PACIFIC JOURNAL OF MATHEMATICS
Vol. 38, No. 3, 1971

CANONICAL DOMAINS AND THEIR GEOMETRY IN C^n

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In this paper we introduce some differential geometric properties of canonical domains of bounded domains in C^n, using our synthetic expression by matrix. In the proofs of the theorems, our formulas of matrix derivatives play the leading part.

In order to construct relatively invariant matrices, the author devised formulas of matrix derivatives and obtained some results ([2]). Here we use these formulas for the calculations on the argument of the theorems of geometry. The constructed matrix $\frac{1}{2}{}_2T_D(\bar{z}, z)$ (see [2]) becomes the *curvature tensor*, and $T_D^{-1}(\bar{z}, z)(\partial T_D(\bar{z}, z)/\partial z)$ becomes *Christoffel symbols* in the Kaehler manifold with the metric $ds_D = dz^* T_D(\bar{z}, z)dz$ where ${}_2T_D(\bar{z}, z) = (E_n \times T_D(\bar{z}, z))(\partial/\partial z^*)(T_D^{-1}(\bar{z}, z)(\partial T_D(\bar{z}, z)/\partial z))$ and $T_D(\bar{z}, z) = \partial^2 log K_D(\bar{z}, z)/\partial z^* \partial z$. We study some differential geometric properties of canonical domains, that is, Bergman representative domains, m-representative domains, homogeneous domains, and our minimal domains of moment of inertia which are defined and investigated in § 2 ([1], [5], [7], [12]).

We calculate Christoffel symbols at the center of canonical domains and give the condition which a geodesic curve through the center of a representative domain satisfies in Theorems 3.4. In Theorems 3.7–12 and Corollaries 3.1–4, we discuss *scalar curvature* and *holomorphic sectional curvature*.

The author wishes to thank Professor S. Ozaki for helpful discussions in the preparation of the present paper.

1. **Preliminaries.** Let D be a domain in C^n which posses a Bergman kernel function $K_D(\bar{t}, z) \equiv \varphi^*(t)\varphi(z)$, $t, z \in D$, where $\varphi(z) \equiv (\varphi_1(z), \varphi_2(z), \cdots)'$ and the marks $'$ and $*$ denote the transposed and transposed conjugate matrices respectively. We consider a vector function $w(z) \equiv (w_1(z), \cdots, w_n(z))'$ in D. If the function $w(z)$ is both holomorphic and locally one-to-one, i.e., $det(dw/dz) \neq 0$, then the function defines a pseudo-conformal mapping of D onto another domain $\Delta \subset C^n$. Further, the inner product of two functions f, g belonging to a class $\mathscr{L}_D^2$ of all holomorphic functions $\zeta(z)$ of D which satisfy $\int_D |\zeta(z)|^2 dv_D \equiv Sp(\int_D \zeta(z)\zeta^*(z)dv_D) < \infty$, as follows:

$$(f, g)_D \equiv \int_D f(z)g^*(z)dv_D ,$$

where dv_D denotes the Euclidean volume element on D. Moreover we define a norm $||f||_D$ of $f(z)$ as

$$(1.1)\qquad ||f||_D^2 \equiv Sp(f, f)_D = \int_D |f(z)|^2 dv_D .$$

We shall define some notations for derivatives of matrix functions with respect to the vector variable $z = (z_1, \cdots, z_n)'$:

$$(1.2)\qquad \frac{\partial^{h+k} w(\bar{t}, z)}{\partial t^{*h} \partial z^k} = \left(\frac{\partial}{\partial t}\right)^{*h}\left(\frac{\partial}{\partial z}\right)^k \times w(\bar{t}, z) ,$$

where $(\partial/\partial t)^{*h}$ and $(\partial/\partial z)^k$ denote h-times and k-times Kronecker product of $(\partial/\partial t)^* = (\partial/\partial \bar{t}_1, \cdots, \partial/\partial \bar{t}_n)'$ and $\partial/\partial z = (\partial/\partial z_1, \cdots, \partial/\partial z_n)$ respectively. If $w(z)$ is a function of z only, the kth derivative is denoted by $d^k w(z)/dz^k$. In particular, if z and t are both fixed, then we shall write the derivatives merely $w_{t^{h*}z^k}$ or $\partial^{h+k}w/\partial t^{*h}\partial z^k$. Hereafter, sometimes we shall write $T_D(\bar{t}_0, t_0) = T_D$, $K_D(\bar{t}_0, t_0) = K_D$, $\partial^2 K_D(\bar{t}_0, t_0)/\partial t^* \partial z = \partial^2 K_D/\partial t^* \partial z = K_{t^*z}$, and so on. Further we denote the following formulas with respect to the matrix derivatives:

$$(1.3)\qquad \frac{\partial F^{-1}}{\partial z} = -F^{-1}\frac{\partial F}{\partial z}(E_n \times F^{-1}) ,$$

(F is a regular $k \times k$ matrix function and E_n is an $n \times n$ unit matrix)

$$(1.4)\qquad \frac{\partial (FG)}{\partial z} = \frac{\partial F}{\partial z}(E_n \times G) + F\frac{\partial G}{\partial z} ,$$

(F and G are $k \times l$, $l \times m$ matrices respectively)

$$(1.5)\qquad \frac{\partial F}{\partial z} = \frac{\partial F}{\partial \zeta}\left(\frac{\partial \zeta}{\partial z} \times E_l\right) + \left(\frac{\partial \zeta^*}{\partial z} \times E_k\right)\left(E_n \times \frac{\partial F}{\partial \zeta^*}\right) ,$$

(F is a $k \times l$ matrix, z, ζ are $n \times 1$ vectors)

$$(1.6)\qquad \frac{\partial (F \times G)}{\partial z} = \frac{\partial F}{\partial z} \times G + \left(F \times \frac{\partial G}{\partial z}\right)(\tilde{E}_{ln} \times E_\nu) ,$$

(F, G are $k \times l$, $\mu \times \nu$ matrices respectively, and

$$\tilde{E}_{ln} = \begin{pmatrix} e_{11}, & \cdots, & e_{l1} \\ & \cdots & \\ e_{1n}, & \cdots, & e_{ln} \end{pmatrix}$$

where e_{ij} are $l \times n$ matrices in which only (i, j) element equal to 1, and others 0). If $\zeta = \zeta(z)$ is a pseudo-conformal mapping of a domain D onto a domain Δ, then we have

$$(1.7)\qquad K_D(\bar{t}, z) = \left(\det \frac{d\tau(t)}{dt}\right)^* K_\Delta(\bar{\tau}, \zeta) \det \frac{d\zeta(z)}{dz} ,$$

$$T_D(\bar{t}, z) = \left(\frac{d\tau(t)}{dt}\right)^* T_\Delta(\bar{\tau}, \zeta) \frac{d\zeta(z)}{dz} , \tag{1.8}$$

and for a matrix $P = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$ with the block subdivisions, it holds that

$$P^{-1} = \begin{pmatrix} K^{-1} + XZ^{-1}Y, & -XZ^{-1} \\ -Z^{-1}Y, & Z^{-1} \end{pmatrix} , \tag{1.9}$$

where K, N are square matrices, and $X = K^{-1}L$, $Y = MK^{-1}$, $Z = N - MK^{-1}L$.

2. Moment of inertia and relative invariant matrix. For the holomorphic mappings $\zeta(z) = A(z-t_0) + (\textit{higher powers})$ with respect to t_0, we define the classes which satisfy respectively the following initial conditions at a fixed point $t_0 \in D$:

$$\mathscr{F}_{E;t_0}\colon A = E , \qquad \mathscr{F}_{|A|;t_0}\colon det\, A = 1 ,$$

$$\mathscr{F}_{|A^*A|;t_0}\colon det\, A^*A = 1 , \qquad \mathscr{F}_{SpA^*A;t_0}\colon \frac{SpA^*A}{n} = 1$$

$$(\mathscr{F}_{E;t_0} \subset \mathscr{F}_{|A|;t_0} \subset \mathscr{F}_{|A^*A|;t_0} \subset \mathscr{F}_{SpA^*A;t_0}) .$$

Bergman representaive and minimal domains were considered for the classes $\mathscr{F}_{E;t_0}$ and $\mathscr{F}_{|A|;t_0}$, respectively. If we define the moment of inertia of Δ which is the image of D by $\zeta(z)$ as

$$mom(\Delta) \equiv ||\zeta||_\Delta^2 = \int_\Delta |\zeta|^2 dv_\Delta = \int_D |\zeta \cdot det\frac{d\zeta}{dz}|^2 dv_D , \tag{2.1}$$

then a minimal domain of moment of inertia with $\zeta(t_0)$ as center which minimizes $mom(\Delta)$ may be considered for the classes of the above four types. But now we treat for the class $\mathscr{F}_{E;t_0}$. First, we deal with the minimum problems following S. Bergman ([1], [5]). The following relations hold for any functions $\zeta(z) = A(z-t_0) + (\textit{higher powers})$, using (1.9),

$$\begin{aligned} ||\zeta(z)||_D^2 \equiv Sp\int_D \zeta\zeta^* dv_D &\geqq Sp(O, A)\begin{pmatrix} K_D & K_z \\ K_{t^*} & K_{t^*z} \end{pmatrix}^{-1} \begin{pmatrix} O \\ A^* \end{pmatrix} \\ &= \frac{1}{K_D} Sp(AT_D^{-1}A^*) , \end{aligned} \tag{2.2}$$

and minimizing function exists uniquely and is expressed as follows:

$$(O, A)\begin{pmatrix} K_D & K_z \\ K_{t^*} & K_{t^*z} \end{pmatrix}^{-1} \begin{pmatrix} K_D(\bar{t}_0, z) \\ \partial K_D(\bar{t}_0, z)/\partial t^* \end{pmatrix} = \frac{K_D(\bar{t}_0, z)}{K_D} AT_D^{-1}\int_{t_0}^z T_D(\bar{t}_0, z)dz , \tag{2.3}$$

where A is an $n \times n$ matrix. If $\zeta(z) \in \mathscr{F}_{E;t_0}$, then $\zeta(z)det(d\zeta(z)/dz)$ also belongs to $\mathscr{F}_{E;t_0}$, hence the mapping $\zeta(z)$ which maps D onto a minimal domain of moment of inertia satisfies

$$\zeta(z) det\frac{d\zeta(z)}{dz} = \frac{K_D(\bar{t}_0, z)}{K_D} T_D^{-1} \int_{t_0}^{z} T_D(\bar{t}_0, z)dz \ , \tag{2.4}$$

and the moment of inertia of this minimal domain is $1/K_D \cdot (SpT_D^{-1})$. (See [9]).

THEOREM 2.1. *A necessary and sufficient condition for a domain D to be a minimal domain of moment of inertia with t_0 as center is*

$$\frac{d}{dz}(K_D(\bar{t}_0, z)\int_{t_0}^{z} T_D(\bar{t}_0, z)dz) \equiv K_D T_D \ . \tag{2.5}$$

In fact, for the identity mapping $\zeta(z) = z$ of D, $\zeta(z)det(d\zeta(z)/dz) = z$, therefore the necessary and sufficient condition is

$$z = \frac{K_D(\bar{t}_0, z)}{K_D} T_D^{-1} \int_{t_0}^{z} T_D(\bar{t}_0, z)dz \ .$$

THEOREM 2.2. *A domain D is a minimal domain of moment of inertia with t_0 as center, if the following condition is fulfiled:*

$$\partial^2 K_D(\bar{t}_0, z)/\partial t^* \partial z \equiv K_D T_D \ . \tag{2.6}$$

Proof. From the hypothesis, we have $\partial K_D(\bar{t}_0, z)/\partial t^* = K_D T_D \cdot (z - t_0)$, therefore $\partial K_D/\partial t^* = 0$. Hence, using the relation

$$K_D(\bar{t}_0, z)\int_{t_0}^{z} T_D(\bar{t}_0, z)dz = \partial K_D(\bar{t}_0 z)/\partial t^* - \frac{K_D(\bar{t}_0, z)}{K_D} \cdot \partial K_D/\partial t^* \ ,$$

we have

$$\frac{\partial}{\partial z}(K_D(\bar{t}_0, z)\int_{t_0}^{z} T_D(\bar{t}_0, z)dz) = K_D T_D \ ,$$

consequently the hypothesis of Theorem 2.1 is fulfiled.

COROLLARY 2.1. *Let D be a minimal domain with center at t_0, then a necessary and sufficient condition for the domain D to be a minimal domain of moment of inertia with the same center t_0 is*

$$\partial^2 K_D(\bar{t}_0, z)/\partial t^* \partial z \equiv K_D T_D \ . \tag{2.7}$$

COROLLARY 2.2. *Let D be a representative domain with center at t_0, then D is a minimal domain of moment of inertia if and only if*

$$\frac{d}{dz}(K_D(\bar{t}_0, z) \cdot (z - t_0)) \equiv K_D \ . \tag{2.8}$$

Proof. By the hypothesis we have $T_D(\bar{t}_0, z) \equiv T_D$, consequently $\int_{t_0}^{z} T_D(\bar{t}_0, z)dz = T_D \cdot (z - t_0)$. Substituting this into (2.5), we obtain (2.8).

COROLLARY 2.3. *Let D be a representative domain with center at t_0, and simultaneously a minimal domain with the same center, then D is also a minimal domain of moment of inertia.*

Proof. We can prove easily from $\partial^2 K_D(\bar{t}_0, z)/\partial t^* \partial z \equiv K_D T_D$ which is a necessary and sufficient condition for a domain D to be a minimal domain with center at t_0 and simultaneously a representative domain with the same center, and (2.6).

Next, we introduce relative invariant matrices which play an important part in Riemannian geometry of a complex n-dimensional manifold.

LEMMA 2.1. *The following relation holds:*

$$(E_n \times T_D(\bar{z}, z))\frac{\partial}{\partial z^*}\left(T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z}\right)$$

$$= \frac{\partial^2 T_D(\bar{z}, z)}{\partial z^* \partial z} - \frac{\partial T_D(\bar{z}, z)}{\partial z^*} T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z} \tag{2.9}$$

$$(\equiv {}_2T_D(\bar{z}, z)) \ ,$$

and for any pseudo-conformal mapping $\zeta = \zeta(z)$ which maps D onto Δ, we have

$$ {}_2T_D(\bar{z}, z) = (d\zeta(z)/dz)^{*2}{}_2T_\Delta(\bar{\zeta}, \zeta)(d\zeta(z)/dz)^2 \ , \tag{2.10}$$

where the power means 2-times Kronecker product. (See [2]).

LEMMA 2.2. *Let ${}_p\psi_D(\bar{z}, z)$ be*

$$ {}_p\psi_D(\bar{z}, z) \equiv \frac{1}{K^p}\left(\frac{\partial^2(K^pT)}{\partial z^* \partial z} - \frac{\partial(K^pT)}{\partial z^*}(K^pT)^{-1}\frac{\partial(K^pT)}{\partial z}\right) , \tag{2.11}$$

where $K^p = (K_D(\bar{z}, z))^p$ and $T = T_D(\bar{z}, z)$, then under any pseudo-conformal mapping we have

$$ {}_p\psi_D(\bar{z}, z) = (d\zeta(z)/dz)^{*2}{}_p\psi_\Delta(\bar{\zeta}, \zeta)(d\zeta(z)/dz)^2 \ . \tag{2.12}$$

REMARK. For $p > 2$ we showed that ${}_p\psi_D(\bar{z}, z)$ are positive definite (see [2]), but in the following (Corollary 3.4), we shall show that, for $p > 1$, these quantities are also positive definite in a bounded domain by the properties of holomorphic sectional curvature.

In fact, Using the formulas (1.4)$\sim$(1.6), we can calculate as follows:

$$\frac{\partial(K^pT)}{\partial z} = pK^{p-1}\Big(\frac{\partial K}{\partial z} \times T\Big) + K^p\frac{\partial T}{\partial z}\,,$$

$$\begin{aligned}\frac{\partial^2(K^pT)}{\partial z^*\partial z} &= p\Big[(p-1)K^{p-2}\frac{\partial K}{\partial z} \times \Big(\frac{\partial K}{\partial z^*} \times T\Big) \\ &\quad + K^{p-1}\Big(\frac{\partial^2 K}{\partial z^*\partial z}\times T + \frac{\partial K}{\partial z^*} \times \frac{\partial T}{\partial z}\Big)\Big] \\ &\quad + pK^{p-1}\frac{\partial K}{\partial z} \times \frac{\partial T}{\partial z^*} + K^p\frac{\partial^2 T}{\partial z^*\partial z}\,,\end{aligned}$$

therefore, we have

$$ {}_p\psi_D(\bar{z}, z) = {}_2T_D(\bar{z}, z) + pT_D(\bar{z}, z) \times T_D(\bar{z}, z)\,. \tag{2.13}$$

From this and (2.10), we obtain (2.12).

3. Curvature in canonical domains. We introduce a positive definite Kaehler metric on D which is invariant under any pseudo-conformal mapping of D

$$ds_D^2 = dz^*T_D(\bar{z}, z)dz\,,$$

and consider a real $2n$-dimensional manifold V_{2n} of the variables $\begin{pmatrix} z \\ \bar{z}\end{pmatrix} = (z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n)'$ and let the metric be

$$\begin{aligned} ds_D^2 &= dz^*T_D(\bar{z}, z)dz \\ &= \begin{pmatrix} dz \\ d\bar{z}\end{pmatrix}^* \begin{pmatrix} \frac{1}{2}T(\bar{z}, z), & 0 \\ 0\quad, & \frac{1}{2}\bar{T}_D(\bar{z}, z)\end{pmatrix}\begin{pmatrix} dz \\ d\bar{z}\end{pmatrix} \\ &= \begin{pmatrix} dz \\ d\bar{z}\end{pmatrix}^*(g_{ij})\begin{pmatrix} dz \\ d\bar{z}\end{pmatrix}\,,\end{aligned} \tag{3.1}$$

then we have $g_{\bar{\alpha}\beta} = \frac{1}{2}T_{\bar{\alpha}\beta} = \bar{g}_{\alpha\bar{\beta}}$, $g_{\alpha\bar{\beta}} = \frac{1}{2}T_{\alpha\bar{\beta}} = \bar{g}_{\bar{\alpha}\beta}$, $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$, where $T_{\bar{\alpha}\beta} = (\partial^2 log K_D(\bar{z}, z)/\partial\bar{z}_\alpha\partial z_\beta)$, and $i, j = 1, \cdots, n, \bar{1}, \cdots, \bar{n}$; $\alpha, \beta = 1, \cdots, n$. If we define a curve in V_{2n} by the functions

$$\begin{pmatrix} z(t) \\ \bar{z}(t)\end{pmatrix} \equiv (z_1(t), \cdots, z_n(t), \bar{z}_1(t), \cdots, \bar{z}_n(t))'$$

with respect to a parameter t, then the infinitesimal distance on this curve is given by $ds = \sqrt{dz^*T_D(\bar{z}(t), z(t))dz}$, and the length of this curve joining two points $A_1 = \begin{pmatrix} z(t_1) \\ \bar{z}(t_1)\end{pmatrix}$ and $A_2 = \begin{pmatrix} z(t_2) \\ \bar{z}(t_2)\end{pmatrix}$ is

$$s = \int_{t_1}^{t_2}\sqrt{\frac{dz^*}{dt}T_D(\bar{z}, z)\frac{dz}{dt}}dt\,.$$

For the function $F \equiv (dz^*/ds)T_D(\bar{z}, z)(dz/ds) \equiv \dot{z}^* T_D(\bar{z}, z)\dot{z}$, we have $(\partial F/\partial z) = \dot{z}^*(\partial T_D(\bar{z}, z)/\partial z)(E_n \times \dot{z})$, therefore substituting this into Euler's equation we obtain

$$\frac{d}{ds}(\dot{z}^* T_D(\bar{z}, z)) - \dot{z}^* \frac{\partial T_D(\bar{z}, z)}{\partial z}(E_n \times \dot{z})$$
$$= \ddot{z}^* T + \dot{z}^*\left(\frac{\partial T}{\partial z}\left(\frac{dz}{ds} \times E\right) + \left(\frac{dz^*}{ds} \times E\right)\frac{\partial T}{\partial z^*}\right) - \dot{z}^* \frac{\partial T}{\partial z}(E \times \dot{z})$$
$$= \ddot{z}^* T + (\dot{z}^* \times \dot{z}^*)\frac{\partial T}{dz^*} = 0 ,$$

hence we have a differential equations of geodesic (see [6], [13])

$$\text{(3.2)} \qquad \begin{aligned} &\ddot{z} + T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z}(\dot{z} \times \dot{z}) = 0 , \\ &\ddot{\bar{z}} + \bar{T}_D^{-1}(\bar{z}, z)\frac{\partial \bar{T}_D(\bar{z}, z)}{\partial \bar{z}}(\dot{\bar{z}} \times \dot{\bar{z}}) = 0 . \end{aligned}$$

Consequently, the Christoffel symbol is expressed as

$$\text{(3.3)} \qquad \begin{aligned} T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z} &= \begin{pmatrix} \Gamma^1_{11}, \cdots, \Gamma^1_{1n}, \Gamma^1_{21}, \cdots, \Gamma^1_{nn} \\ \cdots \\ \Gamma^n_{11}, \cdots, \Gamma^n_{1n}, \Gamma^n_{21}, \cdots, \Gamma^n_{nn} \end{pmatrix} \\ \bar{T}_D^{-1}(\bar{z}, z)\frac{\partial \bar{T}_D(\bar{z}, z)}{\partial \bar{z}} &= \begin{pmatrix} \Gamma^{\bar{1}}_{\bar{1}\bar{1}}, \cdots, \Gamma^{\bar{1}}_{\bar{1}\bar{n}}, \Gamma^{\bar{1}}_{\bar{2}\bar{1}}, \cdots, \Gamma^{\bar{1}}_{\bar{n}\bar{n}} \\ \cdots \\ \Gamma^{\bar{n}}_{\bar{1}\bar{1}}, \cdots, \Gamma^{\bar{n}}_{\bar{1}\bar{n}}, \Gamma^{\bar{n}}_{\bar{2}\bar{1}}, \cdots, \Gamma^{\bar{n}}_{\bar{n}\bar{n}} \end{pmatrix} . \end{aligned}$$

(See [4], [10], [13]).

Now, for any pseudo-conformal mapping $\zeta = \zeta(z)$, we can calculate as follows by virtue of the above mentioned formulas (1.4)~(1.6):

$$T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z}$$
$$= \left(\frac{d\zeta}{dz}\right)^{-1} T_\Delta^{-1}(\bar{\zeta}, \zeta)\left(\frac{\partial T_\Delta(\bar{\zeta}, \zeta)}{\partial \zeta}\left(\frac{d\zeta}{dz}\right)^2 + T_\Delta(\bar{\zeta}, \zeta)\frac{d^2\zeta}{dz^2}\right)$$
$$= \left(\frac{d\zeta}{dz}\right)^{-1} T_\Delta^{-1}(\bar{\zeta}, \zeta)\frac{\partial T_\Delta(\bar{\zeta}, \zeta)}{\partial \zeta}\left(\frac{d\zeta}{dz}\right)^2 + \left(\frac{d\zeta}{dz}\right)^{-1}\frac{d^2\zeta}{dz^2} . \qquad \text{(See [2]).}$$

LEMMA 3.1. *For any pseudo-conformal mapping, we obtain the following relations with respect to the Christoffel symbol*:

$$\text{(3.4)} \qquad \frac{d^2\zeta}{dz^2} = \frac{d\zeta}{dz}\left(T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z}\right) - \left(T_\Delta^{-1}(\bar{\zeta}, \zeta)\frac{\partial T_\Delta(\bar{\zeta}, \zeta)}{\partial \zeta}\right)\left(\frac{d\zeta}{dz}\right)^2,$$

$$\text{(3.4}') \qquad \frac{d^2 z}{d\zeta^2} = \frac{dz}{d\zeta}\left(T_\Delta^{-1}(\bar{\zeta}, \zeta)\frac{\partial T_\Delta(\bar{\zeta}, \zeta)}{\partial \zeta}\right) - \left(T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z}\right)\left(\frac{dz}{d\zeta}\right)^2 .$$

THEOREM 3.1. *The vector* $\varepsilon_D(\bar{z}, z) \equiv \ddot{z} + T_D^{-1}(\bar{z}, z)(\partial T_D(\bar{z}, z)/\partial z)(\dot{z} \times \dot{z})$

is a contravariant vector, and a geodesic curve in D is also a geodesic curve in Δ under any pseudo-conformal mapping $\Delta = \zeta(D)$.

Proof. We have

$$\dot{z} = \frac{dz}{ds} = \frac{dz}{d\zeta}\dot{\zeta} ,$$
$$\ddot{z} = \frac{d^2z}{ds^2} = \frac{d}{ds}\Big(\frac{dz}{d\zeta}\dot{\zeta}\Big) = \frac{d^2z}{d\zeta^2}(\dot{\zeta} \times \dot{\zeta}) + \frac{dz}{d\zeta}\ddot{\zeta} ,$$

hence, substituting (3.4′) into this formula, we have

$$\ddot{z} = \frac{dz}{d\zeta}\Big(T_{\Delta}^{-1}(\bar{\zeta}, \zeta)\frac{\partial T_{\Delta}(\bar{\zeta}, \zeta)}{\partial\zeta}\Big)(\dot{\zeta} \times \dot{\zeta}) - \Big(T_{D}^{-1}(\bar{z}, z)\frac{\partial T_{D}(\bar{z}, z)}{\partial z}\Big)(\dot{z} \times \dot{z}) + \frac{dz}{d\zeta}\ddot{\zeta} .$$

Therefore

$$\varepsilon_D(\bar{z}, z) = \frac{dz}{d\zeta}\Big(T_{\Delta}^{-1}(\bar{\zeta}, \zeta)\frac{\partial T_{\Delta}(\bar{\zeta}, \zeta)}{\partial\zeta}\Big)(\dot{\zeta} \times \dot{\zeta}) + \frac{dz}{d\zeta}\ddot{\zeta} = \frac{dz}{d\zeta}\varepsilon_{\Delta}(\bar{\zeta}, \zeta) .$$

Hence, $\varepsilon_D(\bar{z}, z)$ is a contravariant vector, and $\varepsilon_D(\bar{z}, z) = 0$ implies $\varepsilon_{\Delta}(\bar{\zeta}, \zeta) = 0$.

Next, we consider a contravariant vector $\begin{pmatrix}\lambda_D\\ \bar{\lambda}_D\end{pmatrix}'$ which satisfies the following transformation law: $\lambda_{\Delta} = (d\zeta/dz)\lambda_D$, $(\bar{\lambda}_{\Delta} = (d\bar{\zeta}/d\bar{z})\bar{\lambda}_D)$. Then we have

$$d\lambda_{\Delta} = \frac{d^2\zeta}{dz^2}(dz \times \lambda_D) + \frac{d\zeta}{dz}\frac{\partial\lambda_D}{\partial z}dz + \frac{d}{dz}\frac{\partial\lambda_D}{\partial\bar{z}}d\bar{z} = \frac{d^2\zeta}{dz^2}(dz \times \lambda_D) + \frac{d\zeta}{dz}(d\lambda_D) .$$

Substituting (3.4) for $(d^2\zeta/dz^2)$, we obtain

$$d\lambda_{\Delta} = \frac{d\zeta}{dz}T_{D}^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z}(dz \times \lambda_D) - T_{\Delta}^{-1}(\bar{\zeta}, \zeta)\frac{\partial T_{\Delta}(\bar{\zeta}, \zeta)}{\partial\zeta}(d\zeta \times \lambda_{\Delta}) + \frac{d\zeta}{dz}d\lambda_D ,$$

therefore we have the transformation expression of the covariant differential:

$$\delta\lambda_{\Delta} \equiv d\lambda_{\Delta} + T_{\Delta}^{-1}(\bar{\zeta}, \zeta)\frac{\partial T_{\Delta}(\bar{\zeta}, \zeta)}{\partial\zeta}(d\zeta \times \lambda_{\Delta}) = \frac{d\zeta}{dz}\Big(d\lambda_D + T_{D}^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z}(dz \times \lambda_D)\Big) = \frac{d\zeta}{dz}(\delta\lambda_D) , \tag{3.5}$$

$$(3.5)' \qquad \delta\bar{\lambda}_\Delta \equiv d\bar{\lambda}_\Delta + \bar{T}_\Delta^{-1}(\bar{\zeta}, \zeta)\frac{\partial \bar{T}_\Delta(\bar{\zeta}, \zeta)}{\partial \bar{\zeta}}(d\bar{\zeta} \times \bar{\lambda}_\Delta) = \frac{d\bar{\zeta}}{d\bar{z}}(\delta\bar{\lambda}_D) ,$$

and the *covariant derivative* of a vector $\begin{pmatrix}\lambda_D \\ \bar{\lambda}_D\end{pmatrix}'$ is given by

$$(3.6) \qquad \nabla\lambda_D \equiv \left(\frac{\partial\lambda_D}{\partial z} + T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z}(E \times \lambda_D), \frac{\partial\lambda_D}{\partial \bar{z}}\right) ,$$

$$(3.6)' \qquad \nabla\bar{\lambda}_D \equiv \left(\frac{\partial\bar{\lambda}_D}{\partial z}, \frac{\partial\bar{\lambda}_D}{d\bar{z}} + T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z}(E \times \bar{\lambda}_D)\right) .$$

Now, we have the conditions of the parallel displacement

$$\frac{\delta\lambda_D}{ds} \equiv \frac{d\lambda_D}{ds} + T^{-1}\frac{\partial T}{\partial z}\left(\frac{dz}{ds} \times \lambda_D\right) = 0 ,$$
$$\frac{\delta\bar{\lambda}_D}{ds} \equiv \frac{d\bar{\lambda}_D}{ds} + \bar{T}^{-1}\frac{\partial \bar{T}}{\partial \bar{z}}\left(\frac{d\bar{z}}{ds} \times \bar{\lambda}_D\right) = 0 ,$$

for a contravariant vector $\begin{pmatrix}\lambda_D \\ \bar{\lambda}_D\end{pmatrix}$ on a curve, then substituting the tangent $\begin{pmatrix}\dot{z} \\ \dot{\bar{z}}\end{pmatrix}'$ of a curve for $\begin{pmatrix}\lambda_D \\ \bar{\lambda}_D\end{pmatrix}'$ we obtain a differential equation of geodesic (3.2). Therefore, a curve on which the tangent is displaced parallelly is a geodesic.

THEOREM 3.2. *At the center t_0 of any representative domain D, the Christoffel symbols with respect to the metric $ds_D^2 = dz^* T_D(\bar{z}, z)dz$ are all zero.*

Proof. A necessary and sufficient condition for a domain D to be a representative domain with t_0 as center is $T_D^{-1}T_D(\bar{t}_0, z) = E_n$, therefore $T_D^{-1}(\partial T_D/\partial z) = 0$.

THEOREM 3.3. *The Christoffel symbols at any point t_0 in a bounded domain with Kaehler matric $ds_D^2 = dz^* T_D(\bar{z}, z)dz$ become all zero by the Bergman representative function with respect to t_0*

$$(3.7) \qquad \zeta(z) \equiv T_D^{-1}\int_{t_0}^{z} T_D(\bar{t}_0, z)dz + t_0 .$$

Proof. Substituting $(d\zeta(t_0)/dz) = E$, $(d^2\zeta(t_0)/dz^2) = T_D^{-1}(\partial T_D/\partial z)$ into (3.4), we have $T_D^{-1}(\partial T_D/\partial z) = T_D^{-1}(\partial T_D/\partial z) - T_\Delta^{-1}(\partial T_\Delta/d\zeta)$, therefore $T_\Delta^{-1}(\partial T_\Delta/\partial\zeta) = 0$.

THEOREM 3.4. *Let t_0 be an arbitrary point in D which is bounded domain with the Kaehler metric $ds_D^2 = dz^* T_D(\bar{z}, z)dz$, and let the Bergman representative function with respect to t_0 be*

$$\overset{(1)}{\zeta_D^2}(z; t_0) \equiv T_D^{-1}\int_{t_0}^{z} T_D(\bar{t}_0, z)dz . \qquad \text{(See [1], [2])}.$$

Then the point t_0 *lies on geodesic* (3.2) *if and only if*

$$d^2\overset{(1)}{\zeta}{}^2_D(t_0;\, t_0)/ds^2 = 0 \, .$$

Proof. From Theorem 3.1 and Theorem 3.2, we have

$$d^2\overset{(1)}{\zeta}{}^2_D(t_0;\, t_0)/ds^2 = \varepsilon_D(\bar{z}, z) \, .$$

THEOREM 3.5. *The Christoffel symbols at the center* c_0 *of any* m*-representative domain* $\Delta(m \geqq 2$, *see* [5], [2]) *with respect to* $t_0 \varepsilon D$ *are equal to that at the point* t_0.

Proof. For any m-representative function $\zeta(z)$ with respect to t_0, we have $d\zeta(t_0)/dz = E$, $d^2\zeta(t_0)/dz^2 = 0$. Hence, by (3.4), we obtain

$$T_D(\bar{t}_0, t_0)\frac{\partial T_D(\bar{t}_0, t_0)}{\partial z} = T_\Delta(\bar{c}_0, c_0)\frac{\partial T_\Delta(\bar{c}_0, c_0)}{\partial \zeta} \, .$$

THEOREM 3.6 *At the center* t_0 *of a minimal domain of moment of inertia, if* $\partial K_D(\bar{t}_0, t_0)/\partial z = 0$, *then the Christoffel symbols are all zero.*

Proof. From Theorem 2.1, we have

$$\frac{d^2}{dz^2}\left(K_D(\bar{t}_0, z)\int_{t_0}^{z} T_D(\bar{t}_0, z)dz\right)_{z=t_0}$$
$$= K_D\frac{\partial T_\Delta}{\partial z} + K_z \times T_D + T_D \times K_z = 0 \, ,$$

therefore, $\partial T_D/\partial z = 0$.

REMARK. By theorem 3.4, we may locate the geodesic through a point t_0, that is, doing coordinate transformation

$$\zeta(z) = T_D^{-1}\int_{t_0}^{z} T_D(\bar{t}_0, z)dz + t_0$$

at t_0, the curve through the point t_0 on which $d^2\zeta(t_0)ds^2 = 0$ is geodesic.

Next, according to our method we express *Riemann-Christoffel tensor* as

$$\frac{\partial}{\partial z^*}\left(T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}. z)}{\partial z}\right) = (E \times T_D^{-1}(\bar{z}, z))_2 T_D(\bar{z}, z)$$

(3.8)

$$= \begin{pmatrix} R^1_{11\bar{1}} \cdots R^1_{nn\bar{1}} \\ \cdots \\ R^n_{11\bar{n}} \cdots R^n_{nn\bar{n}} \end{pmatrix} = -\begin{pmatrix} R^1_{1\bar{1}1} \cdots R^1_{n\bar{1}n} \\ \cdots \\ R^n_{1\bar{n}1} \cdots R^n_{n\bar{n}n} \end{pmatrix} .$$

For any pseudo-conformal mapping $\zeta = \zeta(z)$, we have

$$\begin{aligned}(3.9)\qquad &\frac{\partial}{\partial z^*}\Big(T_D^{-1}(\bar z, z)\frac{\partial T_D(\bar z, z)}{\partial z}\Big)\\ &= \Big(\Big(\frac{d\zeta}{dz}\Big)^* \times \frac{dz}{d\zeta}\Big)\frac{\partial}{\partial \zeta^*}\Big(T_\Delta^{-1}(\bar\zeta, \zeta)\frac{\partial T_\Delta(\bar\zeta, \zeta)}{\partial \zeta}\Big)\Big(\frac{d\zeta}{dz}\Big)^2 ,\end{aligned}$$

therefore it is a tensor of contravariant degree 1 and covariant degree 3.

Further, we can express the *curvature tensor* as

$$\begin{aligned}(3.10)\qquad &\frac{1}{2}(E \times T_D(\bar z, z))\frac{\partial}{\partial z^*}\Big(T_D^{-1}(\bar z, z)\frac{\partial T_D(\bar z, z)}{\partial z}\Big) = \frac{1}{2}{}_2T_D(\bar z, z)\\ &\equiv \begin{pmatrix} R_{\bar 1 1 1 \bar 1}, & \cdots, & R_{\bar 1 n n \bar 1} \\ & \cdots & \\ R_{\bar n 1 1 \bar n}, & \cdots, & R_{\bar n n n \bar n}\end{pmatrix} = -\begin{pmatrix} R_{\bar 1 1 \bar 1 1}, & \cdots, & R_{\bar 1 n \bar 1 n} \\ & \cdots & \\ R_{\bar n 1 \bar n 1}, & \cdots, & R_{\bar n n \bar n n}\end{pmatrix} .\end{aligned}$$

And, we can express the contracted Christoffel symbols as

$$(3.11)\qquad \Big(SpT^{-1}\frac{\partial T}{\partial z_1}, \cdots, SpT^{-1}\frac{\partial T}{\partial z_n}\Big) \equiv (\Gamma^{\alpha}_{1\alpha}, \cdots, \Gamma^{\alpha}_{n\alpha}) .$$

By the rule $\partial \log (\det T)/\partial z_i = SpT^{-1}(\partial T/\partial z_i)$, we obtain *Ricci tensor*

$$(3.12)\qquad -\partial^2 \log (\det T)/\partial z^* \partial z \equiv \begin{pmatrix} R_{1\bar 1}, & \cdots, & R_{n\bar 1} \\ & \cdots & \\ R_{1\bar n}, & \cdots, & R_{n\bar n}\end{pmatrix} .$$

Therefore, the *scalar curvature* becomes

$$(3.13)\qquad R_0 = -4Sp\Big(T_D^{-1}(\bar z, z)\frac{\partial^2 \log (\det T_D(\bar z, z))}{\partial z^* \partial z}\Big) ,$$

which is invariant under any pseudo-conformal mapping.

THEOREM 3.7. *At any bounded domain D, $R_0 < 4n(n+1)$.*

Proof. It is known that both $M \equiv (n+1)T + (\partial^2 \log (\det T)/\partial z^* \partial z)$ and T^{-1} are positive definite Hermitian matrices (see [1], [3]), therefore

$$\frac{1}{\lambda_1}(\Sigma \rho_j) , \quad \text{or} \quad \rho_n\Big(\Sigma\frac{1}{\lambda_j}\Big) \leqq Sp(T^{-1}M) \leqq \frac{1}{\lambda_n}(\Sigma \rho_j) , \quad \text{or} \quad \rho_1\Big(\Sigma\frac{1}{\lambda_j}\Big) ,$$

where $\lambda_1 \geqq \cdots \geqq \lambda_n > 0$ and $\rho_1 \geqq \cdots \geqq \rho_n > 0$ are eigenvalues of T and M, respectively. Thus we have

$$n(n+1) - \rho_1 SpT^{-1} ,$$

or

$$n(n+1) - \frac{1}{\lambda_n}SpM \leqq \frac{R_0}{4} \leqq n(n+1) - \rho_n SpT^{-1} ,$$

or

$$n(n+1) - \frac{SpM}{\lambda_1} .$$

THEOREM 3.8. *Let D be a homogeneous domain, then we have always*

$$R_0 = -4n . \tag{3.14}$$

Proof. At the homogeneous domain, it becomes

$$\frac{\partial^2 \log (\det T)}{\partial z^* \partial z} = T ,$$

therefore we have $R_0 = -4Sp(T^{-1}T) = -4n$.

THEOREM 3.9. *In a manifold D with the metric $ds_D^2 = dz^* T_D(\bar{z}, z) dz$, if there exists a fixed point t_0 in D such that $I_D(\bar{z}, z) \underset{(\geqq)}{\leqq} I_D(\bar{t}_0, t_0)$ everywhere in D, and if $-4n \underset{(\geqq)}{\leqq} R_0$, then we must have $I_D(\bar{z}, z) = I_D(\bar{t}, {}_0t_0)$ everywhere in D, and consequently we have $R_0 = -4n$, where $I_D(\bar{z}, z)$ is a real valued (invariant) function defined by $I_D(\bar{z}, z) \equiv K_D(\bar{z}, z)/\det T_D(\bar{z}, z)$.*

Proof. From $T = \partial^2 \log I/\partial z^* \partial z + \partial^2 \log (\det T)/\partial z^* \partial z$, we obtain

$$n = Sp\left(T^{-1} \frac{\partial^2 \log I}{\partial z^* \partial z}\right) - \frac{R_0}{4} .$$

Therefore, by Theorem of E. Hopf (see [13]), our proof is completed.

THEOREM 3.10. *In a bounded domain D, if there exists a fixed point t_0 in D such that $J_D(\bar{z}, z) \leqq J_D(\bar{t}_0, t_0)$ everywhere in D, then we must have $J_D(\bar{z}, z) = J_D(\bar{t}_0, t_0)$ everywhere in D, and consequently $R_0 = 4n(n+1)$, where $J_D(\bar{z}, z) \equiv (K_D(\bar{z}, z))^{n+1} \det T_D(\bar{z}, z)$.*

Proof. From $(n+1)T + \partial^2 \log (\det T)/\partial z^* \partial z = \partial^2 \log J/\partial z^* \partial z$, we obtain

$$(n+1)n - \frac{R_0}{4} = SpT^{-1} \frac{\partial^2 \log J}{\partial z^* \partial z} .$$

Since, by Theorem 3.7, we have $SpT^{-1}(\partial^2 \log J/\partial z^* \partial z) > 0$ everywhere in D, then J is constant by theorem of E. Hopf. Consequently we obtain the following Ricci tensor: $(R_{\alpha\bar{\beta}}) = (n+1) T_D(\bar{z}, z)$. Thus we have $R_0 = 4n(n+1)$.

Next, a *holomorphic sectional curvature* $\kappa(z; u)$ with respect to a contravariant vector u which is invariant under any pseudo-conformal mapping is expressed by our method as follows:

$$\kappa(z; u) = -2\frac{(u \times u)^* {}_2T_D(\bar{z}, z)(u \times u)}{(u \times u)^*(T_D(\bar{z}, z) \times T_D(\bar{z}, z))(u \times u)} . \tag{3.15}$$

THEOREM 3.11. *If D is a homogenous domain with the metric $ds_D^2 = dz^* T dz$, then the holomorphic sectional curvature $\kappa(z; u)$ is constant everywhere in D.*

Proof. Since $\kappa(z; u)$ is invariant, then for arbitrary points z, t in D we have $\kappa(z; u) = \kappa(t; u)$ by a suitable holomorphic automorphism.

THEOREM 3.12. *In a manifold of constant holomorphic curvature κ, for the scalar curvature R_0, we have*

$$R_0 = n(n + 1)\kappa . \tag{3.16}$$

Proof. By the hypothesis, the culvature tensor becomes

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{\kappa}{2}(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}) , \tag{3.17}$$ (see [13])

consequently we have $R_{\alpha\bar{\beta}} = (n + 1)/2 \cdot \kappa g_{\alpha\bar{\beta}}$. Thus we have

$$R_0 = 2g^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}} = n(n + 1)\kappa .$$

COROLLARY 3.1. *The unit hypersphere $|z|^2 < 1$ is a manifold of constant holomorphic curvature κ and we have $\kappa = -4/(n + 1)$.* (See Theorem 4 in [10]).

Proof. Using the formulas (1.3)$\sim$(1.6), we obtain

$$T_D(0, 0) = (n + 1)E , \qquad \partial T_D(0, 0)/\partial z = 0 ,$$
$$\partial^2 T_D(0, 0)/\partial z^* \partial z = (n + 1)(E^2 + \tilde{E}_{nn}) .$$

Then we have

$${}_2T_D(0, 0) = (n + 1)(E^2 + \tilde{E}_{nn}) ,$$

and consequently $\kappa(0; u) = -4(n + 1)(u^*u)^2/(n + 1)^2(u^*u)^2 = -4/(n + 1)$. Therefore, the holomorphic sectional curvature are all the same at origin. Consequently, by Theorem 3.11, we obtain the required results.

REMARK. Since a unit hypersphere is a homogeneous domain, then $R_0 = -4n$. Therefore, by Theorem 3.12, we can compute $\kappa = -4n/n(n + 1) = -4(/n + 1)$.

COROLLARY 3.2. *In a polydisc* $\{|z_j| < r_j, j = 1, \cdots, n\}$, *for the holomorphic sectional curvature* κ, *we have* $-2 \leqq \kappa \leqq -2/n$. (See [10]).

Proof. We may calculate at origin as follows:

$$T_D(0, 0) = 2\begin{pmatrix} \left(\frac{1}{r_1}\right)^2 & & 0 \\ & \ddots & \\ 0 & & \left(\frac{1}{r_n}\right)^2 \end{pmatrix}, \quad {}_2T_D(0, 0) = 4\begin{pmatrix} \left(\frac{1}{r_1}\right)^4 0 & & & 0 \\ & \ddots & & \\ & 0\left(\frac{1}{r_2}\right)^4 0 & & \\ & & \ddots & \\ 0 & & & \left(\frac{1}{r_n}\right)^4 \end{pmatrix}.$$

Thus, we have

$$\kappa = -2\sum_j(|u_j|/r_j)^4/(\sum_j(|u_j|/r_j)^2)^2 ,$$

and consequently $-2 \leqq \kappa \leqq -2/n$, $(n \geqq 2)$.

COROLLARY 3.3. *In a complex spheres*

$$\mathfrak{M}_{(n)} = \{z \mid |z'z| < 1, 1 - 2|z|^2 + |z'z|^2 > 0\} ,$$

for the holomorphic sectional curvature κ, *we have*

$$-\frac{2}{n}\left(2 - \frac{1}{n}\right) < \kappa < -\frac{2}{n} .$$

Proof. Since we have

$$T_D(\bar{z}, z) = \frac{2n}{K_0^2}[K_0(E - 2\bar{z}z') + 2(E - \bar{z}z')zz^*(E - \bar{z}z')]$$

where $K_0 = 1 - 2|z|^2 + |z'z|^2$, in the complex spheres (see [8]), then we have

$$T_D(0, 0) = 2nE ,$$

$${}_2T_D(\bar{0}, 0) = 4n\left[E^2 - \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 & \\ & \ddots & \\ & & 0 \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} 0 & 1 & \\ & \ddots & \\ & & 0 \end{matrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{matrix} 0 & & \\ & \ddots & \\ & 0 & 1 \end{matrix}} \end{pmatrix} + \tilde{E}_{nn}\right]$$

Consequently,

$$\kappa(0; u) = -2\frac{4n[2(u^*u)^2 - (|u_1|^4 + \cdots + |n_n|^4)]}{4n^2(u^*u)^2}$$
$$= \frac{-2}{n}\left[2 - \frac{|u_1|^4 + \cdots + |u_n|^4}{|u|^4}\right]$$

where $u = (u_1, \cdots, u_n)'$, hence we have the required result.

It is known that the holomorphic sectional curvature for a bounded domain in C^n is less than 2 ([1], [3]), therefore we have

COROLLARY 3.4. *Let D be a bounded domain with Kaehler metric $ds_D^2 = dz^*Tdz$, then*

$$(3.18)\qquad \begin{aligned} {}_1\psi_D(\bar{z}, z) &\equiv \frac{1}{K}\left[\frac{\partial^2(KT)}{\partial z^*\partial z} - \frac{\partial(KT)}{\partial z^*}(KT)^{-1}\frac{\partial(KT)}{\partial z}\right] \\ &= {}_2T_D(\bar{z}, z) + T_D(\bar{z}, z) \times T_D(\bar{z}, z) \end{aligned}$$

is relative invariant under any pseudo-conformal mapping and positive definite.

Proof. From $\kappa < 2$, we have $(u \times u)^*({}_2T + T \times T)(u \times u) > 0$.

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Received June 9, 1970.

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