PACIFIC JOURNAL OF MATHEMATICS
Vol. 38, No. 3, 1971

MULTIPLIERS OF QUOTIENTS OF L_1

I. GLICKSBERG AND I. WIK

Let G be a locally compact abelian group with dual Γ. The multiplier problem for $L_1(G)$ has a well known and easy solution, while the corresponding problem for its ideals is subtle. So far as we have been able to determine, the problem for quotient algebras of $L_1(G)$ has not received attention. The purpose of this note is to point that out and to give a condition on the quotient, which ensures that the simplest possible answer holds, and an example, which shows that in general that answer is false.

1. The multiplier problem for ideals is treated in [2]. We denote by $M(G)$ the finite, regular, complex valued measures on G, by $A(\Gamma)$ the Fourier transforms of functions in $L_1(G)$, by $B(\Gamma)$ the Fourier-Stieltjes transforms of measures in $M(G)$ and by $f \mid E$ the restriction of a function f to the set E. Finally $A(E) = A(\Gamma) \mid E$ and $B(E) = B(\Gamma) \mid E$. The problem can be stated in a somewhat greater generality as follows:

Let E be a subset of Γ and φ a (necessarily bounded and continuous) function on E, for which

$$\varphi A(E) \subset A(E) . \tag{1.1}$$

Under what conditions on E does this imply that $\varphi \in B(E)$? If Γ is compact $\hat{f} \equiv 1$ belongs to $A(\Gamma)$ and it follows that (1.1) implies $\varphi \in A(E)(=B(E))$ for every E. In the present section we shall see that if E is sufficiently nice near ∞, all such φ are restrictions of Fourier-Stieltjes transforms, and in the second section we shall characterize the sets E, such that every bounded continuous function on E is a multiplier of $A(E)$. This gives an easy proof that any noncompact I-group contains a set E for which some ϕ satisfying (1.1) is not contained in $B(E)$.

First, however, we make explicit the connection between this problem and that of multipliers of quotients of $L_1(G)$.

If I is any closed ideal in $L_1(G)$ with hull $E \subset \Gamma$, a *multiplier* of $L_1(G)/I$ is a bounded operator T on $L_1(G)/I$ which commutes with translation.

We denote by kE the set $\{f \in L_1(G): \hat{f} = 0, \text{ on } E\}$, which is the largest ideal with hull E (or E^- if E is not assumed closed). $\bar{f}$ will denote the coset $f + kE$. A multiplier of $L_1(G)/I$ then satisfies $T(\bar{f} * \bar{g}) = \bar{f} * T\bar{g} = T\bar{f} * \bar{g}$, for $\bar{f}$ and $\bar{g}$ in $L_1(G)/I$. In terms of Fourier transforms this says that in E, $(T\bar{f})^\wedge/\hat{f} = (T\bar{g})^\wedge/\hat{g}$ near points γ in E

where $(\bar{f})^\wedge$ and $(\bar{g})^\wedge$ are nonzero. Defining φ locally on E as this common ratio yields a continuous function satisfying (1.1). Conversely any such φ is easily seen to define a multiplier in $L_1(G)/kE$.

Our problem is to find necessary and sufficient conditions on E insuring that (1.1) implies $\varphi \in B(E)$. At present we can only give a sufficient condition, the real content of which is in no way clear and mainly arises as just the requirement for our proof to apply.

We pair L_1 and L_∞ via $\langle f, h\rangle = f*h(0)$ and denote by $(kE)^\perp$ the subspace of $L_\infty(G)$ orthogonal to kE. This is precisely the w^* closed span of E (and of $\bar{E}$) in $L_\infty(G)$. With this terminology our positive result is the following.

THEOREM 1.1. *If φ satisfies* (1.1) *while*

$$E\backslash(C_0(G) \cap (kE)^\perp)^- \tag{1.2}$$

has compact closure in Γ, then $\varphi \in B(E)$. (The bar denotes w^* closure.)

COROLLARY 1.2. *The conclusion of the theorem holds if E coincides off some compact set with a set F*

(a) *such that $C_0 \cap k(V \cap F)^\perp \neq \{0\}$ for every open V with $V \cap F \neq \emptyset$*, or

(b) *which is a subset of a discrete subgroup of Γ.*

Our corollary merely exhibits a few situations in which our less than transparent hypothesis in 1.1. obtains. In particular (a) is satisfied if E, off some compact set, locally carries measures with transforms in $C_0(G)$. This is the case if E, off some compact set, is open or locally of positive measure or is the support of a measure with transform $C_0(G)$.

The proof of 1.1 itself is simple enough. The operator

$$T\colon L_1(G)/kE \longrightarrow L_1(G)/kE$$

defined by

$$T(f + kE) = g + kE \text{ iff } \hat{g} = \varphi\hat{f} \text{ on } E\,,$$

is bounded by the closed graph theorem and commutes with translations. Thus the adjoint T^* is a bounded map, of $(kE)^\perp = (L_1(G)/kE)^*$ into itself, which also commutes with translations. Consequently, for $g \in C_0(G) \cap (kE)^\perp$ we have

$$\| T^*g - R_xT^*g\|_\infty = \|T^*(g - R_xg)\|_\infty \leqq \|T^*\|\,\|g - R_xg\|_\infty\,.$$

(R_x is translation by x.) This shows that T^*g is (essentially) continuous and $g \rightarrow T^*g(0)$ is a bounded linear functional on a subspace of $C_0(G)$. By the Hahn-Banach and Riesz theorems we have a measure μ in $M(G)$ with

$$T^*g(0) = \mu*g(0),\ g \in C_0(G) \cap (kE)^\perp .$$

Replacing g by $R_x g$ of course yields

$$T^*g(x) = \mu*g(x) \text{ for all } x \in G$$

and $T^*g = \mu*g$. But now for $f \in L_1(G)$ we have

$$\langle Tf - \mu*f, g\rangle = f*T^*g(0) - f*\mu*g(0) = 0$$

for all $g \in C_0(G) \cap (kE)^\perp$, so that $Tf - \mu*f$ is orthogonal to $(C_0(G) \cap (kE)^\perp)^-$, and in particular its Fourier transform $\varphi\hat{f} - \hat{\mu}\hat{f}$ vanishes on the part of Γ lying in that span, for all $f \in L_1(G)$, so $\varphi = \hat{\mu}$ on that subset of Γ. Thus $\varphi - \hat{\mu}|E$ is supported by (1.2) and by our hypothesis there is an $\hat{f} \in L_1(G)$, with $\hat{f} \equiv 1$ on (1.2). Since $\varphi\hat{f}|E \in A(E)$ and $\hat{\mu}\hat{f}|E \in A(E)$ we have an $h \in L_1(G)$, for which $\varphi - \hat{\mu} = (\varphi - \hat{\mu})\hat{f} = \hat{h}$ on E and our desired measure ν is simply the sum of μ and the measure corresponding to h.

To prove part (a) of the corollary we assume that there are points of F that do not belong to $(C_0(G) \cap (kE)^\perp)^-$. That is, there is a $\gamma_0 \in F$ and a function f in $L_1(G)$, orthogonal to $C_0(G) \cap (kE)^\perp$, but with $\hat{f}(\gamma_0) \neq 0$. Then $|\hat{f}|$ is bounded away from zero on some compact neighborhood V of γ_0 and there is a function $h \in L_1$ such that $\hat{h} = 1/\hat{f}$ near V. By assumption there is a nonzero function $g \in C_0(G) \cap (k(F \cap V))^\perp$. Since $[f*L_1 \perp C_0 \cap (kF)^\perp \supset C_0 \cap k(F \cap V)^\perp$,

$$0 = \langle f*L_1, g\rangle = \langle f*h*L_1, g\rangle = \langle L_1, f*h*g\rangle = \langle L_1, g\rangle .$$

This implies $g = 0$, a contradiction which proves part (a).

To prove part (b) of the corollary we denote by Δ the discrete subgroup in question and by Λ its annihilator. If F lies in Δ then we have a measure μ on G/Λ, with $\hat{\mu} = \varphi$ on F, since F is an *open* subset of Δ. Any measure λ, which maps onto μ under the map induced by $G \to G/\Lambda$ then satisfies $\hat{\lambda} = \varphi$ on F. We obtain a measure ν with $\hat{\nu} = \varphi$ on E as before, since $E \backslash F$ has compact closure.

2. We are indebted to Professor Y. Katznelson for pointing out the following result, which can be used to obtain the promised example.

THEOREM 2.1. *Suppose E is closed. A necessary and sufficient condition for ϕ to be a multiplier of $A(E)$ is that $\|\phi\|_{A(E_0)}$ is uniformly bounded for all compact $E_0 \subset E$.*

Proof. If ϕ is a multiplier then there exists a constant k (by the closed graph theorem), such that

$$\|\phi f\|_{A(E)} \leqq k\|f\|_{A(E)} \text{ for every } f \in A(E) .$$

For f we can take a function $\equiv 1$ on E_0 and with norm less than 2. It follows that

$$\|\phi\|_{A(E)} \leqq 2k \text{ for every compact } E_0 \subset E .$$

To prove that the condition is sufficient we assume that $\|\phi\|_{A(E)} \leqq k$ for every compact $E_0 \subset E$. Let $f \in A(E)$ and $F \in A(\Gamma)$ with $F|E = f$. Write $F = \sum_1^\infty F_j$, where $F_j \in A(\Gamma), F_j$ has compact support σ_j, and $\sum_1^\infty \|F_j\|_{A(\Gamma)} < \infty$. Let Φ_j be in $A(\Gamma)$, $\|\Phi_j\| < 2k$ and $\Phi_j | E \cap \sigma_j = \phi| E \cap \sigma_j$. Now

$$\sum_1^\infty F_j \Phi_j \in A(\Gamma) \text{ and on } E$$

$$\sum_1^\infty F_j \Phi_j = \sum_1^\infty F_j \phi = \phi f .$$

Thus ϕ is a multiplier of $A(E)$.

The following theorem, which we state for closed sets E, can of course be modified to suit any E.

THEOREM 2.2. *The following three conditions on a closed set E are equivalent.*

(a) $C_0(E) = A(E)$.

(b) *Every bounded continuous on E is a multiplier of $A(E)$.*

(c) *There exists a number $\lambda > 0$, such that $f \in A(E_0)$ and*

$$\|f\|_{A(E_0)} \leqq \lambda \|f\|_{C(E_0)}$$

for every $f \in C_0$ and every compact $E_0 \subset E$.

Proof. We prove that (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).

That (a) $\Rightarrow$ (b) is obvious.

To prove that (b) $\Rightarrow$ (a) we note that (b) implies

$$C_0(E) \cdot A(E) \subset A(E) .$$

Since $C_0(\Gamma) \cdot A(\Gamma) = C_0(\Gamma)$ [1] we obtain the following inclusions

$$C_0(E) = C_0(E) A(E) \subset A(E) \subset C_0(E)$$

and thus $C_0(E) = A(E)$.

To prove that (a) $\Rightarrow$ (c) we use the fact that $\|f\|_C \leqq \|f\|_A$ to conclude from the open mapping theorem that there exists a constant λ such that

$$\|f\|_{A(E)} \leqq \lambda \|f\|_{C_0(E)} \text{ for every } f \in C_0(E) .$$

Given any $f \in C(E_0)$ we extend it to a function $f^* \in C_0$ with maximum on E_0. Then

$$\|f\|_{A(E_0)} \leqq \|f^*\|_{A(E)} \leqq \lambda \|f^*\|_{C_0(E)} = \lambda \|f\|_{C(E_0)}$$

and (c) obtains.

To prove that (c) $\Rightarrow$ (a) we use the fact that $A(E) = C_0(E)$ if and only if

$$\|\hat{\mu}\|_\infty \geqq \lambda^{-1} \|\mu\|, \text{ for every } \mu \in M(E) . \tag{2.1}$$

[2. p. 141] Assumption (c) is the same as to say that (2.1) is valid (for one λ) for all compact $E_0 \subset E$. Obviously it is then also valid for E itself.

COROLLARY 2.3. *If E is an independent sequence $\{\gamma_n\}_1^\infty$, then every bounded function on E is a multiplier of $A(E)$.*

Proof. This is an immediate consequence of Theorem 2.1 or 2.2 since every compact subset of E is a Helson set and satisfies (c) for $\lambda = 1$.

Note that if $\phi \in B(E)$, then ϕ is uniformly continuous. Thus if E is an independent sequence $\{\gamma_n\}_1^\infty$ tending to infinity in R, then every bounded function on E is a multiplier, but if $\gamma_{n+1} - \gamma_n \rightarrow 0$ then only few of them will be in $B(E)$. We use this in the following theorem.

THEOREM 2.4. *Every noncompact I-group Γ contains a set E, such that $\phi A(E) \subset A(E)$ does not imply $\phi \in B(E)$.*

Proof. Let V be a neighbourhood of 0 in Γ. Since Γ is noncompact there exists a sequence of disjoint open sets $V_n = \gamma_n + V$, with the property that only a finite number of the sets intersect a given compact set. Furthermore, since Γ is an I-group it contains a closed metric subgroup Γ_1, (with metric d) which is also an I-group. We are interested in the sets $E_n = \gamma_n + V \cap \Gamma_1$., Our set E is constructed as the sequence obtained by taking two points γ_n' and γ_n'' in each E_n, such that $d(\gamma_n' - \gamma_n, \gamma_n'' - \gamma_n) < 1/n$ and $E = \cup_1^\infty \{\gamma_n', \gamma_n''\}$ is independent. The bounded continuous function ϕ defined on E by $\phi(\gamma_n') = 1, \phi(\gamma_n'') = 0$, $n = 1, 2, \cdots$, is a multiplier of $A(E)$ but does not belong to $B(E)$ since it is not uniformly continuous.

REMARK. As has been pointed out by Professor Katznelson, even the condition of uniform continuity is *not* sufficient. If $E = \{\gamma_n\}_{-\infty}^{\infty}$ and $|\gamma_n - n| \to 0$ and E is independent then every bounded function on E is a multiplier and uniformly continuous, but $B(E)$ consists of those functions ϕ for which there exists a measure μ on the circle such that $|\phi(\gamma_n) - \hat{\mu}(n)| \to 0$ as $n \to \infty$. But this of course cannot hold if $\lim_{n \to +\infty} \phi(\gamma_n) = 1$, $\lim_{n \to -\infty} \phi(\gamma_n) = -1$.

REFERENCES

1. P. C. Curtis, and A. Figa-Talamanca, *Factorization theorems for Banach algebras,* Function Algebras pp. 169-185 Atlanta: Scott Foresman and Co. 1966.
2. J. P. Kahane, and R. Salem, *Ensembles Parfaits et Séries Trigonometriques.* Actualités Sci. et Ind. 1301. Paris: Herman et Cie. 1963.
3. Y. Meyer, *Endomorphismes des ideaux fermés de* $L^1(G)$, Ann. Sci. Ecole Norm. Sup. 1. (1968). 499-580.

Received June 2, 1970. The first author was supported in part by the NSF. The second author is a Fellow of the American Scandinavian Foundation.

UNIVERSITY OF WASHINGTON