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CHARACTERIZATIONS OF UNIFORM CONVEXITY

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In this paper, three new characterizations of uniform convexity of a Banach space X are established. The characterization developed in Theorem 1 resembles the definition of the modulus of smoothness given by J. Lindenstrauss. The characterizations developed in Theorems 2 and 3 are interrelated, both involving the duality map of X into X^*. The methods used are adapted to give an abbreviated proof of a recent result of W. V. Petryshyn relating the strict convexity of X to the duality map of X into X^*.

The following definitions are included for reference. For a Banach space X, the *unit sphere of* X, denoted by S_1, is the set of all elements of X having norm 1. A Banach space X is *uniformly convex* if for each t in $(0, 2]$, $2\ \delta(t) = \inf \{2 - \|x + y\|\colon x, y \in S_1, \|x - y\| \geqq t\}$ is positive ([1], [2]) (the function δ is called the *modulus of convexity of* X). A direct consequence of this definition is that each of the following conditions is equivalent to X being uniformly convex:

(i) Whenever $\{a_n\}$ and $\{b_n\}$ are sequences in S_1 such that $\|a_n + b_n\| \to 2$, then $\|a_n - b_n\| \to 0$.

(ii) Whenever $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\|a_n\| \to 1$, $\|b_n\| \to 1$, and $\|a_n + b_n\| \to 2$, then $\|a_n - b_n\| \to 0$.

(see [3, p. 113] or [9, p. 109]). The *modulus of smoothness of* X is the function ρ such that for $t \geqq 0$,

$$2\ \rho(t) = \sup \{\|x + ty\| + \|x - ty\| - 2\colon x, y \in S_1\}$$

([5]). A Banach space X is *strictly convex* if for each x and y in S_1 such that $x \neq y$ and each λ in $(0, 1)$, $\|\lambda x + (1 - \lambda)y\| < 1$ ([1], [6]). A function $J\colon X \to 2^{X^*}$ is a *duality map of* X *into* X^* if for each x in X, $J(x) = \{w \in X^*\colon (w, x)(= w(x)) = \|w\|\,\|x\| \text{ and } \|w\| = \|x\|\}$ (see [6] for notation and a list of pertinent literature).

I would like to thank Professor Tosio Kato for suggesting the following formulation of Theorem 1.

THEOREM 1. *Let ϕ be a strictly convex and strictly increasing function on $[0, 2]$ such that $\phi(1) = 1$. Then X is uniformly convex if and only if for each t in $(0, 1]$, $\alpha(t) = \inf \{\phi(\|x + ty\|) + \phi(\|x - ty\|) - 2\colon x, y \in S_1\}$ is positive.*

Proof. Suppose that X is uniformly convex and that there is a t in $(0, 1]$ such that $\alpha(t) = 0$. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in S_1 such that if we let $a_n = x_n + ty_n$ and $b_n = x_n - ty_n$, then $\phi(||a_n||) + \phi(||b_n||) \to 2$. Since ϕ is convex and nondecreasing and $\phi(1) = 1$, $2 \leqq 2\phi((||a_n|| + ||b_n||)/2) \leqq \phi(||a_n||) + \phi(||b_n||) \to 2$ and thus by the strict convexity of ϕ, we have that $|\,||a_n|| - ||b_n||\,| \to 0$. The preceding inequality and the continuity of ϕ^{-1} at 1 imply that $||a_n|| + ||b_n|| \to 2$ and consequently that $||a_n|| \to 1$ and $||b_n|| \to 1$. For each n, $||a_n + b_n|| = 2$, so the uniform convexity of X implies that $2t = ||a_n - b_n|| \to 0$, which is contradictory.

Now suppose that $\alpha(t)$ is positive for each t in $(0, 1]$. For fixed x and y in S_1, the function $h(t) = \phi(||x + ty||) + \phi(||x - ty||) - 2$ is convex, and since $h(0) = 0$ and $h \geqq 0$ on $[0, 1]$, h is nondecreasing. Therefore, since α is the infimum of a collection of nondecreasing functions, α is nondecreasing on $[0, 1]$. By the definition of α, if $||y|| \leqq ||x|| \neq 0$, then $\phi(||x + y||/||x||) + \phi(||x - y||/||x||) - 2 \geqq \alpha(||y||/||x||)$. Thus if a and b are in S_1 and $||a - b|| \leqq ||a + b||$, we have that

$$(1) \qquad 2\phi(2/||a + b||) - 2 \geqq \alpha(||a - b||/||a + b||) \geqq \alpha(||a - b||/2)\ .$$

Now, let $\{a_n\}$ and $\{b_n\}$ be sequences in S_1 such that $||a_n + b_n|| \to 2$. We may assume that for sufficiently large n, $||a_n - b_n|| \leqq ||a_n + b_n||$. Thus inequality (1) and the continuity of ϕ at 1 imply that $\alpha(||a_n - b_n||/2) \to 0$, so $||a_n - b_n|| \to 0$ and X is uniformly convex.

Inequality (1) above gives a bound on the modulus of convexity, δ, in terms of ϕ^{-1} and α. By considering each of the cases $||a - b|| \leqq ||a + b||$, $1 \leqq ||a + b|| \leqq ||a - b||$, and $||a + b|| < 1$, it follows that $2\delta(||a - b||)$ is not less than the smaller of

$$1,\ ||a - b||\,\{\phi^{-1}(1 + 1/2\,\alpha(1/2)) - 1\}\ ,$$

and $||a - b||\,\{\phi^{-1}(1 + 1/2\,\alpha(||a - b||/2)) - 1\}$.

In Theorem 1, the case when $\phi(t) = t^2$ merits special attention. Note that for each Banach space X and each t in $[0, 1]$, $\alpha(t) \leqq 2t^2$; moreover, X is an inner product space if and only if $\alpha(t) = 2t^2$ for each t in $[0, 1]$. In the same vein, note that X *obeys a weak parallelogram law* (i.e., there is a λ in $(0, 1]$ such that for each x and y in X, $||x + y||^2 + \lambda\,||x - y||^2 \leqq 2\,||x||^2 + 2\,||y||^2$ — see [4]) if and only if there is a μ in $(0, 2]$ such that $\alpha(t) \geqq \mu t^2$ for each t in $[0, 1]$.

THEOREM 2. *A Banach space X is uniformly convex if and only if for each t in $(0, 2]$, $\beta(t) = \inf\ \{1 - (f, y): x, y \in S_1, ||x - y|| \geqq t, f \in J(x)\}$ is positive, where J is the duality map from X into X^*.*

Proof. If X is uniformly convex and $x, y \in S_1$ and $f \in J(x)$, then

$1-(f, y)=2-(f, x+y) \geqq 2-\|x+y\| \geqq 2 \delta(\|x-y\|)$.

Now suppose that $\beta>0$ on $(0,2]$ and that X is not uniformly convex. Then by the definition there exist sequences $\{x_n\}$ and $\{y_n\}$ in S_1 such that $0<\|x_n+y_n\| \rightarrow 2$ and for each n, $\|x_n-y_n\| \geqq t$. For each n, let $a_n=\|x_n+y_n\|^{-1}$, $z_n=a_n(x_n+y_n)$, $h_n \in J(z_n)$, $f_n \in J(x_n)$, and $g_n \in J(y_n)$. Then,

$$\begin{aligned} 2-\|x_n+y_n\| = 1-(h_n, x_n)+1-(h_n, y_n) &\geqq \beta(\|x_n-z_n\|) \\ &+\beta(\|y_n-z_n\|) . \end{aligned}$$

But neither $\|x_n-z_n\|$ nor $\|y_n-z_n\|$ is less than $t a_n-|1-2 a_n|$, so that for sufficiently large n, we have $\|x_n-z_n\| \geqq t / 4$, $\|y_n-z_n\| \geqq t / 4$, and $2-\|x_n+y_n\| \geqq 2 \beta(t / 4)$, which is contradictory.

THEOREM 3. *A Banach space X is uniformly convex if and only if the duality map J of X into X^* is* uniformly monotone-*in the sense that for each t in* $(0,2]$, $\gamma(t)=\inf \{(f-g, x-y): x, y \in S_1, \|x-y\| \geqq t$, $f \in J(x), g \in J(y)\}$ is positive.

Proof. If X is uniformly convex and $x, y \in S_1, f \in J(x), g \in J(y)$, then $(f-g, x-y)=2-(g, x+y)+2-(f, x+y) \geqq 2(2-\|x+y\|)$, so J is uniformly monotone.

Suppose J is uniformly monotone and X is not uniformly convex. By Theorem 2, $\beta(t)=0$ for some t in $(0,2]$; i.e., there exist sequences $\{x_n\}$ and $\{y_n\}$ in S_1 and $\{f_n\}$ in X^* such that for each n,

$$f_n \in J(x_n),\ \|x_n-y_n\| \geqq t ,$$

and $1-(f_n, y_n) \rightarrow 0$. Since $1-(f_n, y_n) \geqq 2-\|x_n+y_n\| \geqq 0$, then $\|x_n+y_n\| \rightarrow 2$ and we may assume that $\|x_n+y_n\|>0$ for each n. As in Theorem 2, let $a_n=\|x_n+y_n\|^{-1}$, $z_n=a_n(x_n+y_n)$, and $h_n \in J(z_n)$. Thus, $(h_n, x_n+y_n)=\|x_n+y_n\| \rightarrow 2$ and since $\|h_n\|=1=\|x_n\|=\|y_n\|$, then $(h_n, x_n) \rightarrow 1$. So,

$$(h_n-f_n, z_n-x_n)=1-a_n-a_n(f_n, y_n)+1-(h_n, x_n) \rightarrow 0 .$$

However, as in Theorem 2, for sufficiently large n, we have that $\|x_n-z_n\| \geqq t / 4$ and $(h_n-f_n, z_n-x_n) \geqq \gamma(t / 4)$, which is contradictory.

Now we turn to the previously mentioned result of Petryshyn [6, Theorem 1, p. 284-287]. We need the following theorem, proved in slightly different form in [8, Theorem, part iii]. We include a proof of it here for completeness. In the sequel, we shall use the following characterization of strict convexity due to Ruston [7]: A Banach space X is strictly convex if and only if for x and y in S_1 such that $x \neq y$, $2-\|x+y\|>0$.

Theorem (Torrance [8]). A Banach space X is strictly convex if and only if for x and y in S_1 such that $x \neq y$ and for f in $J(x)$, $1 - (f, y) > 0$.

Proof. Suppose that X is strictly convex and let x, y, and f be as above. Then, $1 - (f, y) \geqq 2 - \|x + y\| > 0$.

Now suppose that the second condition of the theorem is satisfied and that X is not strictly convex. Then, there exist $x, y \in S_1 (x \neq y)$ such that $\|x + y\| = 2$. Let $z = (x + y)/2$ and $h \in J(z)$. Since $\|h\| = 1 = \|x\| = \|y\|$ and $(h, x + y) = 2$, $(h, x) = 1$, a contradiction, since $z \neq x$.

Theorem (Petryshyn [6]). A Banach space X is strictly convex if and only if the duality map J of X into X^* is *strictly monotone*-in the sense that if $x \neq y$, $f \in J(x)$, and $g \in J(y)$, then $(f - g, x - y) > 0$.

Proof. Suppose that X is strictly convex. Let $x, y \in X$, $f \in J(x)$, and $g \in J(y)$. Then, $\|f\| \|y\| - (f, y) \geqq \|f\| (\|x\| + \|y\| - \|x + y\|)$ and $\|g\| \|x\| - (g, x) \geqq \|g\| (\|x\| + \|y\| - \|x + y\|)$ and by the use of equation ($\sharp$) of [6], we have

$$(f - g, x - y) \geqq (\|x\| - \|y\|)^2 + (\|x\| + \|y\|)(\|x\| + \|y\| - \|x + y\|) .$$

If $x \neq y$ and $\|x\| = \|y\|$, then $\|x\| > 0$ and $\|x\| + \|y\| - \|x + y\| = \|x\| (2 - \|x/\|x\| + y/\|x\| \|)$, which is positive by the strict convexity of X. Consequently, J is strictly monotone.

Now, suppose that J is strictly monotone and that X is not strictly convex. Then by the previous theorem, there exist $x, y \in S_1$ $(x \neq y)$ and an $f \in J(x)$ such that $1 - (f, y) = 0$. As before, $1 - (f, y) \geqq 2 - \|x + y\|$, so $\|x + y\| = 2$. If $z = (x + y)/2$ and $h \in J(z)$, then $(h, x + y) = 2$ and $\|h\| = 1 = \|x\| = \|y\|$, so $(h, x) = 1$. Consequently, $(h - f, z - x) = 1 - (h, x) + 1 - (f, z) = 0$, which contradicts the fact that $z \neq x$.

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