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ON THE COBORDISM OF PAIRS

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The cobordism classification of pairs, introduced by Wall, is generalized to allow additional structure on the submanifold. Emphasis is placed on classification of pairs of stably almost complex manifolds.

1. Introduction. The object of this note is a generalization of the cobordism classification of "pairs" introduced by C. T. C. Wall [3]. Briefly, a "pair" (N, M) consists of a closed manifold M^m embedded in a closed manifold N^n, and the pair (N, M) bounds if there is a pair $W^{m+1} \subset V^{n+1}$ of compact manifolds with boundary with $\partial V^{n+1} = N^n$ and $\partial W^{m+1} = M^m$.

In the situation considered by Wall, it is assumed that the manifold N^n(or V^{n+1}) has additional structure given by a (B, f) structure (i.e., by a class of liftings into the fibration $f: B \rightarrow BO$ of the normal map; e.g., a framing, orientation, or stable almost complex structure) and that the normal bundle ν_M^N(or ν_W^V) of M in N (or W in V) has a chosen reduction to some group G_{n-m} (where $\varphi: G_{n-m} \rightarrow O_{n-m}$ is a homomorphism into the orthogonal group O_{n-m}). Wall then proves that this reduces to separate cobordism problems for N (as (B, f)-manifold) and for M (as $(B \times BG_{n-m}, f \times \varphi')$-manifold).

The generalization considered here is the imposition of a completely unrelated additional structure on M (or W) given by a class of liftings of the normal map into the fibration $g: C \rightarrow BO$. The main result is that this cobordism problem reduces to separate problems for N (as (B, f)-manifold) and for M (as manifold with (D, h)-structure, where $h: D \rightarrow BO$ is a fibration obtained from (B, f), (C, g), and (BG_{n-m}, φ')). In a sense, this generalization is really a special case of Wall's results, requiring only a careful analysis of the "normal structure" of M in N. This analysis is performed in § 2 of this paper.

In § 3 several special cases will be considered. These lead to geometric interpretations of certain homotopy theoretic groups, which unfortunately seem extremely difficult to compute. In § 4, attention is devoted to pairs (N, M) of stable almost complex manifolds. If one considers this case for a moment, one observes that the normal bundle ν_M^N of M in N is a real $(n - m)$-plane bundle with a chosen stable complex structure. The study of the classification problem for such bundles is the main portion of § 4, and is basically the nontrivial portion of this paper.

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2. Normal structure. In order to make precise the notion of additional structure on a manifold, the definition given by Lashof [1] will be used. Denote by $G_{r,n}$ the Grassmann manifold of unoriented r-planes in R^{n+r} and let γ_n^r be the r-plane bundle over $G_{r\,n}$ consisting of pairs: an r-plane in R^{n+r} and a point in that plane. Then $BO_r = \lim_{n\to\infty} G_{r\,n}$ with universal r-plane bundle $\gamma^r = \lim_{n\to\infty} \gamma_n^r$.

DEFINITION. Let $f_r\colon B_r \to BO_r$ be a fibration. If ξ is an r-dimensional vector bundle over the space X classified by the map $\xi\colon X \to BO_r$, then a (B_r, f_r) structure on ξ is a homotopy class of liftings to B_r of the map ξ; i.e., an equivalence class of maps $\hat{\xi}\colon X \to B_r$ with $f_r \circ \hat{\xi} = \xi$, where $\hat{\xi}$ and $\tilde{\xi}$ are equivalent if there is a homotopy $H\colon X \times I \to B_r$ with $f_r \circ H(x, t) = \xi(x)$, $H(x, 0) = \hat{\xi}(x)$, and $H(x, 1) = \tilde{\xi}(x)$.

(*Note*: $A(B_r, f_r)$ structure depends on the specific map into BO_r. There is no way to make (B_r, f_r) structures correspond for equivalent bundles, for the correspondence must depend on the choice of equivalence.)

Let M^m be a compact differentiable (C^∞) manifold (with or without boundary) and $i\colon M^m \to R^{m+r}$ an embedding. Letting $\tau(M)$ denote the tangent bundle of M^m, the normal bundle of i may be identified with the orthogonal complement of $\tau(M)$ in the induced bundle $i^*\,\tau(R^{m+r})$, where R^{m+r} is given the usual Riemannian metric, or the fiber N_m of the normal bundle at m may be identified with the subspace of vectors $x \in R^{m+r}$ such that $(m, x) \in R^{m+r} \times R^{m+r} = \tau(R^{m+r})$ is orthogonal to $i_*\tau(M)_m$. (in $m \times R^{m+r}$). The normal map of i sends m to $N_m \in G_{r,m}$, and composing with the inclusion provides a map $\nu(i)\colon M \to BO_r$ which classifies the normal bundle of i.

For sufficiently large r (depending only on m) there is a well-defined equivalence between the normal bundles of any two embeddings of M, and hence a one-to-one correspondence between (B_r, f_r)-structures on the normal bundles of any two embeddings.

DEFINITION. Suppose one is given a sequence (B, f) of fibrations $f_r\colon B_r \longrightarrow BO_r$ and maps $g_r\colon B_r \longrightarrow B_{r+1}$ so that the diagram

$$\begin{array}{ccc} B_r & \xrightarrow{g_r} & B_{r+1} \\ {\scriptstyle f_r}\downarrow & & \downarrow{\scriptstyle f_{r+1}} \\ BO_r & \xrightarrow{j_r} & BO_{r+1} \end{array}$$

commutes, j_r being the usual inclusion. A (B_r, f_r) structure on the normal bundle of M^m in R^{m+r} defines a unique (B_{r+1}, f_{r+1}) structure via the inclusion $R^{m+r} \subset R^{m+r+1}$. A *$(B, f)$-structure on M* is an equivalence class of sequences of (B_r, f_r) structures $\xi = (\xi_r)$ on the normal bundle of M, two sequences being equivalent if they agree for sufficiently large r. A *(B, f)-manifold* is a pair consisting of a manifold M^m and a (B, f) structure on M.

If W^w is a manifold and M^m is a submanifold of W with trivialized normal bundle, one may imbed M in R^{m+r}, r large, and extend by means of the trivialization to an embedding of a neighborhood of M in W into $R^{w+r} = R^{m+r} \times R^{w-n}$ so that the neighborhood meets R^{m+r} orthogonally along M. This may then be extended to an embedding of W in R^{w+r}. Since the normal map for M is the restriction of that for W, a lifting $\tilde{\nu}$: $W \to B_r$ of the normal map of W gives a lifting $\tilde{\nu} \mid M$ for the normal map of M. Thus a (B, f)-structure on W induces a well-defined (B, f)-structure on M.

In particular, if $M = \partial W$, the trivialization of the normal bundle of M in W by means of the inner normal gives M an induced (B, f) structure, and with this structure M is the boundary of the (B, f) manifold W.

The cobordism group $\Omega_m(B, f)$ is the quotient of the semigroup (under disjoint union) of isomorphism classes of closed m-dimensional (B, f) manifolds by the subsemigroup of isomorphism classes of boundaries. The main result concerning these groups is the Pontryagin-Thom theorem:

Theorem. $\Omega_m(B, f) \cong \lim_{r\to\infty} \pi_{m+r}(Tf_r^*(\gamma^r), \infty)$ where $Tf_r^*(\gamma^r)$ denotes the Thom space of the pull-back of the bundle γ^r to B_r, and ∞ denotes the base point of this Thom space.

Turning now to "pairs", one considers two fibration sequences (B, f) and (C, g) given by fibrations f_r: $B_r \to BO_r$ and g_r: $C_r \to BO_r$, and a fibration φ': $BG_{n-m} \to BO_{n-m}$ (which may arise from a group homomorphism φ: $G_{n-m} \to O_{n-m}$ but is not required to do so in the following). One then considers a "pair" with $((B, f), (C, g), (BG_{n-m}, \varphi'))$ structure as being given by an embedding j: $M^m \to N^n$ (with M and N being compact C^∞ manifolds and j embedding ∂M in ∂N and mapping a tubular neighborhood of ∂M in M orthogonally to ∂N; i.e., in neighborhoods $\partial N \times [0, 1)$ of $\partial N = \partial N \times 0$ and $\partial M \times [0, 1)$ of $\partial M = \partial M \times 0$, j is given by $j(m, t) = (j\mid_{\partial M}(m), t))$ where N is a (B, f)-manifold, M is a (C, g)-manifold, and the normal bundle ν_M^N of M in N is given a (BG_{n-m}, φ')-structure.

Note: Letting i: $N^n \to R^{n+r}$, r large, be an embedding, the normal bundle ν_M^N may be classified by identifying the fiber $(\nu_M^N)_m$

with the space of vectors $x \in R^{n+r}$ such that

$$(m, x) \in (i_* \tau(N_{j(m)})) \subset R^{n+r} \times R^{n+r}$$

and (m, x) is orthogonal to $(ij)_* \tau(M)_m$. This defines a classifying map $\nu_M^N: M \to G_{n-m,r+m} \to BO_{n-m}$ for the bundle. Any two embeddings of N are regularly homotopic (if r is sufficiently large) and any two regular homotopies are homotopic through regular homotopies, rel end points and hence there is as well-defined (preferred) homotopy class of homotopies between any two normal maps $\nu_M^N: M \to BO_{n-m}$ arising in this way. With this classification for the bundle ν_M^N it is meaningful to speak of a (BG_{n-m}, φ') structure on ν_M^N for a one to one correspondence between the (BG_{n-m}, φ') structures on any two normal maps has been established.

Being given a pair (V, W) with $((B, f), (C, g), (BG_{n-m}, \varphi'))$-structure and a pair of submanifolds (N, M) with $M = N \cap W$ with compatible trivialization of the normal bundles (near N, the pair (V, W) is identified with the pair $(N \times D^k, M \times D^k)$, $k = \dim V - \dim N$) one has an induced structure on (N, M). In fact, embedding (V, N) in (R^{v+r}, R^{n+r}) as before, all normal maps are given by restriction and the restrictions of the structure maps give a structure on the pair (N, M). Using the inner normal trivialization, one has a well-defined boundary for $((B, f), (C, g), (BG_{n-m}, \varphi'))$-pairs.

Being given a pair (N, M) with $((B, f), (C, g), (BG_{n-m}, \varphi'))$ structure defined by liftings of the normal maps of an embedding $i: N^n \to R^{n+r}$ one may find an embedding

$$I: N \times [0, 1] \longrightarrow R^{n+r} \times R^+ \; (R^+ = \{x \in R \mid x \geqq 0\})$$

which embeds $N \times 0$ in $R^{n+r} \times 0$ by i, embeds $N \times 1$ in $R^{n+r} \times 0$, and maps neighborhoods of $N \times \{0, 1\}$ orthogonally along these embeddings. The liftings for the normal map i may then be extended by covering homotopy to liftings for the normal map of I, and the induced structure on $(N \times 1, M \times 1)$ is the "inverse" of (N, M).

The cobordism group $\Omega_{n,m}((B, f), (C, g), (BG_{n-m}, \varphi'))$ of pairs with $((B, f), (C, g), (BG_{n-m}, \varphi'))$-structure of dimension (n, m) is the quotient of the semigroup (under disjoint union) of isomorphism classes of closed pairs by the subsemigroup consisting of boundaries. The "inverse" of (N, M) is actually an inverse in this group.

Being given a pair (N^n, M^m) with $((B, f), (C, g), (BG_{n-m}, \varphi'))$ structure defined by liftings for the normal maps of an embedding $i: N \to R^{n+r}$ space, the normal bundle ν_M of M in R^{n+r} is identified as the Whitney sum of the normal bundle of M in N, ν_M^N, and the restriction to M of the normal bundle of N in R^{n+r}, $j^* \nu_N$. This is the choice of a lifting of the normal map of M into the fibration

$BO_{n-m} \times BO_r \longrightarrow BO_{r+n-m}$ obtained from the Whitney sum. The (B, f) structure on ν_N and the (BG_{n-m}, φ') structure on ν_M^N give a further lifting of the normal map of M to $BG_{n-m} \times B_r$. (*Note*: As observed by Lashof, these maps may be converted into fibrations, giving rise to normal structures with isomorphic cobordism groups.)

One now has a commutative diagram

$$\begin{tikzcd}
& D_{r+n-m} \arrow[dl, "\pi_C"'] \arrow[dr, "\pi_B"] \arrow[ddd, "h_{r+n-m}"] & \\
C_{r+n-m} \arrow[ddr, "g_{r+n-m}"'] & & BG_{n-m} \times B_r \arrow[d, "\varphi' \times f_r"] \\
& & BO_{n-m} \times BO_r \arrow[dl, "\oplus"] \\
& BO_{r+n-m} &
\end{tikzcd}$$

in which D_{r+n-m} is the fibration over C_{r+n-m} induced via g_{r+n-m} from the fibration $\oplus \circ (\varphi' \times f_r)$, and the stabilization maps for (B, f) and (C, g) give similar stabilization maps for a new family (D, h). The previous analysis of the normal structure of M indicates a well-defined (D, h) structure on M. Since this analysis may be performed for closed or bounded manifolds, one has defined a homomorphism

$$\Phi\colon \Omega_{n,m}((B, f), (C, g), (BG_{n-m}, \varphi')) \longrightarrow \Omega_n((B, f)) \oplus \Omega_m((D, h))$$

in which Φ sends the class of the pair (N, M) to the class of N as (B, f)-manifold and the class of M as (D, h)-manifold.

Following Wall, one has:

THEOREM: *The homomorphism*

$$\Phi\colon \Omega_{n,m}((B, f), (C, g), (BG_{n-m}, \varphi')) \longrightarrow \Omega_n((B, f)) \oplus \Omega_m((D, h))$$

is an isomorphism.

Proof. Define a homomorphism

$$\Psi\colon \Omega_n((B, f)) \oplus \Omega_m((D, h)) \longrightarrow \Omega_{n,m}((B, f), (C, g), (BG_{n-m}, \varphi'))$$

by letting Ψ send a closed n-dimensional (B, f)-manifold N to the class of the pair $(N, \varnothing)$, where $\varnothing$ is the empty submanifold and sends a closed m-dimensional (D, h)-manifold M^m to the class obtained as follows.

Being given a lifting $\tilde{\nu}\colon M^m \longrightarrow D_{r+n-m}$ for the normal map of M,

$\pi_C \circ \tilde{\nu}$ defines a (C, g) structure on M^m, while the composite

$$M \xrightarrow{\tilde{\nu}} D_{r+n-m} \xrightarrow{\pi_B} BG_{n-m} \times B_r \xrightarrow{\varphi' \times f_r} BO_{n-m} \times BO_r \xrightarrow{\pi_1} BO_{n-m}$$

induces an $n-m$ plane bundle ξ over M, and ξ has a clearly defined (BG_{n-m}, φ') structure (via $\pi_1 \circ \pi_B \circ \tilde{\nu}$), and the composite

$$M \xrightarrow{\tilde{\nu}} D_{r+n-m} \xrightarrow{\pi_B} BG_{n-m} \times B_r \xrightarrow{\varphi' \times f_r} BO_{n-m} \times BO_r \xrightarrow{\pi_2} BO_r$$

induces an r-plane bundle $\hat{\nu}$ over M, and $\hat{\nu}$ has a (B_r, f_r) structure (via $\pi_2 \circ \pi_B \circ \tilde{\nu}$). Further, the normal bundle of M is isomorphic to $\xi \oplus \hat{\nu}$ with an isomorphism induced via the structural maps above.

Now consider the pair $(S(\xi \oplus 1), M)$, where $S(\xi \oplus 1)$ is the sphere bundle of the Whitney sum of ξ and a trivial line bundle and $M \subset S(\xi \oplus 1)$ is the copy of M obtained by

$$M \longrightarrow E(\xi) \times R = E(\xi \oplus 1) \colon m \longrightarrow (O_m, 1)$$

where O_m is the zero vector over m in $E(\xi)$. Then M has a (C, g) structure, the normal bundle of M in $S(\xi \oplus 1)$ is naturally isomorphic to ξ (which has a chosen (BG_{n-m}, φ')-structure) and the normal bundle of $S(\xi \oplus 1)$ has a (B, f) structure under an identification with the pull-back of $\hat{\nu}$. (This being possible for the tangent bundle of $S(\xi \oplus 1)$ is the pull-back of $\tau(M) \oplus \xi \oplus 1$ less a trivial bundle, or extending $i : M \to R^{n+r}$ one has an embedding of $D(\xi)$ inside $D(\nu) = D(\xi \oplus \hat{\nu})$ and hence one may embed $S(\xi \oplus 1)$ in $R^{n+r+1} = R^{n+r} \times R$ as a subset of $D(\xi) \times R$ with normal bundle equal to the pull-back of $\hat{\nu}$ plus a trivial bundle). One then lets $\Psi[M]$ be the class of $(S(\xi \oplus 1), M)$ with these structures.

Since $[\varnothing] = 0$ in $\Omega_m((D, h))$ and $[S(\xi \oplus 1)] = 0$ in $\Omega_n((B, f))$ (being the boundary of $D(\xi \oplus 1)$ with structure induced from $\hat{\nu}$), it is clear that $\Phi \circ \Psi = 1$. To see that $\Psi \circ \Phi = 1$, let (N, M) be a $((B, f), (C, g), (BG_{n-m}, \varphi'))$ pair and form the pair structure on $(N \times I, M \times I)$ used in forming the inverse. Inside $N \times I$ one may imbed a copy of

$$D(\nu_M^N) \times \left[\frac{1}{4}, \frac{3}{4}\right] = U$$

as a neighborhood of $M \times 1/2$, and let T be obtained from $N \times I$ by removing the points of U of the form (x, t) with

$$\|x\|^2 + \left|t - \frac{1}{2}\right|^2 < \frac{1}{16},$$

with $S \subset T$ being the submanifold $M \times [0, 1/4]$. The pair structure on $(N \times I, M \times I)$ induces a pair structure on (T, S) by restriction and the boundary of (T, S) is (N, M) (embedded as $N \times 0$), $-(N, \varnothing)$

(embedded as $N \times 1$) and $-(S(\nu_M^N \oplus 1), M)$ (embedded as the set of $(x, t) \in U$ with $\| x \|^2 + | t - 1/2 |^2 = 1/16$.) Thus (T, S) gives a cobordism of (N, M) and $\Psi \circ \Phi(N, M)$.

3. Some Examples. In this section, several examples will be considered. Notationally, one considers pairs (N^n, M^m) with $((B, f), (C, g), (BG_{n-m}, \varphi'))$ structure.

EXAMPLE 1. To begin, consider the Wall case $(C, g) = (BO, 1)$ where $g_r : C_r \to BO_r$ is the identity map. One then has

$$
\begin{tikzcd}
 & D_{r+n-m} \arrow[dl] \arrow[dr, "\pi_B"] & \\
BO_{r+n-m} \arrow[dr, "1"'] & & BG_{n-m} \times B_r \arrow[dl, "\oplus \circ (\varphi' \times f_r)"] \\
 & BO_{r+n-m} &
\end{tikzcd}
$$

and π_B is an identification. Thus $\Omega_m((D, h))$ is the stable homotopy of the Thom spectrum $\{T((\oplus \circ (\varphi' \times f_r))^* (\gamma^{r+n-m}))\}$. Since the Thom space of the Whitney sum is the smash product of the Thom spaces, one has

$$
\begin{aligned}
\Omega_m((D, h)) &= \lim_{s \to \infty} \pi_{m+s}((T\varphi'^* \gamma^{n-m}) \wedge (Tf^*_{s-n+m} \gamma^{s-n+m}), \infty) \\
&= \widetilde{\Omega}_n(B, f)\, (T\varphi'^* \gamma^{n-m})
\end{aligned}
$$

where $\widetilde{\Omega}_n(B, f)\,(X)$ denotes the usual reduced (B, f)-bordism of X. Combining this isomorphism with Φ gives

$$\Omega_{n\,m}((B, f), (BO, 1), (BG_{n-m}, \varphi')) \cong \widetilde{\Omega}_n(B, f)\, (T\varphi'^* \gamma^{n-m}) .$$

This is really the standard relation of a submanifold of N^n with G_{n-m} normal bundle and a map of N^n into the Thom space $T\varphi'^* \gamma^{n-m}$ of BG_{n-m}, which is the basis of Poincaré duality in bordism-cobordism.

EXAMPLE 2. As an example not covered by Wall's situation consider the case $(B, f) = (BO, 1)$, $(BG_{n-m}, \varphi') = (BO_{n-m}, 1)$ and $(C, g) = (BU, \pi)$ being the classifying space sequence for complex bundles. Thus, one is considering pairs (N^n, M^m) in which M^m is a stably almost complex submanifold of N^n. While this case is not difficult to analyze directly, it is clear that the structure cannot adequately be described by normal structure of N and structure of the bundle ν_M^N.

In order to analyze these groups, one may reformulate the sequence (D, h) by making use of stability. Specifically, being given a sequence (B, f) of fibrations $f_r : B_r \to BO_r$ and compatibility maps $g_r : B_r \to B_{r+1}$ with $j_r \circ f_r = f_{r+1} \circ g_r$, one may let $B = \lim_{r\to\infty} B_r$ and $BO = \lim_{r\to} BO_r$ (using the maps g_r, j_r to define the limits) and the maps f_r define a map $f_r : B \to BO$. Conversely, beginning with a fibration, $f : B \to BO$ one may take the induced fibrations $f'_r : B'_r \to BO_r$ (with obvious compatibility maps) to define a new sequence (B', f'). If $f : B \to BO$ arises from the sequence of maps (B, f), there is a map of sequences ψ: $(B, f) \to (B', f')$, ψ_r: $B_r \to B'_r$ with the obvious compatibility conditions. By stability of the spaces BO_r one sees that ψ induces an isomorphism of cobordism theories.

Applying this limit process to the sequence (B, f), (C, g) and (D, h), one obtains a commutative diagram of fibrations

$$\begin{tikzcd} & D \arrow[dl, "\pi_C"'] \arrow[dr, "\pi_B"] \arrow[dd, "h"] & \\ C \arrow[dr, "g"'] & & BG_{n-m} \times B \arrow[dl, "\oplus \circ (\varphi' \times f)"] \\ & BO\,. & \end{tikzcd}$$

One may also form a commutative diagram of fibrations

$$\begin{tikzcd} & C \times BG_{n-m} \arrow[d, "g \times \varphi'_s"] & E \arrow[l, "p_0"'] \arrow[dd, "p_1"] \\ BO & BO \times BO \arrow[l, "\pi_1"'] \arrow[d, "1 \oplus (-1)"] & \\ & BO & B \arrow[l, "f"'] \end{tikzcd}$$

in which E is the induced fibration (either way), φ'_s is the stabilization of φ' and $1 \oplus (-1) : BO \times BO \to BO$ classifies the Whitney sum of the universal bundle and its "inverse" over the two factors. One has maps $D \xrightarrow{(\pi_1 \circ \pi_B) \times \pi_C} C \times BG_{n-m}$ and $D \xrightarrow{\pi_2 \circ \pi_B} B$ inducing a map $\lambda : D \to E$ and $E \xrightarrow{\pi_1 \circ p_0} C$, $E \xrightarrow{(\pi_2 \circ p_0) \times p_1} BG_{n-m} \times B$ giving $\mu : E \to D$, and λ and μ identifying the spaces E and D. Under this identification the map $h : D \to BO$ is identified with

$$k = \pi_1 \circ (g \times \varphi'_s) \circ p_0 : E \longrightarrow BO\,.$$

One then has

LEMMA. *$\Omega_m((D, h)) \cong \Omega_m((E, k))$ under the identifications λ and μ.*

The identification of D with E is the general formulation of the fact that a space X with a stable bundle ν with C structure and $\hat{\nu}$ with B structure and an $(n-m)$-plane bundle ξ with BG_{n-m} structure for which $\nu \cong \xi \oplus \hat{\nu}$ is the same as X having a stable C bundle ν, an $(n-m)$-plane ξ with BG_{n-m} structure together with a B structure on the stable bundle $\nu - \xi$.

Applying this lemma to any situation in which $(B, f) = (BO, 1)$ one has E identified with $C \times BG_{n-m}$ and hence the associated Thom spectrum is the smash product of BG_{n-m}^{+} (BG_{n-m} with adjoined base point) and the Thom spectrum $Tg_r^*\gamma^r$ of C. Thus

$$\begin{aligned}\Omega_{n,m}((BO, 1), (C, g), (BG_{n-m}, \varphi')) &\cong \mathfrak{N}_n \oplus \tilde{\Omega}_m(C, g)\,(BG_{n-m}^{+}) \\ &\cong \mathfrak{N}_n \oplus \Omega_m(C, g)\,(BG_{n-m})\end{aligned}$$

where $\mathfrak{N}_n$ is the ordinary unoriented cobordism group of Thom.

The special case given in this example is

$$\Omega_{n,m}((BO, 1), (BU, \pi), (BO_{n-m}, 1)) \cong \mathfrak{N}_n \oplus \Omega_m^U(BO_{n-m})$$

where Ω_m^U is ordinary complex bordism. This identification is quite straightforward from the arguments that (N, M) is given by the class of N and the stably almost complex manifold M with an $(n-m)$ plane bundle.

EXAMPLE 3. As a highly non-trivial example one may consider the classification of pairs (N^n, M^m) of framed manifolds. This is the case $(B, f) = (C, g) = (EO, \pi)$ where $\pi : EO \to BO$ is the universal principal O-bundle with contractible total space, and $(BG_{n-m}, \varphi') = (BO_{n-m}, 1)$. For this case one has the fibration diagram

$$\begin{tikzcd} & D \arrow[dl] \arrow[dd, "h"] \arrow[dr] & \\ EO \arrow[dr, "\pi"'] & & BO_{n-m} \times EO \arrow[dl, "\oplus\,(1\times\pi)"] \\ & BO & \end{tikzcd}$$

and identifying $BO_{n-m} \times EO$ with BO_{n-m} (EO being contractible) the inclusion $BO_{n-m} \to BO$ may be considered as the O/O_{n-m} bundle associated with the principal bundle π. Pulling π back over itself gives a trivial bundle, so that the associated bundle is trivial and $\pi_C : D \to EO$ may be identified with the fibration $\pi_1 : EO \times (O/O_{n-m}) \to EO$. Taking Thom spaces gives

$$\Omega_{n,m}((EO, \pi), (EO, \pi), (BO_{n-m}, 1)) \cong \Omega_n^{fr} \oplus \Omega_m^{fr}(O/O_{n-m})$$

where Ω_*^{fr} denotes framed bordism, or stable homotopy. Notice that O/O_{n-m} is the classifying space for $(n-m)$-plane bundles with a stable trivialization. (i.e., $\nu_M^N \cong \nu_M - \nu_N$ has a stable trivialization) and this is obviously compatible with the geometric analysis.

The case $n-m=0$ is quite interesting, giving the group $\Omega_n^{fr} \oplus \Omega_n^{fr}(0)$. The pair (N, M) being a codimension 0 pair, M is a union of components of N, and gives the element of $\Omega_n^{fr}(0)$ represented by the two distinct framings of M (its own framing and that induced from N) with the map into 0 being the "rotation" of one framing to the other.

The computation of the groups $\Omega_*^{fr}(0)$ is a highly unknown quantity, being the stable homotopy of the stable orthogonal group. (Compare I. M. James's "Problem 52" [2; page 587]).

Note: Geometric interpretations of other standard homotopy groups may be obtained similarly by demanding that the normal structures should be compatible in some fibration over BO. (i.e., $j: F \to BO$ is a multiplicative fibration and $f = j \circ f'$, $g = j \circ g'$ with $f': B \to F$, $g': C \to F$ and $\varphi' = j_{n-m} \circ \varphi'': BG_{n-m} \to F_{n-m}$, and one demands that the underlying F structures given by the $\varphi' \times f'$ and g' structures should coincide). For example, the classification of pairs (N^n, M^m) of framed manifolds with $n-m=2s$ and ν_M^N being a complex s plane bundle so that the stable almost complex structure of M coincides with that induced from N with the complex structure of ν_M^N leads to the group $\Omega_n^{fr} \oplus \Omega_m^{fr}(U/U_s)$.

4. Pairs of complex manifolds. As the main example, consider pairs (N^n, M^m) of stably almost complex manifolds, or the case in which $(B, f) = (C, g) = (BU, \pi)$ and $(BG_{n-m}, \varphi') = (BO_{n-m}, 1)$.

$$\begin{tikzcd}
 & BU \times BO_{n-m} \arrow[d, "\pi \times \varphi_s'"] & D \arrow[l, "p_0"'] \arrow[dd, "p_1"] \\
BO & BO \times BO \arrow[l, "\pi_1"'] \arrow[d, "1 \oplus (-1)"] & \\
 & BO & BU . \arrow[l, "\pi"']
\end{tikzcd}$$

Considering the diagram one has over D two stable complex bundles ν and $\hat{\nu}$ and a real $(n-m)$-plane bundle ξ with $\nu \cong \hat{\nu} \oplus \xi$, so ξ has a complex structure stably given by $\nu - \hat{\nu}$. Let F_{n-m} denote the induced fibration

$$\begin{tikzcd} F_{n-m} \arrow[r, "q_1"] \arrow[d, "q_0"'] & BU \arrow[d, "\pi"] \\ BO_{n-m} \arrow[r, "j"] & BO \end{tikzcd}$$

and one has a commutative diagram

$$\begin{tikzcd} BU \times BO_{n-m} \arrow[d, "\pi\times\varphi_s'"] & BU \times F_{n-m} \arrow[l, "1\times q_0"'] \arrow[d, "1\times q_1"] \\ BO \times BO \arrow[d, "1\oplus(-1)"] & BU \times BU \arrow[l, "\pi\times\pi"'] \arrow[d, "1\oplus(-1)"] \\ BO & BU \arrow[l, "\pi"'] \end{tikzcd}$$

inducing a map $r : BU \times F_{n-m} \to D$ while

$$\begin{tikzcd} D \arrow[r, "p_0\times p_1"] \arrow[d, "\pi_2\times p_0"'] & BU \times BO_{n-m} \times BU \arrow[r, "\pi_1\times\pi_3"] & BU \times BU \arrow[r, "1\oplus(-1)"] & BU \arrow[d, "\pi"] \\ BO_{n-m} \arrow[rrr, "j"] & & & BO \end{tikzcd}$$

gives a map $s : D \to F_{n-m}$ and so $\pi_1 \circ p_0 \times s : D \to BU \times F_{n-m}$. The maps r and $\pi_1 \circ p_0 \times s$ are equivalences, and one may then apply the Thom space construction to the map

$$BU \times F_{n-m} \xrightarrow{\pi_1} BU \xrightarrow{\pi} BO$$

which is identifiable with h to obtain

$$\Omega_{n,m}((BU, \pi), (BU, \pi), (BO_{n-m}, 1)) \cong \Omega_n^U \oplus \Omega_m^U(F_{n-m})$$

where F_{n-m} is the induced bundle in

$$\begin{tikzcd} F_{n-m} \arrow[r, "q_1"] \arrow[d, "q_0"'] & BU \arrow[d] \\ BO_{n-m} \arrow[r, "j"] & BO \end{tikzcd}$$

and is the classifying space for real $(n-m)$-plane bundles with a (chosen) stable complex structure.

To complete the analysis of this problem, one analyzes the cohomology structure of F_{n-m}. Since $\pi : BU \to BO$ has fiber O/U

which has two components interchanged under the action of $\pi_1(BO) = Z_2$ on the fiber, it is convenient to factor the above diagram as

$$\begin{array}{ccc} F_{n-m} & \xrightarrow{q_1} & BU \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle \pi'} \\ BSO_{n-m} & \xrightarrow{j'} & BSO \\ {\scriptstyle \pi''_{n-m}}\downarrow & & \downarrow{\scriptstyle \pi''} \\ BO_{n-m} & \xrightarrow{j} & BO \end{array}$$

so that F_{n-m} fibers over BSO_{n-m} with fiber SO/U which is connected. (*Note*: For $n - m = 0$, $F_{n-m} = O/U$ which is two disjoint copies of SO/U).

Considering first the rational or odd primary structure of F_{n-m}, one has:

PROPOSITION 1. *The fibration $\pi' : BU \rightarrow BO$ is totally nonhomologous to zero for $H^*(\; ; K)$, $K = Q$ or $Z_p(p$ odd), so that the fibration $q' : F_{n-m} \rightarrow BSO_{n-m}$ is also totally nonhomologous to zero. In particular, $H^*(F_{n-m}, K)$ is the polynomial algebra over $H^*(BSO_{n-m}, K)$ on the classes $c_{2i+1}(\xi)$* (odd Chern classes of the complex bundle).

Note:

$$\text{If } n - m = 2s + 1, \; H^*(BSO_{n-m}; K) = K[\mathscr{P}_i \mid 1 \leqq i \leqq s] \, ,$$
$$\text{if } n - m = 2s > 0, \; H^*(BSO_{n-m}; K) = K[\mathscr{P}_i, \chi \mid 1 \leqq i \leqq s - 1]$$

and if $n - m = 0$, $H^*(BSO_{n-m}; K) = K \oplus K$, where $\mathscr{P}_i$ is the Pontrjagin class and χ is the Euler class. One may interpret $H^*(BSO_0; K)$ as having base 1, χ with $\chi^2 = 1 = \mathscr{P}_0$.

Proof. $\pi' : BU \rightarrow BSO$ is a multiplicative fibration, with

$$\pi'^*: H^*(BSO \; ; K) = K[\mathscr{P}_i] \longrightarrow H^*(BU \; ; K) = K[c_i]$$

by $\pi'^*(\mathscr{P}_i) = 2c_{2i} +$ decomposables. Thus π'^* is monic, so π' is totally nonhomologous to zero. In fact, one then has $H^*(SO/U \; ; K) \cong K[c_{2i+1}]$ and the induced fibration is also totally nonhomologous to zero.

For the prime 2 one has

$$\pi'^* : H^*(BSO \; ; Z_2) = Z_2[w_i \mid i > 1] \longrightarrow H^*(BU \; ; Z_2) = Z_2[c_i]$$

with

$$\pi'^*(w_{2i}) = c_i, \, \pi'^*(w_{2i+1}) = 0 \, .$$

In the fibration

$$U \xrightarrow{i} SO \xrightarrow{p} SO/U$$

one has $i^*: H^*(SO;Z_2) = Z_2[y_{2i+1}] \to H^*(U\,;\,Z_2) = E[u_{2i+1}]$, where E denotes the exterior algebra with $i^*(y_{2i+1}) = u_{2i+1}$. Then $H^*(SO/U\,;\,Z_2)$ is the Z_2 polynomial ring on generators $z_{4i+2} = y_{2i+1}{}^2$. $H^*(SO/U\,;\,Z_2)$ has an associated graded module isomorphic to the exterior algebra $E[v_{2i}]$ and in the Serre spectral sequence of the fibration

$$SO/U \longrightarrow BU \xrightarrow{\pi'} BSO$$

the class v_{2i} transgresses to w_{2i+1}. (*Note*: One has a simple system of transgressive generators given by monomials in the v_{2i} where $v_{2^k(4j+2)} = z^{2k}_{4j+2}$). In the Serre spectral sequence for the fibration

$$SO/U \longrightarrow F_{n-m} \longrightarrow BSO_{n-m}$$

the classes v_{2i} transgress to w_{2i+1} if $2i + 1 \leqq n - m$, while v_{2i} transgresses to zero for $2i + 1 > n - m$.

One may then observe that the Serre spectral sequence for F_{n-m} is a tensor product of spectral sequences of three types

Type 1: Exterior fiber on Z_{2i}, Polynomial base on its transgression $(2i + 1 \leqq n - m)$

Type 2: Exterior fiber on Z_{2i}, trivial base $(2i + 1 > n - m)$

Type 3: Trivial fiber; Polynomial base on w_{2i}, $(2i \leqq n - m)$.

One may then compute $H^*(F_{n-m}; Z_2)$, which is a polynomial ring on the Stiefel-Whitney classes w_{2i}, $1 < 2i \leqq n - m$, and classes $t_{2^k(4j+2)}$ mapping to $z_{4i+2^{2k}} \in H^*(SO/U; Z_2)$ where k is the least integer for which $2^k(4j + 2) \geqq n - m$. (This is not entirely obvious, but an analysis of the spectral sequence shows that the classes t exist and one then notes that the polynomial ring maps isomorphically onto E^∞ in the Serre spectral sequence, hence coincides with $H^*(F_{n-m}\,;\,Z_2)$).

For the purposes of this paper, it suffices to observe that $H^{2j+1}(F_{n-m}; Z_2) = 0$ for all j since the E^∞ terms of the spectral sequences of types $1 - 3$ contain only even dimensional elements. One then has

THEOREM. *The space F_{n-m} has torsion free homology.*

Proof. If $H_*(F_{n-m}; Z)$ had any torsion, one of the groups $H^*(F_{n-m}; Z_p)$, $p = 2$ or odd, must be nonzero in two consecutive dimensions. Since $H^{2j+1}(F_{n-m}; Z_p) = 0$ for all p, there can be no torsion.

COROLLARY. *$\Omega_*((BU, \pi), (BU, \pi), (BO_{n-m}, 1))$ is a free Ω^U_* module* *and a pair* (N^n, M^m) *bounds if and only if all Chern numbers of* N *and all evaluations on* $[M]$ *of polynomials in the Chern classes of* M *and* N *and the Euler Class of* ν_M^N *are zero.*

Proof. Since F_{n-m} is torsion free, $\Omega_*^U(F_{n-m})$ is a free Ω_*^U module. Further, being given (M, ξ) with ξ a real $(n-m)$ plane bundle with complex structure, its class is determined by generalized Chern numbers $c_\omega(M) \cdot c_\lambda(\xi)\mathscr{P}_{\lambda'}(\xi)\chi(\xi)^k[M]$ given by Chern classes of M and rational cohomology classes of F_{n-m}. Since the Pontryagin class of a complex vector bundle is a polynomial in the Chern classes (with $\mathscr{P}_i = 2c_{2i} +$ decomposables) one may rewrite this using numbers $c_\omega(M)c_\lambda(\xi)\chi(\xi)^k[M]$. If $\xi = \nu_M^N$ then the Chern classes of ξ are polynomials in the Chern classes of M and N (restricted to M), and this gives the result.

Notes. (1) A complex structure on the $(n-m)$-plane bundle ξ imparts an orientation, and hence ξ has an Euler class defined. This class is *not* related to the Chern classes of ξ as stable complex bundle, however. The integral cohomology $H^*(F_{n-m}; Z)$ is embedded in $H^*(F_{n-m}; Q)$ via the rational reduction and contains the integral polynomial ring on the classes $c_i(\xi)$ for i odd or $i = 2k$, $(k \leqq [(n-m)/2]$, if $n-m$ is odd, $k < [(n-m)/2]$ if $n-m$ is even) and $\chi(\xi)$ (if $n-m$ is even) as a subgroup with 2 primary index. [Observe that $\chi(\xi) - c_{(n-m)/2}(\xi)$ is divisible by 2, since both reduce to $w_{n-m}(\xi)$, and that $\chi^2(\xi) = \mathfrak{P}_{(n-m)/2}(\xi) = 2c_{n-m}(\xi) +$ decomposables in the $c_i(\xi)$, and hence this is a proper subgroup for $n - m$ even].

(2) For $n = m$, $M^m \subset N^n$ consists of a union of components of N having two stable almost complex structures. Then $\Omega_m(F_0) = \Omega_m(0/U)$ corresponds to two distinct stable almost complex structures on the same manifold. This group is closely related to the classification of involutions on stable almost complex manifolds (which need not preserve the structure). This relation will be explored in another paper, "Involutions on Complex Manifolds" (to appear).

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