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LIE HOMOMORPHISMS OF OPERATOR ALGEBRAS

C. ROBERT MIERS

A mapping $\phi: M \rightarrow N$ between *-algebras M, N which is *-linear, and which preserves the Lie bracket $[X, Y] = XY - YX$ of elements X, Y in M is called a Lie *-homomorphism or just a Lie homomorphism. The main result of this paper states that if $\phi: A \rightarrow B$ is a uniformly continuous Lie *-homomorphism of the C*-algebra A onto the C^*-algebra B then there exists a central projection D in the weak closure of B such that modulo a center-valued *-linear map which annihilates brackets, $D\phi$ is a *-homomorphism and $(I - D)\phi$ is the negative of a *anti-homomorphism.

Previously we showed that if M is a *factor* then so is N and in this case $\phi = \sigma + \lambda$ where σ is a *-isomorphism or the negative of *a* *-anti-isomorphism of M onto N and λ is a *-linear functional which annihilates brackets in M. This result parallels the algebraic theorems of L. Hua and W. S. Martindale.

The main techniques used in this paper are the algebraic techniques of Martindale [8], [9], and Herstein [3]. Adaptations of them allow us to characterize Lie *-isomorphisms between von Neumann algebras and also ultra-weakly $(= UW)$ closed Lie *-ideals which contain the center. For a complete exposition concerning Lie structures on associative algebras we recommend Herstein [4]. We wish to thank Professor H. A. Dye for many invaluable conversations during the preparation of this paper.

1. Preliminaries and notation. We denote by $\mathscr{L}(H)$ the ring of all linear operators $T: H \rightarrow H$, H a complex Hilbert space with inner product $(.\,,\,.)$, which are bounded in the norm $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$. With this norm, $\mathscr{L}(H)$ is a Banach algebra with identity the identity operator I. In addition to the uniform topology on $\mathscr{L}(H)$ we shall be concerned with (1) the weakest topology making the linear functionals $T \rightarrow (Tx, y)$ continuous for all $x, y \in H$, called the weak (operator) topology and (2) the weakest topology making the linear functionals $T \rightarrow \sum_{n=1}^{\infty} (Tx_n, y_n)$ continuous for all sequences $\{x_n\}$ $\{y_n\}$ such that $\sum_{n=1}^{\infty} \|x_n\|^2, \sum_{n=1}^{\infty} \|y_n\|^2 < \infty$ called the ultra-weak (operator) topology.

To each operator $T \in \mathscr{L}(H)$ there corresponds an operator $T^* \in \mathscr{L}(H)$, called the adjoint of T, defined by $(Tx, y) = (x, T^*y)$ for all $x, y \in H$. If $T = T^*$, T is called self-adjoint; if T is selfadjoint and $T = T^2$, T is called a projection; if $TT^* = T^*T = I$, T is called unitary.

A C^*-algebra, M, is a subalgebra of $\mathscr{L}(H)$ which is closed in the uniform topology and closed under the operation of taking adjoints. A von Neumann algebra is a weakly closed *-subalgebra of $\mathscr{L}(H)$ which contains I. It is a fact that if M is a von Neumann algebra then M is the smallest weakly closed linear subspace of $L(H)$ containing the set $\{P | P \text{ a projection in } M\}$. The set $Z_M = \{S \in M | ST = TS \text{ for all } T \in M\}$ is called the center of M. If P is a projection in M then $M_P = \{PAP | A \in M\}$.

Any associative algebra M can be made into a Lie algebra by defining a new multiplication of elements $A, B \in M$ to be the commutator $[A, B] = AB - BA$ where AB is the associative product. For notation let $[M, M]$ be the linear span of all such commutators. A Lie ideal of M is a linear subspace which is an ideal with respect to the multiplication $[A, B]$. Any associative ideal is a Lie ideal, $[M, M]$ is a Lie ideal, and any linear subspace of Z_M is a Lie ideal.

We use Dixmier [1], [2] as general references.

2. General results.

DEFINITION. Let M be a von Neumann algebra with center Z_M. For each self-adjoint operator $A \in M$ we define the core of A, denoted by $\underline{A}$, to be the LUB $[S \in Z_M | S \text{ self adjoint}, S \leqq A]$. One has $A - \underline{A} \geqq 0$. Further, if $S \in Z_M$ and $A - \underline{A} \geqq S \geqq 0$ then $S = 0$. If P is a projection it is clear that $\underline{P}$ is the largest central projection $\leqq P$. We call a projection core-free if $\underline{P} = 0$.

The following theorem is a generalization of a result of L. Hua [5] and its proof is due to H. A. Dye.

THEOREM 1. *Let M be a von Neumann algebra on the Hilbert space H. If A is a self-adjoint operator in M satisfying the identity $[[[X, A], A] = [X, A]$ for all $X \in M$, then $A - \underline{A}$ is a projection.*

Proof. The identity is true for $A - \underline{A}$ as well so we can assume $\underline{A} = 0$ and $A \geqq 0$. In particular, for each nonzero central projection C, 0 lies in the spectrum of CA (considered as an operator on CH) (otherwise there exists $\varepsilon > 0$ such that $A \geqq \varepsilon C$).

Suppose $\sigma(A) \neq \{0, 1\}$. We can then choose $\mu \in \sigma(A)$ such that $d(\mu, \{0, 1\}) > 2\varepsilon > 0$ where $d(a, S) =$ distance from a to the set S. Let $\delta > 0$ be such that $|t^3 - t| < \delta$ implies $d(t, \{0, 1\}) < \varepsilon$. By the spectral theorem there exists an operator B, commuting with A, of the form $B = \sum_{i=1}^{n} \lambda_i P_i$ where the λ_i are distinct and $\geqq 0$, the P_i are mutually $\perp$ projections of sum I, $0 \leqq B \leqq A$, and $||A - B||$ is so small that $||A - B|| < \varepsilon$ and

(1) $||[[[X, B], B], B] - [X, B]|| < \delta$ for all X in M such that $||X|| \leqq 1$.

Furthermore, 0 occurs among the λ_i and $\bar{P}_i = I$ (if P is a projection $\bar{P}$ is the central carrier of P) for $i = 1, \cdots, n$. Finally, since $\|A - B\| < \varepsilon$, one of the λ_i, say λ_2, will be at distance $> \varepsilon$ from $\{0, 1\}$.

Now we consider the mapping $\psi_u(S) = USU^{-1}$ where U is unitary in M. Taking $X = U$ in (1) and rewriting this relation in terms of ψ_u we have

$$(2) \qquad \|\psi_u(B)^3 - 3B\psi_u(B)^2 + 3B^2\psi_u(B) - B^3 - \psi(B) + B\| < \delta$$

$\bar{P}_i = I$ implies the existence of nonzero projections $Q_1 \leqq P_1, Q_2 \leqq P_2$ such that $Q_1 \sim Q_2$. We choose a unitary operator U such that ψ_u has order 2, $\psi_u(Q_1) = Q_2$ and ψ_u is the identity off $Q_1 + Q_2$. For this U, (2) reduces to $\|(\lambda_2^3 - \lambda_2)Q_1 + (\lambda_2^3 - \lambda_2)Q_1\| < \delta$. In particular, $|\lambda_2^3 - \lambda_2| < \sigma$ so that $d(\lambda_2, \{0, 1\}) < \varepsilon$ which is a contradiction. Hence $\sigma(A) \subsetneqq \{0, 1\}$ and A is a projection.

LEMMA 1. *Let P and Q be commuting core-free projections in M. Then $P + Q \in Z_M$ implies $P \perp Q$, and $P - Q \in Z_M$ implies $P = Q$.*

Proof. If $P + Q \in Z_M$ then $(P + Q)^2 - (P + Q) = 2PQ \in Z_M$. But in this case $PQ \leqq \underline{P} = 0$. If $P - Q \in Z_M$, then $(P - Q)^2 + (P - Q) = 2(P - PQ) \in Z_M$. But $P - PQ \leqq \underline{P} = 0$ so that $P = PQ$. By symmetry $Q = PQ$.

DEFINITION. Two projections P and Q are called parallel $(P \| Q)$ if $\bar{P}\bar{Q} = 0$.

LEMMA 2. *$P \| Q$ implies $[[P, X], Q] = 0$. Conversely, if P and Q are commuting core-free projections such that $[[P, X], Q] = 0$ for all $x \in M$ then $P \| Q$.*

Proof. The first statement is clear from the definition of central carrier.

For the other part, multiplying the relation $[[P, X], Q] = 0$ on the left by PQ gives $PQX(I - P)(I - Q) = 0$ for all $X \in M$ so that $PQ \| (I - P)(I - Q)$. Multiplying the relation on the left by P and on the right by Q gives $P(I - Q)XQ(I - P) = 0$ for all $X \in M$ so that $P(I - Q) \| Q(I - P)$. The first parallelism statement implies $\overline{PQ} \subsetneqq P \cup Q$. Thus $\overline{P(I - Q)PQ}$ lies in $P \cup Q$, is orthogonal to Q, and hence is in P. Since $\underline{P} = 0$ this implies $\overline{P(I - Q)PQ} = 0$. Likewise $\overline{Q(I - P)PQ} = 0$, forcing $\overline{PQ} = PQ$ so that $PQ = 0$. Thus $P = P(I - Q)$ and $Q = Q(I - P)$ which implies $P \| Q$.

LEMMA 3. *Let P and Q be commuting projections in M such that*

$[[[[X, P], Q], P], Q] + [[Y, P], Q] = 0$ *for all* $X \in M$. *Then there exists a projection* $C \in Z_M$ *such that* $PQ(I - C) = 0$, $(I - P)(I - Q)C = 0$.

Proof. Multiplying the bracket identity on the left by PQ gives $PQX(I - P)(I - Q) = 0$ for all X so that $PQ \| (I - P)(I - Q)$. Let $C = \overline{PQ}$.

LEMMA 4. *If the von Neumann algebra* M *has no summands of type* I_1 (= central abelian projection) *then each nonzero central projection* $C \in M$ *is the carrier of a core-free projection in* M.

Proof. Nonzero core-free projections exist in M since if P is a noncentral projection, $\bar{P} - P$ is core-free. By Zorn, let $\{P_\alpha\}$ be a maximal collection of $\|$ core-free projections and let $P = \Sigma P_\alpha$. Note that $\bar{P} = I$. For, otherwise there is a central projection $C \neq 0$ orthogonal to all the P_α, and this C would dominate a nonzero core-free projection thus contradicting maximality. Moreover P is core-free. If C is a central projection and $C \leqq P$, then $\bar{P}_\alpha C \leqq P_\alpha$ so that $\bar{P}_\alpha C = 0$. Finally $\overline{CP} = C\bar{P} = C$ for any projection $C \in Z_M$.

LEMMA 5. *Let* M *be any von Neumann algebra,* P *and* Q *projections in* M *with* $\bar{P} = \bar{Q} \neq 0$. *If* $A \in M$ *commutes with* PXQ *and* QXP *for all* $X \in M$, *then* A *commutes with* PXP *and* QXQ *for all* $X \in M$.

Proof. Since $\bar{P} \cdot \bar{Q} \neq 0$ there exist nonzero projections $P_1 \leqq P$, $Q_1 \leqq Q$ such that $P_1 \sim Q_1$. Let $V_1 \in M$ be such that $V_1^* V_1 = P_1$, $V_1 V_1^* = Q_1$. Since $P_1XP_1 = PP_1XV_1^*Q_1QV_1P_1P$ we have that A commutes with P_1XP_1 for all X. If $P_1 \neq P$, then $\overline{P - P_1}\bar{Q} \neq 0$ so there exist projections $P_2 \leqq P_1 - P_1$, $Q_2 \leqq Q$ with $P_2 \sim Q_2$. Let $V_2 \in M$ be such that $V_2^* V_2 = P_2$, $V_2 V_2^* = Q_2$. As before $P_2XP_2 = PP_2XV_2^*Q_2QV_2P_2P$ so that A commutes with P_2XP_2 for all $X \in M$. Moreover, since $P_1XP_2 = PP_1XV_2^*QV_2P_2P$, A commutes with P_1XP_2 (and similarly with P_2XP_1) for all $X \in M$.

By Zorn, choose a maximal collection $\{P_\alpha\}$ of non-zero mutually orthogonal projections such that (*i*) $P_\alpha \leqq P$, (*ii*) A commutes with $P_\alpha XP_\beta$ for all α, β and all $X \in M$. By maximality $\Sigma P_\alpha = P$. Thus $PXP = (\Sigma P_\alpha)\, X\, (\Sigma P_\alpha)$ and so A commutes with PXP. A similar statement holds for QXQ.

LEMMA 6. *Let* M *be a* C^**-algebra of operators on* H. *If* $X, Y \in M$ *with* Y *self-adjoint, and if* $[X, Y] \in Z_M$ *then* $[X, Y] = 0$.

Proof. By (Singer [13] p. 242), if X, Y skew adjoint, then $[[X,$

$Y], Y] = 0$ implies $[X, Y] = 0$. The same statement holds for X, Y self adjoint since then iX and iY are skew-adjoint.

Now suppose Y self-adjoint. Then $[X, Y] \in Z_M$ implies $[-X^*, Y] \in Z_M$ and hence $[X - X^*, Y] \in Z_M$. By the above $[X - X^*, Y] = 0$. Moreover $-[-X^*, Y] \in Z_M$ so $[X + X^*, Y] \in Z_M$ and is therefore zero. Finally $[X, Y] = 0$.

3. Near-isomorphisms of von Neumann algebras.

DEFINITION. A mapping $\phi: M \rightarrow N$ between von Neumann algebras M and N which is *-linear and which preserves the Lie bracket of operators in M is called a Lie *-homomorphism (or just Lie homomorphism). If $h_M(X) = X + Z_M$ is the natural Lie homomorphism of M onto M/Z_M we call ϕ L-onto if $h_N \circ \phi$ maps M onto N/Z_N. This implies $\phi(Z_M) \subseteqq Z_N$. Thus ϕ (if L-onto) induces a Lie homomorphism of M/Z_M onto N/Z_N. If this induced homomorphism is a Lie isomorphism call ϕ a near-isomorphism.

If $\phi: M \rightarrow N$ is a L-onto Lie homomorphism, and $P \in M$ then by Theorem 1 and Lemma 1 there exists a unique core-free projection $\theta(P) \in N$ and a central element $\lambda(P)$ ($=\underline{\phi(P)}$) such that $\phi(P) = \theta(P) + \lambda(P)$. If we write $\theta'(P) = \overline{\theta(P)} - \theta(P)$, then $\theta'(P)$ is core-free, $\phi(P) = -\theta'(P) + \lambda'(P) + \lambda'(P)$ ($\lambda'(P) \in Z_N$) and this representation is unique. Note that $\overline{\theta(P)} = \overline{\theta'(P)}$ and that $P \in Z_M$ implies $\theta(P) = 0$. We assume from now on that ϕ is a near-isomorphism between the von Neumann algebras M and N.

LEMMA 7. *If Q is a core-free projection in N, then there exists a core-free projection $P \in M$ such that $\theta(P) = Q$.*

Proof. $[[[Y, Q]Q]Q] - [Y, Q] = 0$ for all $Y \in N$. Let $P' \in M$ be such that $\phi(P') - Q \in Z_N$. Then $\phi([[[X, P']P']P'] - [X, P']) = 0$ for all $X \in M$ so that $[[[X, P']P']P'] - [X, P'] \in \ker \phi \subseteqq Z_M$. By Lemma 6 this bracket expression is zero. Hence, by Theorem 1, $P' - \underline{P'} = P$ a core-free projection. $\phi(P') = \theta(P) + \lambda(P) + \phi(\underline{P'}) = Q + Z, z \in Z_M$ and $\lambda(P) + \phi(\underline{P'}) \in Z_N$. Hence by Lemma 1, $\theta(P) = Q$.

LEMMA 8. *Let P and Q be core-free projections in M. Then $P || Q$ if and only if $\theta(P) || \theta(Q)$ and $\bar{P} = \bar{Q}$ if and only if $\overline{\theta(P)} = \overline{\theta(Q)}$.*

Proof. $P || Q$ implies $[[P, X], Q] = 0$ for all $X \in M$. If $Y \in N$ let $X \in M$ be such that $\phi(X) - Y \in Z_N$. Then $\phi([[P, X], Q) = [[\theta(P), Y], \theta(Q)] = 0$. Hence $\theta(P) || \theta(Q)$. On the other hand if $\theta(P) || \theta(Q)$ then $[[P, X], Q] \in \ker \phi \subseteqq Z_M$ for all $X \in M$. By Lemma 6, $[[P, X]Q] = 0$

for all X.

For the other part, if $\bar{P} = \bar{Q}$ but $\overline{\theta(P)} \neq \overline{\theta(Q)}$ then there exists a projection $C \in Z_N$ such that $C\theta(Q) = 0$, $C\theta(P) \neq 0$. There exists a core-free projection R in M such that $\theta(R) = C\theta(P)$. Hence $\theta(R) \| \theta(Q)$ which implies $R \| Q$. But $\theta(R) \| \theta(P)$ so that $R \| P$ a contradiction. Similarly $\overline{\theta(P)} = \overline{\theta(Q)}$ implies $\bar{Q} = \bar{P}$.

LEMMA 9. *θ and θ' are additive on parallel core-free projections.*

Proof. Let $P_1, \cdots, P_n$ be parallel core-free projections in M. By Lemma 8 the $\theta(P_1), \cdots, \theta(P_n)$ are parallel (and core-free) so that $\theta(P_1) + \cdots + \theta(P_n)$ is a projection. It is also core-free by parallelism. One has

$$\theta(P_1 + \cdots + P_n) - \sum_{i=1}^{n} \theta(P_i) = \sum_{i=1}^{n} \lambda(P_i) - \lambda(P_1 + \cdots + P_n) \in Z_N$$

By Lemma 1

$$\theta\left(\sum_{i=1}^{n} P_i\right) = \sum_{i=1}^{n} \theta(\mathrm{P}_i) \left(\text{and} \sum_{i=1}^{n} \lambda(P_i) = \lambda\left(\sum_{i=1}^{n} P_i\right)\right).$$

A similar argument gives the result for θ'.

LEMMA 10. *If M and N have no central summands of type I_1 then there exists a unique *-isomorphism ψ of Z_M onto Z_N such that $\theta(CP) = \psi(C)\theta(P)$ for all projections P and all central projections C. One has $\psi(\bar{P}) = \overline{\theta(P)}$. A similar statement holds with θ replaced by θ'.*

Proof. We first define ψ for central projections. For each central projection C, choose a core-free projection P such that $\bar{P} = C$ (Lemma 4). Define $\psi(C) = \overline{\theta(P)}$. If Q is any other core-free projection such that $\bar{Q} = C$ then $\overline{\theta(Q)} = \overline{\theta(P)}$ by Lemma 8 so that the mapping is well defined. If D is a central projection in N, choose a core-free projection $R \in N$ such that $\bar{R} = D$. There exists a core-free projection $P \in N$ such that $\theta(P) = R$ so that $\psi(\bar{P}) = \overline{\theta(P)} = \bar{R} = D$. Hence ψ is onto. If $\psi(\bar{P}) = \psi(\bar{Q})$ for core-free projections P and Q then $\overline{\theta(P)} = \theta(Q)$ and by Lemma 8, $\bar{P} = \bar{Q}$. If C and D are central projections in M with $CD = 0$ let P and Q be core-free projections in M with $\bar{P} = C$, $\bar{Q} = D$. Then $CD = 0$ iff $\bar{P}\bar{Q} = 0$ iff $\bar{P} \| \bar{Q}$ iff $\theta(P) \| \theta(Q)$ iff $\overline{\theta(P)}\overline{\theta(Q)} = 0$ iff $\psi(C)\psi(D) = 0$. Thus ψ is a projection orthoisomorphism of Z_M on Z_N and as such is implemented by a unique *-isomorphism (also called ψ) of Z_M on Z_N.

To show $\theta(CP) = \psi(C)\theta(P)$ choose a core-free projection Q such that $\bar{Q} = C(I - \bar{P})$. Then $PC + Q$ has carrier C, and $\psi(C) = \overline{\theta(PC + Q)}$ $= \overline{\theta(PC)} + \overline{\theta(Q)}$ by Lemma 9. Now $\theta(P) = \theta(PC) + \theta(P(I - C))$ and both terms on the right are orthogonal to $\overline{\theta(Q)}$. Multiplying the two relations together we have $\psi(C)\theta(P) = \theta(CP)$.

Finally we show that $\theta(CP) = \psi(C)\theta(P)$ implies $\psi(\bar{P}) = \overline{\theta(P)}$. Put $C = \bar{P}$ to get $\theta(P) = \psi(\bar{P})\theta(P)$ or $\theta(P) \leqq \psi(\bar{P})$. So $\overline{\theta(P)} \leqq \psi(\bar{P})$. Now if $\bar{P} + \bar{R} = I$ and $\bar{P} \perp \bar{R}$ we have $\overline{\theta(P)} + \overline{\theta(R)} = 1$.

DEFINITION. Two projections P, Q are called co-orthogonal (co $\perp$) if $\bar{P} - P \perp \bar{Q} - Q$.

LEMMA 11. *Let $P_1, \cdots, P_n$ be commuting core-free projections, each pair of which satisfy the identity of Lemma 3. Then there exists a central projection C such that the P_i are orthogonal on C, co-orthogonal on $I - C$.*

Proof. For each pair P_i, P_j $(i \neq j)$ there exists a central projection C_{ij} such that $P_iP_jC_{ij} = 0$ and $(I - P_i)(I - P_j)(I - C_{ij}) = 0$. Let $\mathscr{B}$ be the boolean algebra generated by the C_{ij}. If C is an atom in $\mathscr{B}$ then $P_iP_jC = 0$ or $(I - P_i)(I - P_j)\, C = 0$ $(i \neq j)$.

Index so that $P_1C, \cdots, P_mC$ are the non-zero terms of the form P_iC (we can leave 0 out since it is both $\perp$ and co $\perp$ to all projections). We claim that these projections are either $\perp$ or co $\perp$. It suffices to take $m \geqq 3$. If $P_1P_2C = 0$ then all the P_iC are mutually $\perp$. For if, say, $P_1P_3C \neq 0$ then $(I - P_1)(I - P_3)\, C = 0$ so that $(I - P_1)\, C \leqq P_3$. The two relations give $P_2C \leqq P_3$. Now if $CP_2P_3 = 0$ we have $P_2C = 0$ a contradiction. If $(I - P_2)(I - P_3)C = 0$ then $C(I - P_2) \leqq P_3$ so that $C \leqq P_3$ contradicting the fact that $\underline{P}_3 = 0$. The same argument shows $P_1P_iC = 0$ for $i \geqq 3$. Applying this reasoning to each P_iC in turn gives their mutual perpendicularity.

In a similar way if, say, $(I - P_1)(I - P_2)\, C = 0$ then all the $(I - P_i)C$ are mutually orthogonal. If, for example, $P_1P_3C = 0$ then $(I - P_1)C \leqq P_2$ and we have $P_3C \leqq P_2CP_2P_3 = 0$ implies $P_3 = 0$ and $(I - P_2) \times (I - P_3)C = 0$ implies $C \leqq P_2$, both contradictions.

COROLLARY. *Let M and N be von Neumann algebra with no central summands of type I_1, and let $P_1, \cdots, P_n$ be a collection of mutually $\perp$ projections in M. There exists a projection $D \in Z_M$ such that the P_jD have $\perp$ image under θ, the $P_i(I - D)$ have $\perp$ image under θ'.*

Proof. Apply the above to the $\theta(P_i)$ and get the projection $C \in Z_N$. One has $C = \psi(D)$ for some $D \in Z_M$ and $\theta(DP_i) = C\, \theta(P_i)$, $\theta(DP_i) = C\theta'(P_i)$.

LEMMA 12. *Let $P_1, \cdots, P_n$ be projections in M with $\bar{P}_{i_0} = I$, $D \in Z_M$ such that* (i) *P_iD are mutually $\perp$,* (ii) *$\Sigma P_iD \in Z_M$, then $\sum_{i=1}^n P_iD = D$.*

Proof. Say $\Sigma P_iD = C \in Z_M$. Then $P_{i_0}D = P_{i_0}CD$ so that $D = CD$. Hence $D \leqq C$. Obviously $C \leqq D$.

LEMMA 13. *Let $P_1, \cdots, P_n$ and D as in Corollary to Lemma* 11 *with $\bar{P}_{i_0} = I$. Then $\psi(D)\sum_{i=1}^n \theta(P_i) = \psi(D)$, $\psi(I - D)\sum_{i=1}^n \theta(P_i) = \psi(I - D)$.*

Proof. $\phi(D) = \sum_{i=1}^n \theta(P_iD) + \sum_{i=1}^n \lambda(P_iD) \in Z_N$. Hence $\sum_{i=1}^n \theta(P_i)\psi(D) \in Z_N$ and the $\theta(P_i)\psi(D)$ are mutually $\perp$. By Lemma 12, $\sum_{i=1}^n \theta(P_i)\psi(D) = \psi(D)$. A similar argument works for θ' and $I - D$.

LEMMA 14. *If M is any von Neumann algebra and P a core-free projection in M, then M_P $(= \{PAP \mid A \in M\}) \cap Z_M = \{0\}$.*

Proof. Suppose $PAP \in Z_M$ where A is self-adjoint and $A \leqq I$. Then $((P - PAP)x, x) = ((I - A)Px, Px) \geqq 0$ so that $P \geqq PAP$. Since $\underline{P} = 0$ we have $PAP = 0$. In general, if $A = A_1 + iA_2$ where A_1, A_2 are (nonzero) self-adjoint and $PAP \in Z_M$ then $PAP + PA^*P = 2PA_1P \in Z_M$. Therefore $P(1/\|A_1\|)\ A_1P \in Z_M$ so that $PA_1P = 0$ by the first part of the argument. Similarly $PA_2P = 0$.

4. The decomposition theorem.

The following arguments are, in part, adaptations of those of Martindale [8], [9]. These adaptations are of sufficient technical complexity to merit their entire conclusion.

We consider first the case where $\phi: M \to N$ is a near isomorphism of M to N, and M is a type I_2 von Neumann algebra. Let P_1, P_2 be equivalent, orthogonal, abelian projections of sum I in M. We have that $\underline{P}_i = 0$, $\bar{P}_i = I$ for $i = 1, 2$, and that $\theta(P_1)\theta(P_2) = 0$ since $\theta(P_1) + \theta(P_2) \in Z_N$. Moreover, $I = \psi(I) = \psi(\bar{P}_1) = \overline{\theta(P_1)} \leqq \theta(P_1) + \theta(P_2) = \theta(P_1) + \theta(P_2) \leqq I$ and so $\theta(P_1) + \theta(P_2) = I$. Notice that $\theta(P_1) = \theta'(P_2)$ and $\theta(P_2) = \theta'(P_1)$. For notation let $M_{ij} = \{P_iAP_j \mid A \in M\}$, $N_{ij} = \{\theta(P_i)A\theta(P_j) \mid A \in N\}$.

LEMMA 15. $\phi^{-1}(N_{11} + N_{22}) = M_{11} + M_{22}$.

Proof. If $X \in M_{11}$ let $Y = \phi(X)$. Since $[X, P_2] = 0$ we have $0 = [Y, \theta(P_2)]$ or $Y\theta(P_2) = \theta(P_2)Y$. This implies $\theta(P_2)Y\theta(P_1) = \theta(P_1)Y\theta(P_2) = 0$ which in turn implies $Y \in N_{11} + N_{22}$. Similarly if $X \in M_{22}$.

Suppose $Y \in N_{11}$ and $X \in M$ is such that $\phi(X) - Y \in Z_N$. Then $[Y, \theta(P_2)] = 0$ so that $[X, P_2] \in \ker\ \phi \subseteqq Z_M$ which implies $[X, P_2] = 0$

by Lemma 6.

LEMMA 16. $\phi^{-1}(N_{ij}) = M_{ij}$ $(i \neq j)$.

Proof. For example, suppose $i = 1, j = 2$ and let $X \in M_{12}$. $X = [Q_1, [X, Q_2]] = [Q_1, X]$. Hence $\phi(X) = [\theta(P_1), [\phi(X), \theta(P_2)]] = \theta(P_1)\phi(X)\theta(P_2) + \theta(P_2)\phi(X)\theta(P_1) = \theta(P_1)[\theta(P_1), \phi(X)]\theta(P_2) + \theta(P_2)[\theta(P_1), \phi(X)]\theta(P_1) = \theta(P_1)\phi(X)\theta(P_2) - \theta(P_2)\phi(X)\theta(P_1)$ since $\theta(P_1) \perp \theta(P_2)$. Therefore $\phi(X) = \theta(P_1)\ \phi(X)\ \theta(P_2)$. So $\phi(M_{12}) \subseteqq N_{12}$.

If $Y \in N_{12}$, then $Y = [\theta(P_1), Y] = [\phi(P_1), Y]$ and also $Y = [\theta(P_1), [Y, \theta(P_2)]] = [\phi(P_1), [Y, \phi(P_2)]]$. Let $X \in M$ be such that $\phi(X) - Y \in Z_N$. Then $[P_1, X] - [P_1, [X, P_1]] \in \ker \phi$ and so equals zero, or $P_1X - XP_1 = P_1XP_2 + P_2XP_1$. This gives $P_2XP_1 = 0$. Now $\phi(X) = \phi(P_1XP_1) + \phi(P_1XP_2) + \phi(P_2XP_2)$ so that $\phi(P_1XP_1) + \phi(P_1XP_2) + \phi(P_2XP_2) - Z = Y$ for some $Z \in Z_N$. By the first part of the argument we then have $\phi(P_1XP_1) + \phi(P_2XP_2) - Z \in N_{12}$. By Lemma 15, $\phi(P_1XP_1) + \phi(P_2XP_2) = \theta(P_1)S\theta(P_1) + \theta(P_2)T\theta(P_2)$. Hence $\theta(P_1)S\theta(P_1) + \theta(P_2)T\theta(P_2) - Z = \theta(P_1)W\theta(P_2)$ for some $W \in N$. Multiplying on the left and right by $\theta(P_1)$ gives $\theta(P_1)S\theta(P_1) - Z\theta(P_1) = 0$. Similarly $\theta(P_2)T\theta(P_2) - Z\theta(P_2) = 0$. Hence $\theta(P_1)S\theta(P_1) + \theta(P_2)T\theta(P_2) - Z = 0$. This implies $\phi(P_1XP_2) = Y$.

COROLLARY. *ϕ is onto.*

LEMMA 17. $\phi(M_{ii}) \subseteqq N_{ii} + Z_N$.

Proof. M_{11} and M_{22} are abelian von Neumann algebras in their own right and so, therefore are $M_{11} + M_{22}$ $(= M_{11} \oplus M_{22})$ and by Lemma 15, $N_{11} + N_{22}$ $(= N_{11} \oplus N_{22})$. The latter implies that N_{11}, N_{22} are abelian.

Let $X \in M_{11}$ and $Y = \phi(X)$. Then $Y \in N_{11} + N_{22} \subseteqq Z_{N_{11}} + Z_{N_{22}}$, (since $N_{11} + N_{22}$ abelian) $= Z_{N11} + Z_{N22} = Z_{\theta(P_1)} + Z_{\theta(P_2)}$. Thus $Y = S\ \theta(P_1) + T\theta(P_2)$ where $S, T \in Z_N$. But $\theta(P_1) = I - \theta(P_2)$ so that $Y = (S - T)\theta(P_1) + T$.

We define mappings $\sigma: M \rightarrow N$ and $\lambda: M \rightarrow Z_N$ in the following manner: if $A \in M_{ij}$ $(i \neq j)$ then $\sigma(A) = \phi(A)$, and if $A \in M_{ii}$ by Lemma 17, $\phi(A) = \sigma(A) + Z$ where $\sigma(A) \in N_{ii}$ and $Z \in Z_N$. Extend σ to all of M by linearity and define $\lambda(A) = \phi(A) - \sigma(A)$. σ and λ are well defined, for if $A_1 + Z_1 = A_2 + Z_2$ where $A_i \in N_{ii}$, $Z_i \in Z_N$ then $A_1 - A_2 \in N_{ii} \cap Z_N = \{0\}$.

LEMMA 18. *σ and λ are linear mappings.*

Proof. We show $\sigma(\alpha A + \beta B) = \alpha\sigma(A) + \beta\sigma(B)$ for $A, B \in M$ and

$\alpha, \beta \in \mathbf{C}$. It suffices to assume $A, B \in M_{ii}$. Then $\sigma(\alpha A + \beta B) - \alpha\sigma(A) - \beta\sigma(B) = -\lambda(\alpha A + \beta B) + \alpha\lambda(A) + \beta\lambda(B) \in N_{ii} \cap Z_N = \{0\}$.

LEMMA 19. *σ and λ preserve adjoints.*

Proof. Again assume $A \in M_{ii}$. Then $\sigma(A^*) - \sigma(A)^* = \lambda(A)^* - \lambda(A^*) \in N_{ii} \cap Z_N = \{0\}$.

LEMMA 20. *If $A \in M_{ii}$, $B \in M_{ij}$ $(i \neq j)$ then $\sigma(AB) = \sigma(A)\sigma(B)$.*

Proof. In this case, $\sigma(AB) = \phi(AB) = \phi(AB - BA) = [\phi(A), \phi(B)] = [\sigma(A), \sigma(B)] = \sigma(A)\sigma(B)$.

LEMMA 21. *If $A, B \in M_{ij}$ then $\sigma(AB) = \sigma(A)\sigma(B)$.*

Proof. Let $S \in M_{ij}$ $(i \neq j)$. Then $\sigma(AB)\sigma(S) = \sigma(ABS) = \sigma(A)\sigma(BS) = \sigma(A)\sigma(B)\sigma(S)$. Hence, $[\sigma(AB) - \sigma(A)\sigma(B)]\sigma(S) = \sigma(S)\sigma(AB) - [\sigma(A)\sigma(B)] = 0$. Specifically, $\sigma(AB) - \sigma(A)\sigma(B)$ commutes with N_{ij}. Similarly $\sigma(AB) - \sigma(A)\sigma(B)$ commutes with N_{ji}. Applying Lemma 5, $\sigma(AB) - \sigma(A)\sigma(B)$ commutes with N_{ii} and N_{jj}, so that, $\sigma(AB) - \sigma(A)\sigma(B) \in Z_N \cap N_{ii} = \{0\}$.

LEMMA 22. *If $A \in M_{ij}$, $B \in M_{ji}$ $(i \neq j)$ then $\sigma(AB) = \sigma(A)\sigma(B)$.*

Proof. Applying ϕ to the identity $[[A, B], A] = 2ABA$ we have $\phi(ABA) = \phi(A)\phi(B)\phi(A)$. But since $\phi = \sigma$ on M_{ij} $(i \neq j)$ we have $\sigma(ABA) = \sigma(A)\sigma(B)\sigma(A)$. Moreover, $\phi[A, B] = [\phi(A), \phi(B)] = [\sigma(A), \sigma(B)]$ and $\phi[A, B] = \phi(AB) - \phi(BA) = \sigma(AB) + \lambda(AB) - \sigma(BA) - \lambda(BA)$ so that (1) $\sigma(A)\sigma(B) - \sigma(AB) + \sigma(BA) - \sigma(B)\sigma(A) = C \in Z_N$. Multiplying this last relation on the right by $\sigma(A)$ we get $C\sigma(A) = \sigma(A)\sigma(B)\sigma(A) - \sigma(AB)\sigma(A) = \sigma(A)\sigma(B)\sigma(A) - \sigma(ABA) = 0$. Similarly $C\sigma(B) = 0$. Multiplying (1) by C, and using the preceding, we have (2) $C(\sigma(BA) - \sigma(AB)) = C^2$. Hence $C^3 = C^2(\sigma(BA) - \sigma(AB)) = C(\sigma(BA) - \sigma(AB))^2 = C(\sigma(BA)\sigma(BA) - \sigma(AB)\sigma(AB))$. Multiplying (1) by $\theta(P)$ gives $C\theta(P) = \sigma(A)\sigma(B) - \sigma(AB)$, and by $I - \theta(P)$ gives $C\theta(1 - P) = \sigma(BA) - \sigma(B)\sigma(A)$. Hence $C^3 = C((\sigma(B)\sigma(A) + C(I - \theta(P)))^2 - (\sigma(A)\sigma(B) - C\theta(P))^2) = C(\sigma(B)\sigma(A)\sigma(B)\sigma(A) + C^2(I - \theta(P)) - (\sigma(A)\sigma(B)\sigma(A)\sigma(B) + C^2\theta(P))) = C^3(1 - \theta(P)) - C^3\theta(P) = C^3 - 2C^3\theta(P)$. Thus $C^3\theta(P) = 0$. In particular the set $NC^2\theta(P)N = \{XC^2\theta(P)\,Y \mid X, Y \in N\}$ is a nilpotent ideal in M so that $C^2\,\theta(P) = 0$. By the same reasoning $C\theta(P) = 0$.

LEMMA 23. *σ is a *-isomorphism of M onto N.*

Proof. (i) $1-1$: If $\sigma(A)=0$ then $\phi(AB)=\sigma(AB)+\lambda(AB)=\lambda(AB)\in Z_N$ for all $B\in M$. Thus, by the near one-one-ness of ϕ, $AB\in Z_M$ for all $B\in M$. Thus A, AP_1, and AP_2 all are in Z_M. This implies $AP_i\in Z_M\cap M_{ii}$ $(i=1,2)$ so that $A=AP_1+AP_2=0$.

(ii) onto: $\sigma=\phi$ on M_{ij} $(i\neq j)$ and $\phi^{-1}(N_{ij})=M_{ij}$. Let $Y\in N_{11}$. There exists $X\in M_{11}+M_{22}$ such that $\phi(X)=Y$. Hence $X=SP_1+TP_2=(S-T)P_1+T$ where $S,T\in Z_M$. Thus $Y=\phi(P_1(S-T)P_1+T)=\sigma(P_1(S-T)P_1)+\lambda(P_1(S-T)P_1)+\phi(T)$ so that $Y-\sigma(P_1(S-T)P_1)\in N_{11}\cap Z_N=\{0\}$.

THEOREM 2. *Let $\phi: M\rightarrow N$ be a near-isomorphism of the type I_2 von Neumann algebra M onto the von Neumann algebra N. Then* (i) *ϕ is onto, and* (ii) *$\phi=\sigma+\lambda$ where σ is a *-isomorphism of M onto N and λ is a *-linear mapping of M into Z_N which annihilates brackets. N is then of type I_2 with $\theta(P_1)$, $\theta(P_2)$ equivalent, orthogonal, abelian projections of sum I.*

In what follows we assume M and N have no summands of type I_1. We now turn our attention to the case when M is not of type I_2 but has a summand of type I_2. Choose non-zero orthogonal projections P_1, P_2, P_3 such that $\sum_{i=1}^{3} P_i=I$, $\bar{P}_1=\bar{P}_2=I$, $I-\bar{P}_3$ is the I_2 summand, $I-\bar{P}_3\leqq P_1+P_2$, and $P_1-P_1\bar{P}_3$, $P_2-P_2\bar{P}_3$ are the equivalent abelian projections comprising $I-\bar{P}_3$. There exists a central projection D such that $\theta(P_iD)$ are mutually $\perp$, and $\theta'(P_i(I-D))$ are mutually $\perp$. For notation, let $Q_i=P_iD$, $R_i=P_i(I-D)$, $T_i=Q_i\bar{P}_3$, $T'_i=R_i\bar{P}_3$ all for $i=1,2,3$, and let $S_i=Q_i(I-\bar{P}_3)$, $S'_i=R_i(I-\bar{P}_3)$ for $i=1,2$. Furthermore let $M_{ij}=Q_iMQ_j$, $M_{ij}=R_iMR_j$, $N_{ij}=\theta(Q_i)M(Q_j)$, $N_{ij}=\theta'(R_i)N\theta'(R_j)$.

LEMMA 24. *S_1, S_2 are equivalent abelian projections. A similar statement is true for S'_1, S'_2.*

Proof. $P_1(1-\bar{P}_3)\sim P_2(1-\bar{P}_3)$ and are abelian. Hence $S_1=DP_1(1-\bar{P}_3)\sim DP_2(I-\bar{P}_3)=S_2$. Moreover $S_1MS_1\subseteqq P_1(1-\bar{P}_3)MP_1(1-\bar{P}_3)$ which is abelian.

LEMMA 25. *The S_i, S'_i, T_j, T'_j $i=1,2, j=1,2,3$ are mutually $\perp$, core-free projections. A similar statement is true for $\theta(S_i)$, $\theta'(S'_i)$, $\theta(T_j)$, $\theta'(T'_j)$. Moreover*

$$\sum_{i=1}^{3} T_i=D\bar{P}_3,\ \sum_{i=1}^{3} T'_i=(I-D)\bar{P}_3,\ \sum_{i=1}^{2}\theta(S_i)+\sum_{i=1}^{3}\theta(T_i)$$
$$=\psi(D),\ \sum_{i=1}^{2}\theta'(S'_i)+\sum_{i=1}^{3}\theta'(T'_i)=\psi(I-D).$$

Finally $\overline{\theta(S_1)}=\overline{\theta(S_2)}$, $\overline{\theta(T_i)}=\overline{\theta(T_j)}$, $\overline{\theta'(S'_1)}=\overline{\theta'(S'_2)}$, $\overline{\theta'(T'_i)}=\overline{\theta'(T'_j)}$.

Proof. $\theta(S_1) = \theta(Q_1)\psi(I - \bar{P}_3)$, $\theta(S_2) = \theta(Q_2)\psi(I - \bar{P}_3)$ $\sum_{i=1}^3 \theta(T_i) = \sum_{i=1}^3 \psi(\bar{P}_3)\theta(Q_i) = \psi(\bar{P}_3) \sum_{i=1}^3 \theta(Q_i)$. Hence $\theta(S_1) + \theta(S_2) + \sum_{i=1}^3 \theta(T_i) = \theta(Q_1) + \theta(Q_2) + \psi(\bar{P}_3)\,\theta(Q_3) = \theta(Q_1) + \theta(Q_2) + \theta(Q_3\bar{P}_3) = \sum_{i=1}^3 \theta(Q_i) = \psi(D)$ by Lemma 13. $\bar{S}_1 = \overline{P_1 D}\,(I - \bar{P}_3) = D(I - \bar{P}_3) = \overline{P_2 D}(1 - \bar{P}_3) = \bar{S}_2$. Hence by Lemma 8, $\overline{\theta(S_1)} = \overline{\theta(S_2)}$.

LEMMA 26. $\phi^{-1}\left(\sum_{i=1}^{3} N_{ii} + \sum_{i=1}^{3} N_{ii}\right) = \sum_{i=1}^{3} M_{ii} + \sum_{i=1}^{3} M_{ii}$.

Proof. If $X \in M_{11}$ then $0 = [X, Q_2] = [X, Q_3] = [X, R_i]$ $i = 1, 2, 3$. Hence, $0 = [\phi(X), \theta(Q_2)] = [\phi(X), \theta(Q_3)] = [\phi(X), \theta'(R_i)]$ $i = 1, 2, 3$. Thus, $0 = \theta(Q_1)\phi(X)\theta(Q_2) = \theta(Q_2)\phi(X)\theta(Q_1) = \theta(Q_3)\phi(X)\theta(Q_2) = \theta(Q_2)\phi(X)\theta(Q_3)$, and $\theta'(R_i)\,\phi(X)\theta'(R_j) = 0$ $i \neq j$.

On the other hand for $Y \in N_{11}$, then $0 = [Y, \theta(Q_2)] = [Y, \theta(Q_3)] = [Y, \theta'(R_i)]$ $i = 1, 2, 3$. Thus if $X \in M$ is such that $\phi(X) - Y \in Z_N$ we have $0 = [X, Q_2] = [X, Q_3] = [X, R_i]$ $i = 1, 2, 3$ by Lemma 6. Hence $X \in \sum_{i=1}^3 M_{ii} + \sum_{i=1}^3 M_{ii}$.

LEMMA 27. *$\phi^{-1}(N_{ij}) = M_{ij}$ and $\phi^{-1}(N_{ji}) = M_{ij}$ for $i \neq j$.*

Proof. $\phi(M_{ij}) \subseteqq N_{ij} (i \neq j)$ as in the I_2 case. Suppose $Y \in N_{12}$ and let X be such that $\phi(X) - Y \in Z_N$. $Y = [\theta(Q_1), Y] = [\theta(Q_1), [Y, \theta(Q_2)]]$. Hence, as before $[Q_1, X] = [Q_1, [X, Q_2]]$ so that $Q_2 \times Q_1 = 0$. Moreover, $0 = [Y, \theta(Q_3)] = [Y, \theta'(R_i)]$ for $i = 1, 2, 3$ so that $0 = [X, Q_3] = [X, R_i]$ for $i = 1, 2, 3$ by Lemma 6. Writing $X = \sum_{1 \leqq i, j \leqq 3} X_{ij} + \sum_{1 \leqq i, j \leqq 3} \underline{X}_{ij}$ where $X_{ij} \in M_{ij}$, $\underline{X}_{ij} \in \underline{M}_{ij}$ we have $X = \sum_{i=1}^3 X_{ii} + \sum_{i=1}^3 X_{ii} + X_{12}$. Hence $\phi(X) = \sum_{i=1}^3 X_{ii} + \sum_{i=1}^3 \underline{X}_{ii} + X_{12} - Z = Y \in N_{12}$ for some $Z \in Z_N$. By Lemma 26, $\phi\,(\sum_{i=1}^3 X_{ii} + \sum_{i=1}^3 \underline{X}_{ii}) \in \sum_{i=1}^3 N_{ii} + \sum_{i=1}^3 \underline{N}_{ii}$ and so, as in Lemma 16, $\phi\,(\sum_{i=1}^3 X_{ii} + \sum_{i=1}^3 \underline{X}_{ii}) = Z$ or $\phi(X_{12}) = Y$.

For the other part, let $X \in \underline{M}_{12}$. As before, $X = [R_1, [X, R_2]] = [R_1, X]$. But now, since $\phi(R_i) = -\theta'(R_i) + \lambda'(R_i)$ and the $\theta'(R_i)$ are mutually $\perp$, we have $\phi(X) = [-\theta'(R_1), [\phi(X), -\theta(R_2)]] = \theta'(R_2) \times \phi(X)\theta'(R_1) + \theta'(R_1)\phi(X)\theta'(R_2) = \theta'(R_2)\phi(X)\theta'(R_1) - \theta'(R_1)\phi(X)\theta'(R_2)$. Hence $\phi(X) = \theta'(R_2)\phi(X)\theta'(R_1)$. This shows $\phi(\underline{M}_{ij}) \subseteqq N_{ji}$. An argument similar to that above shows $\phi^{-1}(\underline{N}_{ji}) = \underline{M}_{ij}$.

COROLLARY. *ϕ is onto.*

LEMMA 28. $\phi(Z_{M_{11}}) \subseteqq N_{11} + Z_N$.

Proof. $A \in Z_{M_{11}}$ implies $[A, X] = 0$ for all X in $M_{23} + M_{32} + \sum_{i=1}^3 M_{ii} + \sum_{i=1}^3 \underline{M}_{ii}$. Hence if $B = \phi(A)$, $[B, X] = 0$ for all X in $N_{23} + N_{32} + \sum_{i=1}^3 N_{ii} + \sum_{i=1}^3 \underline{N}_{ii}$ by Lemma 26 and Lemma 27. That is, $[B, X] = 0$ for all X in $N_0 = \{STS \mid T \in N, S = \theta(Q_2) + \theta(Q_3) + \psi(I - D)\}$. Now by Lemma 26, $B = B_1 + C$ where $B_1 \in N_{11}$, $C \in N_0$. Thus, $0 =$

$[B, X] = [B_1 + C, X] = [C, X]$ for all $X \in N_0$, and $C \in Z_{N_0} = \{SZS \mid Z \in Z_N\}$. Thus, since $S = I - \theta(P_1)$, $B = B_1 + C = B_1 + Z(I - \theta(P_1)) = B_1 - Z\theta(P_1) + Z$ where $Z \in Z_N$.

COROLLARY. $\phi(S_1MS_1) \subseteqq N_{11} + Z_N$.

Proof. S_1MS_1 is an abelian algebra, and so $S_1MS_1 \subseteqq Z_{S_1MS_1} = \{ZS_1 \mid Z \in Z_N\} \subseteqq Z_{M_{11}}$ since $S_1 = P_1D(I - \bar{P}_3)$.

LEMMA 29. $\phi(T_1MT_1) \subseteqq \sum_{i=1}^{2} N_{\theta(S_i)} + \sum_{i=1}^{2} N_{\theta'(S_i')} + \sum_{i=1}^{3} N_{\theta(T_i)} + \sum_{i=1}^{3} N_{\theta'(T_i')}$.

Proof. Since $I = \sum_{i=1}^{2} \theta(S_i) + \sum_{i=1}^{2} \theta'(S_i') + \sum_{i=1}^{3} \theta(T_i) + \sum_{i=1}^{3} \theta'(T_i')$ we apply the method of Lemma 15.

LEMMA 30. $\phi(T_iMT_j) = \theta(T_i)N\theta(T_j)$, $\phi(S_iMS_j) = \theta(S_i)N\theta(S_j)$, $\phi(S_i'MS_j') = \theta'(S_j')N\theta'(S_i')$, $\phi(T_i'MT_j') = \theta'(T_j')N\theta'(T_i')$ *for* $i \neq j$.

Proof. As before.

LEMMA 31. $\phi(T_1MT_1) \subseteqq \theta(T_1)N\theta(T_1) + Z_N$.

Proof. If $A \in T_1MT_1$ then $[A, X] + 0$ for all X in $T_2MT_3 + T_3MT_2 + \sum_{i\neq j} S_iMS_j + \sum_{i\neq j} S_i'MS_j' + \sum_{i=j} T_i'MT_j'$. Hence, if $B = \phi(A)$, $[B, X] = 0$ for all X in $\theta(T_2)N\theta(T_3) + \theta(T_3)N\theta(T_2) + \sum_{i\neq j} \theta(S_i)N\theta(S_j) + \sum_{i\neq j} \theta'(S_i')\ N\theta'(S_j') + \sum_{i\neq j} \theta'(T_i')\ N\theta'(T_j')$. Since the corresponding projections (e.g., $\theta(S_i), \theta(S_j)$) have the same central carrier, we can apply Lemma 5 to conclude that $[B, X] = 0$ for all X in $\{STS \mid T \in N, S = \theta(T_2) + \theta(T_3) + \sum_{i=1}^{n} \theta(S_i) + \sum_{i=1}^{2} \theta'(S_i') + \sum_{i=1}^{3} \theta'(T_i')\} = N_0$. As before we have $B = B_1 + C$ where $B_1 \in \theta(T_1)N\theta(T_1)$, $C \in Z_{N_0}$. Hence $B \in \theta(T_1)N\theta(T_1) + Z_N$.

LEMMA 32. $\phi(M_{11}) \subseteqq N_{11} + Z_N$.

Proof. $M_{11} = P_1MP_1 = (S_1 + T_1)M(S_1 + T_1) = S_1MS_1 + T_1MT_1$. Similar arguments show that $\phi(M_{ii}) \subseteqq N_{ii} + Z_N$, $\phi(\underline{M}_{ii}) \subseteqq \underline{N}_{ii} + Z_N$ for $i = 1, 2, 3$.

We define mappings $\sigma: M_D \rightarrow N_{\psi(D)}$ and $\lambda: M_D \rightarrow Z_N$ in the following manner: if $A \in M_{ij} (i \neq j)$ then $\sigma(A) = \phi(A)$, and if $A \in M_{ii}$ by Lemma 30 $\phi(A) = \sigma(A) + Z$ where $\sigma(A) \in N_{ii}$ and $Z \in Z_N$. Extend σ to all of M_D by linearity and define $\lambda(A) = \phi(A) - \sigma(A)$ for $A \in M_D$. σ and λ are well defined, for if $T_1 + Z_1 = T_2 + Z_2$ where $T_i \in N_{ii}$, $Z_i \in Z_N$, then $T_1 - T_2 \in N_{ii} \cap Z_n = \{0\}$.

We can analogusly define mappings $\sigma'\colon M_{I-D} \rightarrow N_{\psi(I-D)}$ and λ': $M_{I-D} \rightarrow Z_N$ using the corresponding facts for $\underline{M}_{ij}$. Note that $\sigma'(\underline{M}_{ij}) = \underline{N}_{ji}$.

As before, σ, λ, σ', and λ' are *-linear mappings.

LEMMA 33. *If* $A \in M_{ii}$, $B \in M_{ij}(i \neq j)$ *then* $\sigma(AB) = \sigma(A)\sigma(B)$. *If* $A \in \underline{M}_{ii}$, $B \in \underline{M}_{ij}(i \neq j)$ *then* $\sigma'(AB) = -\sigma'(B)\sigma'(A)$.

Proof. $A \in \underline{M}_{ii}$, $B \in \underline{M}_{ij}$ then $\sigma'(AB) = \phi(AB) = \phi[A, B] = [\phi(A), \phi(B)] = [\sigma'(A), \sigma'(B)] = -\sigma'(B)\sigma'(A)$ since $\sigma(B) \in \underline{M}_{ji}$.

LEMMA 34. *If* $A, B \in M_{ii}$ then $\sigma(AB) = \sigma(A)\sigma(B)$. *If* $A, B \in \underline{M}_{ii}$ then $\sigma'(AB) = -\sigma'(B)\sigma'(A)$.

Proof. The proof for $A, B \in M_{ii}$ is similar to Lemma 21. If $A, B \in \underline{M}_{ii}$ and $S \in \underline{M}_{ij}(i \neq j)$ then $\sigma'(S)\sigma'(B)\sigma'(A) = -\sigma'(BS)\sigma(A) = \sigma'(ABS) = -\sigma'(S)\sigma'(AB)$. Hence $0 = [\sigma'(AB) + \sigma'(B)\sigma'(A)]\sigma'(S) = \sigma'(S)[\sigma'(AB) + \sigma'(B)\sigma'(A)]$.

LEMMA 35. *If* $A \in M_{ij}$, $B \in M_{ji}(i \neq j)$ *then* $\sigma(AB) = \sigma(A)\sigma(B)$. *If* $A \in \underline{M}_{ij}$, $B \in \underline{M}_{ji}$ *then* $\sigma'(AB) = \sigma'(B)\sigma'(A)$.

Proof. Similar to Lemma 22.

THEOREM 3. σ(resp. σ') *is a *-isomorphism of* M_D *onto* $N_{\psi(D)}$ (resp. is the negative of a *-anti-isomorphism of M_{I-D} onto $N_{\psi(I-D)}$) *and* λ(resp. λ') *is a *linear map from* M_D *into* Z_N (resp. from M_{I-D} into Z_N) *which annihilates brackets.*

Proof. Similar to the I_2 case.

REMARK 1. If M has no summand of type I_2, then a proof much like the one above gives the same Theorem 3 for such an M. Thus, including the I_2 case, Theorem 3 holds for all von Neumann algebras M which have no abelian summands of type I.

REMARK 2. A result of Sunouchi [15] (see remarks following Theorem 3 below) shows, in part, that if M is an infinite von Neumann algebra, then $[M, M] = M$. Thus in the above, if $\phi\colon M \rightarrow N$ is a near-isomorphism and M infinite then $\phi = \sigma + \sigma'$ and is automatically bounded.

REMARK 3. Martindale [10] proved that if L is a Lie derivation of a primitive ring R into itself, where R has a nonzero idempotent and is not of characteristic 2, then $L = D + T$ where D is an ordinary derivation and T a center-valued additive map which annihilates commutators. By slightly altering Martindale's proof we can show the same result if R is a von Neumann algebra.

4. Lie *-homomorphisms of C*-algebras.

We now turn our attention to characterizing uniformly continuous Lie *-homomorphisms between C^*-algebras. In order to do this we first investigate ultra weakly (UW) closed Lie *-ideals in von Neumann algebras.

DEFINITION. A Lie *-ideal U in a *-algebra M is a *-linear subspace of M such that if $X \in U$ and $Y \in M$ then $[X, Y] \in U$.

LEMMA 36. *Let M be a C^*-algebra, U a Lie *-ideal in M such that if $A, B \in U$ then $[A, B] = 0$. Then $U \subseteqq Z_M$.*

Proof. Since U is closed with respect to the *-operation it is generated, as a linear space, by its self-adjoint elements. Suppose $A = A^* \in U$, and B any self-adjoint element in M. Then $[B, A] \in U$ so that $[[B, A], A] = 0$ since U commutative. By Lemma 6, $[B, A] = 0$.

LEMMA 37. *If U is a Lie *-ideal in a C^*-algebra M which is at the same time an associative subring of M then either $U \subseteqq Z_M$ or U contains a* (two-sided, associative) *ideal of M.*

Proof. We follow Herstein [3: Theorem 2, p. 281]. If U is commutative then $U \subseteqq Z_M$ by Lemma 36. If U is not commutative then there exist $X, Y \in U$ such that $[X, Y] \neq 0$. In this case Herstein proves that the ideal $M[X, Y]M$ (= all finite linear combinations of elements of the form $S[X, Y]T$ where $S, T \in M$) is contained in U.

LEMMA 38. *If U is any Lie ideal in an associative ring M, then $T(U) = \{T \in M \mid [T, X] \in U \text{ for all } X \in M\}$ is a Lie ideal and subring of M. Moreover $U \subseteqq T(U)$. If U is a Lie *ideal and ultra-weakly closed, so is $T(U)$.*

Proof. See Herstein [3: Theorem 2, p. 282].

COROLLARY. *If M is a von Neumann algebra, U a ultra-weakly closed Lie *-ideal in M, then $T(U) = Z_M$ or there exists a nonzero* *central projection C in M such that $M_C \subseteqq T(U)$.*

Proof. By Lemma 37 either $T(U) \subseteqq Z_M$ (in which case $T(U) = Z_M$ since $Z_M \subseteqq T(U)$), or there exists a nonzero two-sided ideal $I \subseteqq T(U)$. In the latter case, $I^{-UW} \subseteqq T(U)$ and $I^{-UW} = M_C$ for some nonzero central projection C.

THEOREM 4. *If M is a von Neumann algebra and U an ultra-weakly closed Lie *-ideal in M, then $T(U) = Z_M$ or there exists a nonzero central projection C such that $T(U) = Z_M + M_C$.*

Proof. By the corollary to Lemma 38, $T(U) = Z_M$ or there exists a nonzero central projection C' such that $M'_C \subseteqq (U)$. Let C be a maximal such projection. Then $M_C \subseteqq T(U)$ so that $Z_M + M_C \subseteqq T(U)$.

Suppose $S \in T(U)$ and $S \notin Z_M + M_C$. We can assume $S = S^*$. Writing $S = SC + S(I - C)$ we see that $S(I - C) \in T(U)$ (since $S, SC \in T(U)$) and that $S(I - C) \neq 0$ (since $S \notin Z_M + M_C$). Since $S(I - C) \notin Z_M$ there exists $S_1 = S_1^* \in M$ such that $[S_1, S(I - C)] \neq 0$. Moreover $[S_1, S(I - C)] \in T(U)$ since $T(U)$ is a Lie ideal, and $[[S_1, S(I - C)], S(I - C)] \neq 0$ by Lemma 6. Applying the techniques of Herstein [3: Theorem 2, p. 281] with $X = [S_1, S(I - C)]$, $Y = S(I - C)$, the set $M[X, Y]M$ is a non-zero two sided ideal contained in $T(U)$. Or, $0 \neq (I - C)M[[S_1, S], S]M \subseteqq U$. Let $J = M[[S_1, S], S]M$. Then $((I - C)J)^{-UW} = (I - C)J^{-UW} = (I - C)M_D$, where D is a nonzero central projection, is a nonzero, two-sided, ultraweakly closed ideal in $T(U)$. Hence $(I - C)M_D + M_C$ is an ultraweakly closed two-sided ideal in $T(U)$ properly containing M_C, a contradiction. Thus $T(U) = Z_M + M_C$.

Sunouchi [14] proved that if M is a von Neumann algebra, and $[M, M]$ = all finite linear combinations of brackets from M, then M infinite implies $[M, M] = M$, M finite implies that $[M, M]$ is uniformly dense in the null space of the center-valued trace, $\natural$, on M. Using this fact and Theorem 3 we can prove the following corollary:

COROLLARY. *If U is an ultra-weakly closed Lie *-ideal in a von Neumann algebra M, and if $Z_M \subseteqq U$, then either $U = Z_M$ or there exists a nonzero central projection C, in M, such that $U = Z_M + M_C$.*

Proof. If $T(U) = Z_M$ then $Z_M \subseteqq U \subseteqq T(U) \subseteqq Z_M$. Otherwise $T(U) = Z_M + M_C$ for some non-zero central projection $C \in M$. From the definition of $T(U)$ this implies $C[M, M] = [M_C, M_C] \subseteqq U$.

Case 1. M infinite. The $[M, M] = M$ so that $C[M, M] = M_C \subseteqq U$. Thus $Z_M + M_C \subseteqq U \subseteqq T(U) \subseteqq Z_M + M_C$.

Case 2. M finite. Then $[M, M]^{-\| \|} = \{X \in M \mid X^\# = 0\}$ where $\#$ is the center-valued trace on M. For notation let $N_C = \{XC \mid (XC)^\# = 0\}$. We have $[M_C, M_C]^{-\| \|} = N_C$ so that $N_C \subseteqq U$. This implies that $Z_M + N_C \subseteqq U \subseteqq T(U) \subseteqq Z_M + M_C$. But for arbitrary $X \in M$, $XC = (XC)^\# + XC - (XC)^\# \in Z_M + N_C \subseteqq U$. Thus $M_C \subseteqq U$.

THEOREM 5. *If $\phi: M \to N$ is an UW-continuous L-onto Lie *-homomorphism between the Neumann algebras M and N then there exists a central projection $C \in M$ and another UW-continuous L-onto Lie *-homomorphism ψ from M to N such that*

(1) *the difference of ψ and ϕ is a λ-map (i.e.,* *-linear from M *to* Z_N, annihilates brackets),
(2) *M_C lies in ker ψ, and*
(3) *the restriction of ψ to M_{I-C} is a near isomorphism of M_{I-C} to N.*

Proof. Let D be the maximal finite central projection in M and set $\psi(X) = \phi(X) - \phi((XD)^\#)$. There exists, by the preceding corollary, a central projection $C \in M$ such that $\phi^{-1}(Z_N) = M_C + Z_M$. It is easy to see that $\phi^{-1}(Z_N) = \psi^{-1}(Z_N) = T(\ker \psi)$ where $T(U)$, U a Lie ideal, is as above. The definition of $T(\ker \psi)$ implies that $[M_C, M_C] \subseteqq \ker \psi$. By Sunochi's result, $(I - D)\,[M_C, M_C] = M_{C(I-D)} \subseteqq \ker\ \psi$. Hence $\psi(XC(I - D)) = 0$ or $\psi(XC) = \psi(XCD)$. Moreover, again by Sunouchi, $[M_{CD}, M_{CD}]^{-\| \|} = \{XCD \mid (XCD)^\# = 0\} \subseteqq \ker \psi$. Thus $\psi(XCD - (XCD)^\#) = 0$. Finally, $\psi(XC) = \psi(XCD) = \psi((XCD(^\#) = \phi((XCD)^\#) - \phi((XCD)^\#) = 0$.

If $Y \in N$ then, since ϕ is L-onto, there exists $X \in M$ such that $\phi(X) - Y = Z \in Z_N$. Hence $\psi(X) = \phi(X) - \phi((XD)^\#) = Y + Z - \phi((XD)$. Since $\phi(Z_M) \subseteqq Z_N$ we have $\psi(X) - Y \in Z_N$.

If $\psi(X(1 - C)) = \psi(Y(1 - C))$ then $\phi(X(1 - C) - Y(1 - C)) \in Z_N$ or $X(1 - C) - Y(1 - C) = WC + Z$ for $W \in M$, $Z \in Z_M$. Thus $X(1 - C) - Y(1 - C) = Z(1 - C) \in Z_M$.

We are now in a position to prove the main result. Note that if $\phi: M \to N$ is a near isomorphism of von Neumann algebras M, N and if C_M (resp C_N) is the maximal central abelian projection in M (resp N) then $\psi(X) = (I - C_N)\, \phi(X(I - C_M))$ is a near isomorphism of M_{I-C_M} to N_{I-C_N}.

THEOREM 6. *If ϕ is a uniformly continuous Lie*-homomorphism of a C*-algebra $\mathscr{A}$ onto a C*-algebra $\mathscr{B}$, there exists a central projection D in the weak closure of B such that* (modulo a λ-map) *$D\phi$ is a *-homomorphism and $(1 - D)\phi$ is the negative of a *-antihomomorphism.*

Proof. Let $M = \mathscr{A}^{**}$, the second dual of $\mathscr{A}$, and $N = \mathscr{B}^{-W}$. The theorem of Stormer [14: Theorem 3.1, p. 443] for Jordan *-homomorphisms can be altered to give, in the present case, the existence of an UW-continuous extension $\tilde{\phi}: M \rightarrow N$ of ϕ which is also a Lie *-homomorphism onto.

Let $\psi: M \rightarrow N$ be an L-onto Lie *-homomorphism such that there exists a central projection C in M such that $\psi - \tilde{\phi}$ is a λ-map, $M_C \subseteqq \ker \psi$, and $\psi|_{M_{I-C}}$ is a near isomorpism of M_{I-C} to N. We can assume, by the above remark, that M_{I-C} and N have no type I summands. Applying Theorem 3 to $\psi|_{M_{I-C}}$ we have the desired result.

REMARK 1. Theorem 6 would be an exact analog of Stormer's generalization [13, Theorem 3.3, p. 445] of the Kadison result [6, Theorem 10, p. 334] on Jordan homomorphisms of C^*-algebras were it not for the assumptions of uniform continuity and ontoness. In particular a Jordan *-homomorphism is automatically uniformly continuous. The question of continuity of Lie *-homomorphisms of von Neumann algebras, because of the presence of λ-maps, is closely connected with the problem of determining the linear span of commutators in rings of operators.

We must assume ontoness in our theorem because of the lack of an analog for the Lie case, to a theorem of Jacobsen and Rickart [16, Theorem 7, p. 487] which states that any Jordan homomorphism of an $m \times n$ matrix ring into another ring is the sum of a homomorphism and an anti-homomorphism.

REMARK 2. Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras with I, $\mathscr{A}_u$, $\mathscr{B}_u$ their respective unitary groups, and $\rho: \mathscr{A}_u \rightarrow \mathscr{B}_u$ a uniformly continuous group homomorphism. As in [12] there exists uniformly continuous map $\phi: \mathscr{A} \rightarrow \mathscr{B}$, which is in particular a Lie *-homomorphism ϕ of $\mathscr{A}$ onto $\mathscr{B}$. Thus the statement of Theorem 6 applies to this ϕ.

REFERENCES

1. J. Dixmier, *Les Algebres D'Operateurs dans L'Espace Hilbertian*, Cahiers Scientifiques, Fas. XXV, Gauthier-Villars, Paris (1957).

2. ———, *Les C*-algebres et Leurs Reprsentations*, Cahiers Scientifiques, Fas. XXIX, Gauthier-Villars, Paris (1964).

3. I. N. Herstein, *On the Lie and Jordan rings of a simple associative ring*, Ameican J. Math., **77** (1955), 279-285.

4. ———, *Topics in Ring Theory*, The University of Chicago Press, Chicago (1969).

5. L. Hua, *A theorem on matrices over a field and its applications*, J. Chinese Math. Soc., (N. S.) **1** (1951), 110-163.

6. N. Jacobsen and C. E. Rickart, *Jordan homomorphisms of rings*, Trans. Amer. Math. Soc., **69** (1950), 479-502.

7. R. V. Kadison, *Isometries of operator algebras*, Annals of Math., **54**, No. 2 (1951), 325-338.
8. W. S. Martindale 3rd. *Lie isomorphisms of primitive rings*, Proc. Amer. Math. Soc., **14** (1963), 909-916.
9, ———, *Lie isomorphisms of simple rings*, J, London Math. Soc., **44** (1969), 213-221.
10. ———, *Lie derivations of primitive rings*, Michigan Math. J. **11** (1964), 183-187.
11. C. Robert Miers, *Lie isomorphisms of factors*, Trans, Amer. Math. Soc., **147**, January (1970), 55-63.
12. S. Sakai, *On the group isomorphism of unitary groups in AW^*-algebras*, Tohoku Math. J. **7** (1955), 87-95.
13. I. M. Singer, *Uniformly continuous representations of Lie groups*, Annals of Math., **56** (1952), 242-247.
14. E. Størmer, *On the Jordan struture of C^*-algebras*, Trans. Amer. Math. Soc., **120** (1965), 438-447.
15. H. Sunouchi, *Infinite Lie Rings*, Tohoku Math. J. **8** (1956), 291-307.

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UNIVERSITY OF VICTORIA