\zeta, a) d\mu(a)$.

LEMMA 7. (Local Maximum Principle). *Let F be a compact subset of R^n containing S_μ. For any $\zeta' \in F$*

$$\lim_{\zeta \in R^n - F} \sup_{\zeta \to \zeta'} g_\mu(\zeta) \leqq \lim_{\zeta \in F} \sup_{\zeta \to \zeta'} g_\mu(\zeta) .$$

LEMMA 8. (Frostman's Maximum Principle). *$g_\mu \mid S_\mu \leqq M$ implies $g_\mu \leqq M$ on R^n.*

LEMMA 9. (Equilibrium Principle). *For any compact subset K of R^n, there exists a unique measure ν on K, called the equilibrium measure of K, such that $g_\nu \leqq 1$, $g_\nu = 1$ a.e., on K and $C(K) = \nu(K) = \|\nu\|^2$, where $C(K)$ is the capacity of K and $\|\nu\|^2$ is the energy of g_ν.*

LEMMA 10. (Ninomiya [7]). *Let Ω be a locally compact Hausdorff space and $k(x, y)$ a continuous positive function on $\Omega \times \Omega$ with $k(x, x) = \infty$ and $k(x, y) = k(y, x)$. If the potential*

$$k_\mu(x) = \int k(x, y) d\mu(y)$$

satisfies Frostman's maximum principle and the unicity principle, then

$$\int k(x, y)d\sigma(x)\, d\sigma(y) > 0$$

for any nonzero signed measure σ.

With these preparations we shall now establish various principles of potential theory with respect to Sario potentials. Since s_μ is subharmonic in $R^n - S_\mu$, its magnitude is determined by its behavior at the ideal boundary of R^n and at S_μ, in view of the maximum principle for subharmonic functions (Ito [2]).

THEOREM 2. (Local Maximum Principle). *For a compact subset* F *of* R^n *containing* S_μ *and any point* $\zeta' \in F$,

$$\lim_{\zeta \in R^n - F} \sup_{\zeta \to \zeta'} s_\mu(\zeta) \leqq \lim_{\zeta \in F} \sup_{\zeta \to \zeta'} s_\mu(\zeta) . \tag{6}$$

Proof. Take a parametric ball V with center at ζ'. Set $\mu' = \mu | V$, i.e., $\mu'(\cdot) = \mu(\cdot \cap V)$, and $\mu'' = \mu - \mu'$. Then $s_\mu = s_{\mu'} + s_{\mu''}$ and $s_{\mu''}$ is continuous in V. Without loss of generality we may assume that $F \subset V$. Hence it suffices to prove (6) for $s_{\mu'}$ and $F \subset V$. Let $v(\zeta, a) = s(\zeta, a) - 2g_V(\zeta, a)$ on $V \times V$. By Theorem 1 $\int v(\zeta, a) d\mu(a)$ is finitely continuous on $V \times V$. Thus the proof of (6) is reduced to

$$\lim_{\zeta \in R^n - F} \sup_{\zeta \to \zeta'} \int g_V(\zeta, a) d\mu(a) \leqq \lim_{\zeta \in F} \sup_{\zeta \to \zeta'} \int g_V(\zeta, a) d\mu(a) ,$$

which is valid by Lemma 7.

As a consequence of the local maximum principle we have:

THEOREM 3. (Continuity Principle). *If* $s_\mu | S_\mu$ *is continuous* (resp. finitely continuous) *on* S_μ, *the same is true of* s_μ *on* R^n.

The linear operator $\mu \to s_\mu$ from the measure space into the function space determined by the potential s_μ is injective:

THEOREM 4. (Unicity Principle). $s_\mu = s_\nu$ *implies* $\mu = \nu$. *More generally, if* $s_\mu = s_\nu + u$ *with* $u \in H(R^n)$, *then* $\mu = \nu$.

Sketch of the Proof. For any point $b \in R^n$, let V_ε be its parametric ball of radius ε and ∂V_ε be the clockwise oriented boundary sphere of V_ε. Applying Green's formula to the functions $s(a, b)$ and $f \in C_0^n(R^n)$ and the region $R - V_\varepsilon$, and letting $\varepsilon \to 0$, we obtain

$$f(\mathrm{b}) = \int f(a)\lambda^2(a) dV_a - \int s(a, b) \Delta_a f(a) dV_a ,$$

where $C_0^n(R^n)$ is the space of nth order continuously differentiable functions with compact supports in R^n. For a signed measure $\sigma = \mu - \nu$ and $f \in C_0^\infty(R^n)$, we obtain

$$(7) \qquad \int f(b)d\sigma(b) = \sigma(S) \int f(a)\lambda^2(a)dV_a - \int s_\sigma(a)\Delta_a f(a)dV_a$$

from the above equation. Since $s_\sigma = u$, i.e., $\Delta_a s_\sigma(a) = 0$, and since $\int s_\sigma(a)\Delta_a f(a)dV_a = \int s_\sigma(a) * d_a f = 0$, (7) implies that

$$\int f(b)(d_\sigma(b) - \sigma(R^n)\lambda^2(b)dV_b) = 0$$

for any $f \in C_0^\infty(R^n)$. Thus we obtain

$$d\sigma(b) = \sigma(R)\lambda^2(b)dV_b \,.$$

The fact that σ is a signed measure implies $\sigma = \mu - \nu = 0$.

Next we shall show that s_μ satisfies Frostman's maximum principle:

THEOREM 5. (Frostman's Maximum Principle). *$s_\mu | S_\mu \leqq M$ implies $s_\mu \leqq M$ on R^n.*

We prove this theorem by dividing it into three lemmas. Let K be any compact subset of R^n and β the ideal boundary of R^n. Define

$$M(K) = \sup_{a \in K} \lim_{\zeta \to \beta} \sup s(\zeta, a)$$

if $\beta \neq \varnothing$; and otherwise $M(K) = 0$. Set $B(M, \mu) = \max\{M, \mu(S_\mu)\}$.

LEMMA 11. *$M(K)$ is finite and the following maximum principle is valid: $s_\mu | S_\mu = M$ implies $s_\mu \leqq B(M, \mu)$ on R^n.*

Since the proof parallels that of the case $n = 2$, we omit it here (cf. Nakai [3; p. 232], Rodin-Sario [8; p. 314]).

If R^n is compact, Theorem 5 is true by Lemma 8. If R^n is parabolic, the theorem follows from Lemma 8, Theorem 2 and the subharmonicity of s_μ on $R^n - S_\mu$. Thus we have only to prove Theorem 5 for the case in which R^n is hyperbolic, i.e., Green's kernel $g(\zeta, a)$ exists on R^n.

By the unicity of $t_0(\zeta)$ and $t(\zeta, a)$, (8)-(10) hold with a suitable constant k.

$$(8) \qquad t_0(\zeta) = 2g(\zeta, \zeta_0) - 2g(\zeta, \zeta_1) + k \,.$$

$$s_0(\zeta) = \log(e^{-2g(\zeta,\zeta_0)} + e^{-2g(\zeta,\zeta_1)+k}) + 2g(\zeta, \zeta_0) . \tag{9}$$

$$t(\zeta, a) = 2g(\zeta, a) - 2g(\zeta, \zeta_0) + s_0(a) - k . \tag{10}$$

Setting

$$u(\zeta) = \log(e^{-2g(\zeta,\zeta_0)} + e^{-2g(\zeta,\zeta_1)+k}) - \log(1 + e^k) , \tag{11}$$

the Sario kernel $s(\zeta, a)$ has the expression (12) with a suitable constant m;

$$s(\zeta, a) = 2g(\zeta, a) + u(\zeta) + u(a) + m . \tag{12}$$

Let μ be a unit measure and set $M' = M - m - \int u(a)d\mu(a)$. Then it is easily seen that

$$2g_\mu(\zeta) + u(\zeta) \leqq M' \text{ on } S_\mu . \tag{13}$$

With these preparations we show

LEMMA 12. $M' \geqq 0$.

Proof. By Lemma 9, there exists a unique equilibrium distribution ν_0 on S_μ. Set $\nu = V_g(S_\mu)|\nu_0$. Since $\int g_\mu d\mu d\mu < \infty$ by (13), the property of ν implies that the μ-measure of the set $\{\zeta \in S_\mu \,|\, g_\nu(\zeta) \neq V_g(S_\mu)\}$ is zero. Here $V_g(S_\mu) = \inf \int g_\mu d\theta d\theta$ with the unit measures θ such that $S_\theta \subset S_\mu$. On integrating (13) with respect to ν, and using $\int g_\mu d\nu = V_g(S_\mu)$, we obtain

$$2V_g(S_\mu) + \int u(\zeta)d\nu(\zeta) \leqq M' . \tag{14}$$

By (11) we see that

$$\int u(\zeta)d\nu(\zeta) = -2g_\nu(\zeta_0) + \int \varphi(\zeta)d\nu(\zeta) \tag{15}$$

with

$$\varphi(\zeta) = \psi(2g(\zeta, \zeta_0) - 2g(\zeta, \zeta_1)) ,$$
$$\psi(\xi) = \log(1 + e^{\xi+k}) - \log(1 + e^k) .$$

Since $\psi(\xi)$ is a convex function, applying Jensen's inequality to (15) with $\xi(\zeta) = 2g(\zeta, \zeta_0) - 2g(\zeta, \zeta_1)$, we obtain

$$-2V_g(S_\mu) = \log\{(e^{-2g_\nu(\zeta_0)} + e^{-2g_\nu(\zeta_1)+k})/(1 + e^k)\} \leqq \int u(\zeta)d\nu(\zeta) .$$

This with (14) implies $M' \geqq 0$.

LEMMA 13. *Let* $\mathscr{F}$ *be the family* $\{\{\zeta_n\}_{n=1}^{\infty} \subset R^n \mid \zeta_n \to \beta$ *as* $n \to \infty\}$ *and* $\mathscr{F}^+$ *be the subfamily* $\{\{\zeta_n\} \subset \mathscr{F} \mid \liminf_n g(\zeta_n, a) > 0$ *for all* $a \in R^n\}$. *Then there exists a nonnegative superharmonic function* v *on* R^n *such that* $\lim_{n\to\infty} v(\zeta_n) = \infty$ *for* $\{\zeta_n\} \in \mathscr{F}^+$.

By the monotone compactness of $H(R^n)$ and the solvability of the Dirichlet problem for regular regions and continuous boundary functions in R^n, the method in Constantinescu-Cornea [1; p. 48] is valid in R^n. We omit the proof.

We now prove Theorem 5 in the hyperbolic case. Without loss of generality we may assume that $\mu(S_\mu) = 1$. By (12) and (13) it suffices to prove that $2g_\mu(\zeta) + u(\zeta) \leqq M'$ on S_μ implies $2g_\mu(\zeta) + u(\zeta) \leqq M'$ on R^n. For the function v in Lemma 13 and $m = 1, 2, 3, \cdots$, define superharmonic function W_m on $R^n - S$ as

$$W_m(\zeta) = M' - (2g_\mu(\zeta) + u(\zeta)) + v(\zeta)/m .$$

Then by Lemma 7, we obtain

$$\lim_{\zeta \in R^n - S_\mu} \inf_{\zeta \to \zeta'} W_m(\zeta) \geqq 0 \tag{16}$$

for $\zeta' \in \partial S_\mu$. Also for $\{\zeta_n\} \in \mathscr{F}^+$ we have

$$\lim_n \inf W_m(\zeta_n) \geqq 0 . \tag{17}$$

By (16), (17) and the minimum principle for superharmonic functions we see that $W_m(\zeta) \geqq 0$ on $R^n - S_\mu$. On letting $m \to \infty$, we have $2g_\mu(\zeta) + u(\zeta) \leqq M'$ on $R^n - S_\mu$ and hence on R^n.

From the above statement it follows that Sario potentials enjoy both Frostman's maximum principle and the unicity principle.

Applying Lemma 10 we obtain

THEOREM 6. (Energy Principle). *For measures* μ *and* ν *with* $\sigma = \mu - \nu \neq 0$,

$$\int s(\zeta, a) d\sigma(\zeta) d\sigma(a) > 0 .$$

3. Sario capacity and the fundamental theorem. We define a set function $V(K)$ first for compact sets $K \subset R^n$ by

$$V(K) = \inf_\mu \int s(\zeta, a) d\mu(\zeta) d\mu(a)$$

where μ runs over all unit measures with $S_\mu \subset K$. For general sets $X \subset R^n$ we set

$$V(X) = \sup_K V(K)$$

where K runs over all compact sets $K \subset X$.

The quantity

$$c(X) = c_s(X) = 1/V(X)$$

will be referred to as the (inner) Sario capacity of X. For Borel sets X, $c(X) = 0$ is equivalent to

(18) $\qquad \mu(X) = 0$ for each μ with $\int s(\zeta, a) d\mu(\zeta) d\mu(a) = \infty$.

Using this we can prove:

THEOREM 7. *A set X in R^n is of Sario capacity zero if and only if X is locally of Newtonian capacity zero.*

Proof. We may suppose that X is compact. By virtue of (5) and (18), $c(X) = 0$ is characterized by $\mu(X \cap V) = 0$ for every μ in each parametric ball V with

$$\int g_V(\zeta, a) d\mu(\zeta) d\mu(a) = \infty .$$

Since $g_V(\zeta, a) = O(|\zeta - a|^{2-n})$, $c(X) = 0$ means that $X \cap V$ has Newtonian capacity zero for each parametric ball V.

As a consequence of Theorem 7, we obtain a solution of problem (10) in Rodin-Sario [8] and Sario [12].

THEOREM 8. *Let P be the equilibrium potential of a unit mass distribution $d\mu$ on a compact set K of R^n defined by*

$$P(x) = \int \frac{|x - y|^{2-n}}{(n-2)\omega_n} d\mu(y) .$$

Let Ω be a regular region of R^n which contains the set K and let p_{Ω_β} be the capacity function for Ω with $p_{\Omega_\beta} | \partial\Omega = k_{\Omega_\beta}$ (c.f. Sario [12], Rodin-Sario [8]). *Then we have locally*

$$P(x) = k_{\Omega_\beta} - \int p_{\Omega_\beta}(x, y) d\mu(y) + M ,$$

with a suitable positive constant M.

Proof. Without loss of generality we may assume that K is contained in some local coordinate system (φ, V) of R^n, that is, we may regard K as a compact set in the parametric ball V and the point y as the center a of V. Thus it suffices to show that

$$\int \frac{|\zeta - a|^{2-n}}{(n-2)\omega_n} d\mu(a) = k_{\Omega_\beta} - \int p_{\Omega_\beta}(\zeta, a) d\mu(a) + M$$

in the parametric ball V, with μ a unit measure such that $S_\mu = K \subset V$. Since $g_\Omega(\zeta, a) = k_{\Omega_\beta} - p_{\Omega_\beta}$ (Rodin-Sario; p. 253) and $g_\Omega | \bar{V} = g_V(\zeta, a) + h(\zeta)$ with $h(\zeta) \in H(\bar{V})$, $g_V(\zeta, a) = k_{\Omega_\beta} - p_{\Omega_\beta}(\zeta, a) + h(\zeta)$ in V. Integrating both sides with respect to μ, we obtain the desired equation with $M = \int h(\zeta) d\mu(\zeta)$.

Let K be a compact set in R^n with $c(K) > 0$. Since $s(\zeta, a)$ is jointly continuous, by the selection theorem for a sequence of measure in R^n we obtain the capacity measure μ, that is, the unit measure with $S_\mu \subset K$ such that $\int s d\mu d\mu = V(K)$. Our final aim is to obtain the capacity principle for the Sario kernel,

THEOREM 9. (Fundamental Theorem of Potential Theory). *Let K be a compact subset of R^n with $c(K) > 0$ and μ its capacity measure. Then $s_\mu \leqq V(K)$ on R^n and $s_\mu = V(K)$ except for an F_σ-set of Sario capacity zero. Furthermore, this capacity measure is unique.*

Proof. First we shall show that $s_\mu \geqq V(K)$ on K except for an F_σ-set of Sario capacity zero. Let A and A_n be the subsets of K on which $s_\mu < V(K)$ and $s_\mu \leqq V(K) - 1/n$ $(n = 1, 2, \cdots)$ respectively. Then the A_n's are compact sets with $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$, and $A = \bigcup_1^\infty A_n$. Hence A is an F_σ-set and we need only show that $c(A) = 0$. Suppose, to the contrary, that for a suitable $\varepsilon > 0$ there exists $A_n = K_1 \subset K$ with $s_\mu | K_1 < V(K) - 2\varepsilon$ and $c(K_1) > 0$. The equality $\int s_\mu d\mu d\mu = V(K)$ implies the existence of a point $\zeta_0 \in S_\mu$ with $s_\mu(\zeta_0) > V(K) - \varepsilon$. Thus $\zeta_0 \notin K_1$. Therefore we may take an open ball U concentrated at ζ_0 with $\bar{U} \cap K = \emptyset$ and $s_\mu | U > V(K) - \varepsilon$. Moreover since $\zeta_0 \in S_\mu$, $\mu(U) > 0$. The fact $c(K_1) > 0$ implies the existence of a measure ν with $S_\nu \subset K_1$,

$$\nu(K_1) = \mu(U), \text{ and with } \int s d\nu d\mu < \infty .$$

Construct a signed measure ν_1 by

$$\nu_1 | K_1 = \nu | K_1, \nu_1 | U = -\mu | U, \nu_1 | R^n - K_1 \cup U = 0 .$$

Clearly $\mu_t = \mu + t\nu_1$ is a unit measure for every $t \in (0, 1)$ with $S_{\mu_t} \subset K$. Therefore

$$\int s d\mu_t d\mu_t \geqq \int s d\mu d\mu = V(K) . \tag{19}$$

A simple calculation shows that

$$\int s d\mu_t d\mu_t < - t\left\{\mu(U)\varepsilon - t \int s d\nu_1 d\nu_1\right\} < 0$$

for sufficiently small t. This violates (19) and we have $c(A) = 0$.

If we show that $s_\mu | S_\mu \leqq V(K)$, by virtue of Theorem 5 the proof is complete except for the uniqueness of μ. Contrary to the assertion, assume that $s_\mu(\zeta_1) > V(K)$ for a $\zeta_1 \in S_\mu$. Choose an open ball U_1 about ζ_1 such that

$$s_\mu | U_1 > V(K) + \varepsilon, \varepsilon > 0 .$$

Then $\mu(U_1) > 0$ and we see that

$$V(K) = \int_{U_1} s_\mu d\mu + \int_{R^n - U_1} s_\mu d\mu > V(K) + \varepsilon\mu(U_1) > V(K) ,$$

a contradiction.

The unicity of the capacity measure follows from Theorem 6 in the same manner as in Nakai [6], or Rodin-Sario [8; p. 332].

As an application of the fundamental theorem, we obtain the subadditivity of the Sario capacity. Since the method is similar to that of Nakai [4; no. 7], we state this without proof.

THEOREM 10. *If X_n $(n = 1, 2, \cdots)$ are sets in R^n and $X = \bigcup_{n=1}^\infty X_n$, then*

$$c(X) \leqq \sum_{n=1}^{\infty} c(X_n) .$$

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