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WIENER'S COMPACTIFICATION AND Φ-BOUNDED HARMONIC FUNCTIONS IN THE CLASSIFICATION OF HARMONIC SPACES

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The class $H\Phi$ of Φ-bounded harmonic functions on Riemann surfaces first investigated by Parreau for the special case where Φ is increasing and convex, was later characterized by Nakai in its complete generality by assuming only that Φ was a nonnegative real-valued function on $[0, \infty)$. In this paper we show that Nakai's theory can be presented in the axiomatic setting of Brelot. The theory of Wiener compactifications which is indispensable in the study of potential theory on Riemann surfaces is extended to harmonic spaces and shown to be equally useful in the potential theory there.

In particular we obtain a classification scheme for the theory of harmonic spaces for the class $O_{H\Phi}$ of spaces for which $H\Phi$ consists only of constants. In this scheme it is shown that boundedness properties such as positiveness, boundedness in absolute value, quasi-boundedness, and essential positiveness can all be considered as special cases of Φ-boundedness. A similar classification is briefly given for subdomains.

2. Let $\mathfrak{X}$ be a locally compact Hausdorff space which is connected and locally connected. Suppose that to each open set Ω in $\mathfrak{X}$ there corresponds a linear space $H(\Omega)$ of finitely-continuous real-valued functions defined on Ω. This in turn defines a family $H = \{H(\Omega)\}_\Omega$ of functions with domains in $\mathfrak{X}$. If Ω is an open subset of $\mathfrak{X}$ then by $\partial\Omega$ we will always mean the boundary of Ω relative to $\mathfrak{X}$. A relatively compact open set Ω is said to be *regular* for H if for every continuous real-valued function f defined on $\partial\Omega$ there is a unique continuous function h_f defined on $\bar{\Omega}$ such that $h_f|\partial\Omega = f$, $h_f|\Omega \in H(\Omega)$ and $h_f \geqq 0$ if $f \geqq 0$.

By a *harmonic space* we mean a pair $(\mathfrak{X}, H)$ where $\mathfrak{X}$ and H are as above and in addition H satisfies the following axioms:

AXIOM I. A function g with open domain $\Omega \subset \mathfrak{X}$ is a member of H if for each $X \in \Omega$ there is a function $h \in H$ and an open set $\hat{\Omega} \ni g|\hat{\Omega} = h|\hat{\Omega}$.

AXIOM II. The regular regions in $\mathfrak{X}$ for H form a basis for the topology of $\mathfrak{X}$.

AXIOM III. If $\mathscr{T}$ is a subset of $H(\Omega)$, $\Omega \subset \mathfrak{X}$ a subregion, and $\mathscr{T}$ is an upper-directed family, then the upper envelope of $\mathscr{T}$ is either ∞ or a function in $H(\Omega)$.

AXIOM IV. $1 \in H(\mathfrak{X})$.

When H is well understood we will simply refer to $\mathfrak{X}$ itself as the harmonic space. Axioms I-III were introduced by Brelot [1], while Axiom IV is similar to Axiom IV of Loeb [10].

3. If Ω is regular and $x \in \Omega$ then $h_f(x)$ as a function of f, is a bounded positive linear functional on the set $C(\partial\Omega)$ of continuous functions on $\partial\Omega$. Hence there exists a finite positive Radon measure $\mu(.,x,\Omega)$ on $\partial\Omega$ such that $h_f(x) = \int_{\partial\Omega} f(y)d\mu(y,x,\Omega)$.

A lower semicontinuous function s with open domain $\Omega \subset \mathfrak{X}$ is said to be in the class $\bar{H}$ if

(i) $s(x) < \infty$ for some x in each component of Ω.

(ii) for each $x_0 \in \Omega$ such that $s(x_0) < \infty$ and for every neighborhood $\Omega_1 \subset \Omega$ of x_0 there is a regular region Ω_0 with $\bar{\Omega}_0 \subset \Omega_1$ such that s is integrable on $\partial\Omega_0$ and $s(x_0) \geqq \int_{\partial\Omega_0} s(y)\, d\mu\,(y, x_0, \Omega_0)$. An upper semicontinuous function t will belong to the class $\underline{H}$ if $-t$ belongs to $\bar{H}$. We call H the class of *harmonic* functions and $\bar{H}$ (resp. $\underline{H}$) the class of *superharmonic* (resp. *subharmonic*) functions associated with H. We denote by $\bar{H}(\Omega)$ (resp. $\underline{H}\ (\Omega)$) the functions in $\bar{H}$ (resp. $\underline{H}$) with domain Ω. The harmonic space $\mathfrak{X}$ is said to be *parabolic* (denoted $\mathfrak{X} \in O_G$) provided there does not exist any nonconstant positive superharmonic functions on $\mathfrak{X}$.

Let Ω be an open set in $\mathfrak{X} \notin O_G$ and f a bounded real-valued function on $\partial\Omega$. Consider the family $\mathscr{C}(\Omega,f)$ of superharmonic functions $s \in \bar{H}(\Omega)$ with $\liminf_{x \in \Omega, x \to x_0} s(x) \geqq f(x_0)$ for all $x_0 \in \partial\Omega$ and $\liminf_{x \in \Omega \to \beta} s(x) \geqq 0$ (β = ideal boundary of $\mathfrak{X}$). Then $C(\Omega,f)$ is a Perron family and by Perron's Theorem (see e.g. Brelot [1]), $\bar{h}(f,\Omega)\ (x) = \inf\ \{s(x) \mid s \in \mathscr{C}(\Omega,f)\}$ and $\underline{h}(f,\Omega)\ (x) = -\bar{h}(-f,\Omega)\ (x)$ are harmonic on Ω with $\bar{h} \geqq \underline{h}$. If $\bar{h} = h$ we denote the common function by $h(f,\Omega)$ and call f *resolutive* on $\partial\Omega$. A point $x_0 \in \partial\Omega$ is said to be *regular* (for the Dirichlet problem) if $\lim_{x \in \Omega, x \to x_0} h(f,\Omega) = f(x_0)$ for every resolutive function f on $\partial\Omega$ which is continuous at x_0. In particular every boundary point of a regular open set is regular. By a result of Loeb [10] a point x_0 on the relative boundary $\partial\Omega$ of an open set is regular if there exists a barrier for Ω at x_0; that is, if there exists a positive harmonic function s defined in the intersection of Ω and an open neighborhood of x_0 and such that $\lim_{x \in \Omega, x \to x_0} s(x) = 0$. An open subset $\Omega \subset \mathfrak{X}$ is said to be *normal* if each $x_0 \in \partial\Omega$ is regular.

Let K be a compact subset of $\mathfrak{X}$ and $\mathscr{C}$ the family of all regular regions Ω containing K. Then by a theorem of Loeb [10] $\mathscr{E}$ is an exhaustion of $\mathfrak{X}$. We will always assume that $\mathfrak{X}$ is countable at the ideal boundary and it is therefore possible to obtain a countable exhaustion of $\mathfrak{X}$ by regular regions $\{\Omega_n\}$ such that $\bar{\Omega}_n \subset \Omega_{n+1}$ and $\mathfrak{X} = \cup_{n=1}^{\infty} \Omega_n$.

4. Let $G \notin O_G$ be a normal subregion of $\mathfrak{X}$ and f a real-valued function on $\mathfrak{X}$. Denote by $\bar{\mathfrak{A}}(G,f)$ (resp. $\underline{\mathfrak{A}}(G,f)$) the class of superharmonic (resp. subharmonic) functions s on G for which there exists a compact set $K_S \subset G$ with $s \geqq f$ (resp. $s \leqq f$) on G-K_S. If neither $\bar{\mathfrak{A}}(G,f)$ nor $\underline{\mathfrak{A}}(G,f)$ is empty, then being Perron families, we have

$$\bar{W}_f^G(p) = \inf_{s \in \bar{\mathfrak{A}}(G,f)} s(p) \quad \text{and} \quad \underline{W}_f^G(p) = \sup_{s \in \underline{\mathfrak{A}}(G,f)} s(p)$$

are harmonic on G with $\bar{W}_f^G \geqq \underline{W}_f^G$. If $\bar{W}_f^G \equiv \underline{W}_f^G$ we denote their common value by W_f^G. A function f is *harmonizable* on $\mathfrak{X}$ if W_f^G exists for every subregion $G \notin O_G$ which is regular for the Dirichlet problem. If $G \in O_G$ we define $W_f^G \equiv 0$. The idea of harmonizability is due to Constantinescu-Cornea (2).

5. Consider the family $\mathfrak{W}(\mathfrak{X})$ of real-valued, bounded, continuous functions f on $\mathfrak{X}$ which are harmonizable on $\mathfrak{X}$. Then $\mathfrak{W}(\mathfrak{X})$ forms an algebra with respect to addition, multiplication, and scalar multiplication of functions and is called the *Wiener algebra* of $\mathfrak{X}$. The subclass $\mathfrak{W}_\Delta(\mathfrak{X}) = \{f \mid f \in \mathfrak{W}(\mathfrak{X}),\ W_f^{\mathfrak{X}} = 0\}$ is an ideal of $\mathfrak{W}(\mathfrak{X})$ called the *potential subalgebra* of $\mathfrak{W}(\mathfrak{X})$. Both $\mathfrak{W}(\mathfrak{X})$ and $\mathfrak{W}_\Delta(\mathfrak{X})$ are closed under the lattice operations $\cap$ and $\cup$, i.e., $f \cup g = \max(g,f)$ and $f \cap g = \min(f,g)$. In addition $\mathfrak{W}(\mathfrak{X})$ is a Banach algebra with norm $\|f\|_{\infty\,\mathfrak{X}} = \sup_{p \in \mathfrak{X}} |f(p)|$ and $\mathfrak{W}_\Delta(\mathfrak{X})$ is a closed subset. For a complete account see the monograph of Sario-Nakai (18).

The *Wiener compactification* $\mathfrak{X}^*$ of $\mathfrak{X}$ is the unique compact Hausdorff space such that $\mathfrak{X}$ is dense in $\mathfrak{X}^*$, every $f \in \mathfrak{W}(\mathfrak{X})$ has a continuous extension to $\mathfrak{X}^*$, and $\mathfrak{W}(\mathfrak{X})$ separates points in $\mathfrak{X}^*$. See Loeb (9) for the existence of such compactifications. The compact set $\Gamma = \mathfrak{X}^* - \mathfrak{X}$ is called the *Wiener boundary* of $\mathfrak{X}$. Since the subset $\mathfrak{W}_0(\mathfrak{X})$ of $\mathfrak{W}(\mathfrak{X})$ consisting of functions with compact supports in $\mathfrak{X}$ is a subset of $\mathfrak{W}(\mathfrak{X})$ the set $\Delta = \{p \in \mathfrak{X}^* \mid f(p) = 0$ for all $f \in \mathfrak{W}_\Delta(\mathfrak{X})\}$, called the *Wiener harmonic boundary* of $\mathfrak{X}$, is a compact subset of Γ. The set $\mathfrak{W}_\Delta(\mathfrak{X})$ can be characterized in terms of Δ as follows: $\mathfrak{W}_\Delta(\mathfrak{X}) = \{f \in \mathfrak{W}(\mathfrak{X}) \mid f(\Delta) = 0\}$. From this we see that $\mathfrak{X} \in 0_G$ if and only if $\Delta = \phi$. Hereafter we will use topological notions with respect to $\mathfrak{X}^*$ only. For example if $\Omega \subset \mathfrak{X}^*$ then $\bar{\Omega}$ means the closure of Ω in $\mathfrak{X}^*$. However we will still retain the symbol $\partial\Omega$ for the boundary of Ω relative to $\mathfrak{X}$. Wiener's algebra was first introduced by S. Mori (14), Hayashi (4), Kusunoki (6), and Constantinescu-Cornea (2). Different treatments of the theory of the Wiener harmonic boundary, also presented in Brelot's axiomatic setting, can be found in the works of Constantinescu-Cornea [3], Loeb-Walsh [12], and Lumer-Naim [13].

6. We shall denote by $HP(\mathfrak{X})$, $HB(\mathfrak{X})$ the classes of functions on $\mathfrak{X}$ which are nonnegative harmonic, and bounded harmonic, respectively; and by O_{HP} (resp. O_{HB}) the class of harmonic spaces $\mathfrak{X}$ for which the class $HP(\mathfrak{X})$ (resp. $HB(\mathfrak{X})$) consists only of constants. Note that $O_G \subset O_{HP} \subset O_{HB}$. A harmonic function u on $\mathfrak{X}$ is called *essentially positive* if u can be represented as a difference of two HP functions on $\mathfrak{X}$, or equivalently, if $|u|$ has a harmonic majorant on $\mathfrak{X}$. The space $HP'(\mathfrak{X})$ of essentially positive harmonic functions on $\mathfrak{X}$ forms a vector lattice with lattice operations $\vee$ and $\wedge$, where for two functions u and v in $HP'(\mathfrak{X})$ we denote by $u \vee v$ (resp. $u \wedge v$) the least harmonic majorant (resp.the greatest harmonic minorant) of u and v. Clearly $HP(\mathfrak{X}) \subset HP'(\mathfrak{X})$. If $\mathfrak{X} \in O_G$ we define $HP'(\mathfrak{X}) = \{0\}$.

For any $u \in HP(\mathfrak{X})$ we define the function Bu by $Bu(p) = \sup \{v(p) \mid v \in HB(\mathfrak{X}),\ v \leqq u \text{ on } \mathfrak{X}\}$. Next for $u \in HP'(\mathfrak{X})$ we define $Bu = Bu_1 - Bu_2$ where $u = u_1 - u_2$, with $u_1, u_2 \in HP(\mathfrak{X})$. This last definition is independent of the particular decomposition of u since B is additive on HP. One can verify that B is order preserving, linear, and satisfies $B^2u = Bu$ on HP'. Moreover $B(u_1 \vee u_2) = Bu_1 \vee Bu_2$ and $B(u_1 \wedge u_2) = Bu_1 \wedge Bu_2$. An HP' function u is called *quasi-bounded* (resp. *singular*) if $Bu = u$ (resp. $Bu = 0$). We denote the class of quasi-bounded (resp. singular) functions on $\mathfrak{X}$ by $HB'(\mathfrak{X})$ (resp. $HP''(\mathfrak{X})$). Since $B^2 = B$ and $I = B + (I - B)$, where I is the identity operator on HP' we have the direct sum decomposition of Parreau (17):

$$HP'(\mathfrak{X}) = HB'(\mathfrak{X}) + HP''(\mathfrak{X}).$$

7. We now state the maximum principle for HB' functions with respect to the Wiener harmonic boundary (S. Mori (14), Hayashi (5), Kusunoki (6)).

THEOREM 1. *Let G be a subregion of a harmonic space $\mathfrak{X}$ and $u \in HB'(G)$ such that*

$$m \leqq \liminf_{z \in G, z \to p} u(z) \leqq \limsup_{z \in G, z \to p} u(z) \leqq M$$

for each $p \in \partial G \cup (\varDelta \cup \bar{G})$. Then $m \leqq u \leqq M$ on G.

A compact set K of $\mathfrak{X}^*$ will be called *distinguished* is (i) $\overline{K \cap \mathfrak{X}} = K$, (ii) $\partial(K \cap \mathfrak{X})$ consists of regular points, and (iii) each component of $\partial(K \cap \mathfrak{X})$ has nonempty interior.

If $\mathfrak{X} \notin O_G$ we denote by $\mathscr{H}(\mathfrak{X})$ the vector lattice of continuous harmonizable functions f on $\mathfrak{X}$ such that there exists a continuous superharmonic function s_f for which the set $\{p \in \mathfrak{X} \mid s(p) = \infty\}$ is discrete and $s_f \geqq |f|$ on $\mathfrak{X}$. Note that $\mathfrak{W}(\mathfrak{X}) \subset \mathscr{H}(\mathfrak{X})$ and that if $f \in \mathscr{H}(\mathfrak{X})$ and G is a subregion of $\mathfrak{X}$ then $f|G \in \mathscr{H}(G)$. The following decomposition theorem is valid for harmonic spaces:

THEOREM 2. *Let $f \in \mathscr{H}(\mathfrak{X})$ and K be a distinguished compact set. Then f can be uniquely expressed in the form $f = u + v$ where $u \in HB'(\mathfrak{X} - K) \cap \mathscr{H}(\mathfrak{X})$ and $v \in \mathscr{H}(\mathfrak{X})$ with $v = 0$ on $K \cup \Delta$. Moreover $\|u\|_{\infty,\mathfrak{X}} \leqq \|f\|_{\infty, K\cup\Delta}$.*

Both Theorem 1 and Theorem 2 may be proved by methods similar to that in Sario-Nakai (18).

8. As a consequence of Theorem 2 we may define a projection function $\pi_K f \in \mathscr{H}(\mathfrak{X})$ for an $f \in \mathscr{H}(\mathfrak{X})$ and K a distinguished compact subset of $\mathfrak{X}^*$: $\pi_K f \mid \mathfrak{X} - K = u \in HB'(\mathfrak{X} - K)$ and $\pi_K f \mid K \cup \Delta = f \mid K \cup \Delta$. The following theorem illustrates the function-theoretic smallness of $\Gamma - \Delta$.

THEOREM 3. *Suppose $\mathfrak{X} \notin O_G$ and F is any compact subset of $\Gamma - \Delta$. Then there exists a finitely continuous positive superharmonic function s_F on $\mathfrak{X}$ which is continuous on $\mathfrak{X}^*$ such that $s_F \mid \Delta = 0$ and $s_F \mid F = \infty$.*

Proof. Let $V \supset F$ be an open neighborhood of $\mathfrak{X}^*$ such that $\bar{V}$ is a distinguished compact set of $\mathfrak{X}^*$ satisfying $\bar{V} \cap \Delta = \emptyset$. For a regular exhaustion $\{\Omega_n\}_1^\infty$ of $\mathfrak{X}$ choose an $f \in \mathfrak{W}_\Delta(\mathfrak{X})$ such that $f \mid \bar{V} = 1$. Let $u_n = \pi_{K_n} f$ where $K_n = \bar{V} - \Omega_n$. By Theorem 1 $\{u_n\}_1^\infty$ is a decreasing sequence and $u = lim_n\ u_n \in HB'(\mathfrak{X})$. Since $0 \leqq u \leqq u_n$ and $u_n \mid \Delta = 0$ we conclude $u \equiv 0$ by Theorem 1. For a $z_0 \in \Omega_1$ we may choose a subsequence again denoted $\{u_n\}$ such that $u_n(z_0) < 2^{-n}$ $(n = 1,2,\cdots)$. Then $s_F = \sum_{n=1}^\infty u_n$ is a finitely continuous positive superharmonic function on $\mathfrak{X}$ with $s_F \mid F = \infty$.

We note that if s is any continuous superharmonic function on $\mathfrak{X}$ bounded from below and if $m = \min_\Delta s$ then $s \geqq m$ on $\mathfrak{X}$. For let $m > c > -\infty$ and $K = \{p \in \Gamma \mid s(p) \leqq c\}$. Then for s_K defined just as s_F above we have for any $\varepsilon > 0$ $\lim_{z \in \mathfrak{X}, z \to p} (s(z) + \varepsilon s_K(z)) \geqq c$ for all $p \in \Gamma$. It follows that $s + \varepsilon S_K \geqq c$ on $\mathfrak{X}$ and consequently $s \geqq m$. Hence if $v \in HB'(\mathfrak{X})$ and $s_F \geqq v$ then $0 \geqq v$ on $\mathfrak{X}$ since $u_n \mid \Delta = 0$, $(n = 1,2,\cdots)$. Now if $s_F|\Delta \not\equiv 0$ then consider $v = \pi_\phi(s_F \cap 1) \in HB'(\mathfrak{X})$ where $v > 0$. Observing $(s_F - v) \mid \Delta \geqq 0$ we have $s_F \geqq v > 0$ on $\mathfrak{X}$ contradicting the fact that $s_F \geqq v$ implies $v \leqq 0$.

9. We have the following maximum principle for superharmonic functions:

THEOREM 4. *Let s be a superharmonic function on a subregion G of a harmonic space $\mathfrak{X} \notin O_G$ such that s is bounded from below and* $\liminf_{z \in G, z \to p} s(z) \geqq m$ *for every* $p \in (\Delta \cap \bar{G}) \cup \partial G$. *Then* $s \geqq m$ *on* G.

Proof. Define a lower semicontinuous function $\hat{s}$ on $\gamma = \hat{G} - G$ by $\hat{s}(p) = \lim_{z \in G, z \to p} \inf s(z)$. For any real number c such that $c < m$ $\mathfrak{A} = \{p \in \gamma \mid \hat{s}(p) > c\}$ is an open set in γ containing $(\bar{G} \cap \Delta) \cup \partial G$. Thus $F = \gamma - U \subset \Gamma - \Delta$ is compact and we may apply Theorem 3 to obtain a function s_F. For each n set $W_n = s + s_F/n$. Then W_n is a superharmonic function on G, bounded from below, satisfying $\liminf_{z \in G, z \to p} W_n(z) > c$ for all $p \in \gamma$. Hence $W_n(z) > c$ on G for each n. In the limit as $n \to \infty$ we obtain $s \geqq m$ on G.

10. A function $u > 0$ in $HB(\mathfrak{X})$ will be called HB-minimal on $\mathfrak{X}$ if the following is true: whenever $v \in HB(\mathfrak{X})$ and $u \geqq v \geqq 0$ then $v = c_v u$ for some constant c_v. We denote by U_{HB} the class of harmonic spaces $\mathfrak{X}$ on which there exists at least one HB-minimal function. Note that $U_{HB} \cap O_G = \emptyset$ since $HB(\mathfrak{X}) = \{0\}$ for $\mathfrak{X} \in O_G$. Also $O_{HB} - O_G \subset U_{HB}$. The following HB-minimal criterion for Riemann surfaces is due to S. Mori (14) and Hayashi (5).

THEOREM 5. *A function $u \in HB(\mathfrak{X})$ is HB-minimal if and only if there exists an isolated point $p \in \Delta$ for which $u(p) > 0$ and $u \mid (\Delta - p) = 0$.*

Proof. Assume that $p \in \Delta$ is isolated and there exists a $u \in HB(\mathfrak{X})$ such that $u(p) > 0$ and $u = 0$ on $\Delta - p$. Since $HB \subset HB'$ we have $u > 0$ by Theorem 1. By the Stone-Weierstrass theorem $\mathfrak{W}(\mathfrak{X})$ coincides with the class $B(\mathfrak{X}^*)$ of bounded continuous functions on $\mathfrak{X}^*$. Hence by Urysohn's lemma there exists an $f \in \mathfrak{W}(\mathfrak{X})$ such that $f(p) = 1$ and $f = 0$ on $\Delta - p$. Now $\hat{u} = \pi_\phi f \in HB'(\mathfrak{X})$ with $\hat{u}(p) = 1$ and $\hat{u} = 0$ on $\Delta - p$. For $c \in (0, u(p))$ the function $u - c\hat{u} \in HB'(\mathfrak{X})$ is nonnegative on Δ. Hence by Theorem 1 $u \geqq c\hat{u}$ on $\mathfrak{X}$. For $z_0 \in \mathfrak{X}$ we have $c \leqq u(z_0)/\hat{u}(z_0)$ and hence $0 < u(p) < \infty$. If $v \in HB(\mathfrak{X})$ and $u \geqq v \geqq 0$ on $\mathfrak{X}$ then $v = 0$ on $\Delta - p$ and $0 \leqq v(p) < \infty$. Setting $c_v = v(p)/u(p)$ it follows that $c_v u - v$ vanishes identically on Δ. Consequently $v = c_v u$ and u is an HB-minimal function.

Conversely let u be HB-minimal on $\mathfrak{X}$. Now there exists a $p \in \Delta$ with $u(p) > 0$. If there is a $q \in \Delta$, $q \neq p$ and $u(q) > 0$ then pick a $f \in \mathfrak{W}(\mathfrak{X})$ such that $f(p) = 1$, $f(q) = 0$, and $0 \leqq f \leqq 1$ on $\mathfrak{X}^*$. Then $v = \pi_\phi(fu)$ satisfies the relation $0 \leqq v \leqq u$ on Δ and hence on $\mathfrak{X}$. Thus there exists a $c_v \neq 0$ such that $v = c_v u$. But then we have a contradiction $0 = v(q) = c_v u(q) > 0$. Hence $u = 0$ on $\Delta - p$, $u(p) > 0$ and moreover p is isolated due to the continuity of u on Δ.

COROLLARY 1. *$\mathfrak{X} \in O_{HB} - O_G$ if and only if Δ consists of a single point.*

Proof. Assume $\mathfrak{X} \in O_{HB} - O_G$. Then since any $u \in HB(\mathfrak{X})$ is continuous at Δ and $\mathfrak{X} \in U_{HB}$ we conclude by Theorem 5 that Δ consists of a single point.

Conversely if Δ consists of a single point then the continuity of any $u \in HB(\mathfrak{X})$ at Δ together with Theorem 1 implies that $\mathfrak{X} \in O_{HB} - O_G$.

11. Let $G \subset \mathfrak{X}$ be a subregion such that each point of $\partial G \neq \emptyset$ is regular. We say that $G \in SO_{HB}$ if every HB-function on G which vanishes continuously on ∂G is identically zero on G. More generally suppose $G = \cup_j G_j$ is a union of subregions $G_j \subset \mathfrak{X}$ such that $\partial G_j \neq \emptyset$ is regular for the Dirichlet problem (we will refer to G simply as a regular open set). Then $G \in SO_{HB}$ if each subregion $G_j \in SO_{HB}$, where the $G_j \cup \partial G_j$ are assumed to be disjoint. We have the following theorem for harmonic spaces whose counterpart for Riemann surfaces is due to Kusunoki-Mori (7,8), Hayashi (5), and S. Mori (14).

THEOREM 6. *Let $G \subset \mathfrak{X}$ be a subregion such that each point of $\partial G \neq \emptyset$ is regular. Then $G \in SO_{HB}$ if and only if $(\bar{G} - \partial\bar{G}) \cap \Delta = \emptyset$.*

Proof. Assume first that $(\bar{G} - \partial\bar{G}) \cap \Delta = \emptyset$. Suppose $u \in HB(G \cup \partial G)$ and $u | \partial G = 0$. Define $\hat{u}(p) = \liminf_{z \in G, z \to p} u(z)$ for $p \in \bar{G} - G$. Let $F = \{p \in \Gamma \cap \bar{G} \mid \hat{u}(p) \leqq m < 0\}$. If $F = \emptyset$ then $u \geqq m$. If $F \neq \emptyset$ then there exists a function s_F as in Theorem 3. Now $\hat{u} > m - \varepsilon s_F$ on $\bar{G} - G$ for each $\varepsilon > 0$. Hence $u \geqq m - \varepsilon s_F$ on G. Letting $\varepsilon \to 0$ it follows that $u \geqq m$, and consequently $u \geqq 0$. Applying a similar argument to $-u$ we get $u \leqq 0$ and $u \equiv 0$ on G.

Conversely assume $G \in SO_{HB}$. If $(\bar{G} - \partial\bar{G}) \cap \Delta \neq \emptyset$ then choose $f \in \mathfrak{W}(\mathfrak{X})$ such that $f \equiv 0$ on $\mathfrak{X} - G$ and $f \not\equiv 0$ on Δ. Then $\pi_{\overline{\mathfrak{X} \setminus G}} f$ is an HB-function which vanishes continuously on ∂G but is not identically zero on G. From this contradiction we conclude $(\bar{G} - \partial\bar{G}) \cap \Delta = \emptyset$.

12. In this section we shall define a new class of functions called Φ-bounded harmonic functions and show their relation to the classes HB and HP. Denote by $\Phi(t)$ a nonnegative real-valued function defined on $[0, \infty)$. A harmonic function u on a harmonic space $\mathfrak{X}$ is called Φ-bounded if the composite function $\Phi(|u|)$ possesses a

harmonic majorant on $\mathfrak{X}$. The family of all Φ-bounded harmonic functions on $\mathfrak{X}$ will be denoted $H\Phi(\mathfrak{X})$ and $O_{H\Phi}$ the totality of harmonic spaces on which every Φ-bounded harmonic function reduces to a constant. We define

$$\bar{d}(\Phi) = \limsup_{t\to\infty} \frac{\Phi(t)}{t} \qquad \text{and} \qquad \underline{d}(\Phi) = \liminf_{t\to\infty} \frac{\Phi(t)}{t} .$$

We note first that if Φ is bounded on $[0, \infty)$ then any nonconstant harmonic function on $\mathfrak{X}$ is a nonconstant $H\Phi$-function. Hence $O_{H\Phi}$ must consist only of trivial harmonic spaces. On the other hand if $\Phi(t)$ is completely unbounded on $[0, \infty)$, that is, if $\Phi(t)$ is not bounded in any neighborhood of any point of $[0, \infty)$ then for any nonconstant harmonic function u on $\mathfrak{X}$, $\Phi(|u|)$ is completely unbounded on $\mathfrak{X}$. Hence $O_{H\Phi}$ consists of all harmonic spaces. Having dispensed with these simpler cases we have the following theorem first established on Riemann surfaces by Nakai (14).

THEOREM 7. *If Φ is not bounded nor completely unbounded on $[0, \infty)$ then $O_{H\Phi} = O_{HP}$* (resp. $O_{H\Phi} = O_{HB}$) *if $\bar{d}(\Phi)$ is finite* (resp. infinite).

Proof. First assume $\bar{d}(\Phi) < \infty$. Then there exists a $c > 0$ such that $\Phi(t) \leqq ct$ for $t \geqq t_0$. If u is a nonconstant HP function on $\mathfrak{X}$ so is $v = u + t_0$. Since $v \geqq t_0 \geqq 0$ we have $\Phi(|v|) \leqq cv$. Hence v is a nonconstant $H\Phi$-function and so $O_{H\Phi} \subset O_{HP}$.

Conversely suppose u is a nonconstant $H\Phi$-function on $\mathfrak{X}$. We must show there exists a nonconstant HP-function $\mathfrak{X}$. Now there exists an HP-function v on $\mathfrak{X}$ such that $\Phi(|u|) \leqq v$ on $\mathfrak{X}$. If v is nonconstant or u is bounded we are done. So we may exclude these cases. The set $A = \{|u(\mathrm{p})|; p \in \mathfrak{X}\}$ is an open connected subset of $[0, \infty)$ not containing 0. For if $0 \in A$ then $A = [0, \infty)$ and this would contradict the fact that $\Phi(|u|) \leqq v = \text{const.}$ on $\mathfrak{X}$. Thus $0 \notin A$ and so either u or $-u$ is a nonconstant HP-function on $\mathfrak{X}$. This proves $O_{HP} \subset O_{H\Phi}$ and consequently $O_{H\Phi} = O_{HP}$.

Now consider the case where $\bar{d}(\Phi) = \infty$. Suppose u is a nonconstant HB-function on $\mathfrak{X}$. By hypothesis Φ is bounded in some interval $(a, b) \subset [0, \infty)$ within which $\Phi(t) \leqq c = \text{const.}$ So the range of $v = c_1 u + c_2$ is contained in (a, b) if c_1, c_2 are suitably chosen constants. It follows that v is a nonconstant $H\Phi$-function on $\mathfrak{X}$ and so $O_{H\Phi} \subset O_{HB}$.

Conversely assume that u is a nonconstant $H\Phi$-function on $\mathfrak{X}$. Suppose to the contrary that $\mathfrak{X} \in O_{HB}$. Now $\Phi(|u|) \leqq v$ on $\mathfrak{X}$ for some HP-function v. This implies $\mathfrak{X} \notin O_{HP}$ since otherwise from the fact that $\Phi(|u|) \leqq v = \text{const.}$ and $\bar{d}(\Phi) = \infty$ we would get that u is bounded, contrary to our assumption $\mathfrak{X} \in O_{HB}$. Thus $\mathfrak{X} \notin O_{HP}$ and hence

$\mathfrak{X} \notin O_G$. Since $\mathfrak{X} \in O_{HB} - O_G$ by Corollary 1 Δ consists of a single point. From the fact that $\bar{d}(\Phi) = \infty$ there is a strictly increasing sequence $\{t_n\}_1^\infty$ of positive numbers for which $\lim_{n\to\infty} \Phi(t_n)/t_n = \infty$ and $\lim_{n\to\infty} t_n = \infty$. Now each set $G_n = \{p \in \mathfrak{X} \mid |u(p)| < t_n\}$ is a regular open set. For if $p_0 \in \partial G_n$ and $u(p_0) = t_n$ (resp. $-t_n$) then $t_n - u$ (resp. $t_n + u$) is a barrier function at p_0 with respect to G_n. Also since u is unbounded $G \uparrow \mathfrak{X}$. Now $G_n \notin SO_{HB}$ for some n, and hence for all sufficiently large n. For if not consider the function $a_n v - |u|$ where $a_n = t_n/\Phi(t_n)$. Then $a_n v - |u|$ is superharmonic, bounded from below on G_n, continuous on $G_n \cup \partial G_n$ and nonnegative on ∂G_n. Hence $a_n v - |u| \geqq 0$ on G_n. Since $a_n \to 0$ and $G_n \uparrow \mathfrak{X}$ we have $u \equiv 0$ on $\mathfrak{X}$, a contradiction. Hence $G_n \notin SO_{HB}$ for $n \geqq n_1$ say, and so we may as well assume $G_n \notin SO_{HB}$ for $n = 1, 2, \cdots$. Now by Theorem 6 $\Delta \in \bar{G}_1 - \partial\bar{G}_1$ and so

$$\limsup_{p \in \mathfrak{X} \to \Delta} |u(p)| = \limsup_{p \in G_1 \to \Delta} |u(p)| \leqq r_1 .$$

The function $a_n v + r_1 - |u|$ is superharmonic, bounded from below on G_n, continuous on $G_n \cup \partial G_n$, and nonnegative on ∂G_n. Hence by Theorem 4, $a_n v + r_1 - |u| \geqq 0$ on G_n. Since $a_n \to 0$ we get $|u| \leqq r_1$ on $\mathfrak{X}$ which contradicts our assumption $\mathfrak{X} \in O_{HB}$. Hence we conclude $\mathfrak{X} \notin O_{HB}$ and so $O_{HB} \subset O_{H\Phi}$.

13. We now give relations between the classes $H\Phi$, HB' and HP'. The following theorems due to Nakai (16) for Riemann surfaces are also valid for harmonic spaces.

THEOREM 8. *If* $\underline{d}(\Phi) > 0$ *then* $H\Phi(\mathfrak{X}) \subset HP'(\mathfrak{X})$.

Proof. Set $\underline{d}(\Phi) = 2c > 0$ and choose $t_0 \in (0, \infty)$ so that $\Phi(t) > ct$ for $t > t_0$. If $u \in H\Phi(\mathfrak{X})$ then $\Phi(|u|)$ has a harmonic majorant v on $\mathfrak{X}$. It follows, that $v + ct_0 \geqq \Phi(|u|) + ct_0 \geqq c|u|$ on $\mathfrak{X}$ and $|u|$ possesses a harmonic majorant on $\mathfrak{X}$. Thus $u \in HP'(\mathfrak{X})$.

THEOREM 9. *If* $\bar{d}(\Phi) = \infty$ *then* $H\Phi(\mathfrak{X}) \cap HP'(\mathfrak{X}) \subset HB'(\mathfrak{X})$.

Proof. For $u \in H\Phi(\mathfrak{X}) \cap HP'(\mathfrak{X})$ there exists an HP-function v on $\mathfrak{X}$ with $\Phi(|u|) \leqq v$. Define $Mu = u \vee 0 + (-u) \vee 0$. Since B commutes with the operations M, $\vee$, and $\wedge$ we need only show $BMu = Mu$. Since $\bar{d}(\Phi) = \infty$ there is an increasing sequence $\{t_n\}_1^\infty$ of positive numbers with $\Phi(t_n) > 0$ and $a_n = t_n/\Phi(t_n) \to 0$. Setting $G_n = \{p \in \mathfrak{X} \mid |u(p)| < t_n\}$ we have $G_n \uparrow \mathfrak{X}$. Let $\{\Omega_m\}_1^\infty$ be an exhaustion of $\mathfrak{X}$. Observe that the boundary points of $\Omega_m \cap G_n$ are regular. Let w_m be harmonic on $\Omega_m \cap G_n$ with $w_m \mid (\partial\Omega_m) \cap G_n = \min(Mu - BMu, t_n)$ and $w_m \mid (\partial G_n) \cap \Omega_m = 0$. Furthermore if we define $w_m \mid (\Omega_m - G_n) = 0$ then w_m

is subharmonic on Ω_m, and $w_m \geqq w_{m+1}$ on Ω_m. Also let w'_m be harmonic on Ω_m with boundary values $w'_m \mid (\partial\Omega_m) \cap G_n = \min(Mu - BMu, t_n)$ and $w'_m \mid (\partial\Omega_m - G_n) = 0$. Then $\{w'_m\}$ is a bounded sequence and $0 \leqq w'_m \leqq Mu - BMu$, $m = 1, 2, \cdots$. It follows from a theorem of Loeb-Walsh [11] that if $\Omega \subset \mathfrak{X}$ is a regular region and the family $\psi = \{h \in H(\Omega) \mid 0 \leqq h\}$ is bounded then Ψ is equicontinous on Ω. Consequently by the Arzelà-Ascoli theorem Ψ is a normal family. Hence $\{w'_m\}$ has a convergent subsequence with limit function w'. We obtain $0 \leqq Bw' \leqq B(Mu - BMu) = 0$. Since w' is bounded and nonnegative, $w' \equiv Bw' \equiv 0$ on $\mathfrak{X}$. In addition $w'_m \geqq w_m \geqq 0$ implies $\lim_{m\to\infty} w_m = 0$ on $\mathfrak{X}$. Now on $(\partial\Omega_m) \cap G_n$ we have $|u| \leqq t_n$ and $|u| \leqq Mu = BMu + (Mu - BMu)$. Hence on $(\partial\Omega_m) \cap G_n$, $|u| - BMu \leqq \min(Mu - BMu, t_n) = w_m$. On ∂G_n, $|u| = t_n = a_n \ \Phi(|u|) \leqq a_n v$, and so $|u| \leqq a_n v + BMu + w_m$ on $\partial(\Omega_m \cap G_n)$ and hence on $\Omega_m \cap G_n$. Upon letting $m \to \infty$ and then $n \to \infty$ we obtain $|u| \leqq BMu$ on $\mathfrak{X}$. Since Mu is the least harmonic majorant of $|u|$ on $\mathfrak{X}$ we must have $Mu \leqq BMu$ and hence $BMu = Mu$ as desired.

Combining Theorem 8 and Theorem 9 we have the following

COROLLARY 2. *If* $\bar{d}(\Phi) = \infty$ *and* $\underline{d}(\Phi) > 0$ *then* $H\Phi(\mathfrak{X}) \subset HB'(\mathfrak{X})$.

14. Finally we briefly mention something about relative classes. Let $F \subset \mathfrak{X}$ be a regular open subset. We denote by $H_0\Phi(\mathfrak{X}, F)$ the class of harmonic functions u on F vanishing continuously on ∂F and such that $\Phi(|u|)$ has a harmonic majorant on F. The corresponding null class $SO_{H_0\Phi}$ will consist of subregions F for which $H_0\Phi(\mathfrak{X}, F) = \{0\}$. Note that $SO_{H_0\Phi}$ consists of all relatively compact regular open subsets of harmonic spaces if $\Phi(t)$ is bounded on $[0, \infty)$. If on the other hand $\Phi(t)$ is not bounded at $t = 0$ then $SO_{H_0\Phi}$ consists of all regular open subsets of harmonic spaces. The remaining case is treated in the following theorem of Nakai (15).

THEOREM 10. *If* $\Phi(t)$ *is bounded at* $t = 0$ *but unbounded in* $[0, \infty)$ *and* $\bar{d}(\Phi) = \infty$ *then* $SO_{H_0\Phi} = SO_{HB}$.

The proof follows by an argument similar to that in Theorem 7.

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