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A GEOMETRIC APPROACH TO THE FIXED POINT INDEX

ROGER D. NUSSBAUM

J. Leray defined a local fixed point index for functions defined in what he called convexoid spaces. From the standpoint of analysis, the most important example of a convexoid space is a compact subset $C \subset X$, X a locally convex topological vector space, such that $C = \cup_{i=1}^{n} C_i$, where C_i are compact, convex subsets of X or a homeomorphic image of such a C. In this paper a simple geometric approach is given (see Lemma 2 below) by means of which a fixed point index can be defined for functions with domain in a class of spaces $\mathscr{F}$ which contains the spaces C mentioned above and also the compact metric ANR's. The usual properties of the fixed point index are established, and it is shown that they axiomatically determine the index for the class of spaces $\mathscr{F}$.

As far as we know, none of the approaches to the fixed point index which have been published since Leray's work have been shown to apply to spaces C of the type above. For instance, A. Granas [9] has remarked that if a compact space C is r-dominated by an open subset of an lctvs, then the Leray-Schauder index for compact maps on open subsets of an lctvs gives a fixed point index for maps of open subsets of C into C. But without metrizability the spaces considered here are not necessarily r-dominated by open subsets of an lctvs. F. Browder [3] has shown that if a compact Hausdorff space admits a "semicomplex structure," then a fixed point index can be defined for functions with domain in the space. However, to show that given topological spaces admit semicomplex structures, the metrizability of the spaces has almost invariable been used. Thus Browder has shown that compact, metric ANR's admit semicomplex structures, and Thompson [19] has established the same thing for metric HLC* spaces. But a finite union of compact, convex sets in an lctvs need not be metrizable. In any event, we shall avoid questions about semicomplex structure and obtain our fixed point index from the classical one for compact, finite dimensional polyhedra.

1. Let us begin with some notation. Let C be a compact subset of a locally convex topological vector space (lctvs) X, always assumed Hausdorff. We shall write $C \in \mathscr{F}_0$ if there exists a finite closed covering $\{C_i: 1 \leqq i \leqq n\}$ of C by compact, convex sets $C_i \subset C$, ie if $C = \bigcup_{i=1}^{n} C_i$, C_i a compact convex subset of X. If $C \in \mathscr{F}_0$, $G \subset C$

is an open subset of C, and $f: G \to C$ is a continuous map such that $\{x \in G: f(x) = x\}$ is compact (possibly empty), then we shall define in this section a fixed point index $i_C(f, G)$ for f.

We take as a starting point Dold's development of the classical fixed point index [6]. If Y is a compact Hausdorff space, Dold calls Y a Euclidean neighborhood retract (ENR) if there exists an open subset 0 of some Euclidean space $\mathbf{R}^n$, a continuous map $i: Y \to 0$, and a continuous map $r: 0 \to Y$ such that $r \circ i =$ identity on Y. Notice that since any *finite dimensional* lctvs F (always assumed Hausdorff) is linearly homeomorphic to $\mathbf{R}^n$ for some n, we may as well assume that Y is imbedded in F in the above definition of ENR. For our work here, the most important example of an ENR will be a compact subset C of a *finite dimensional* lctvs F such that $C = \bigcup_{i=1}^n C_i$ for some compact, convex subsets $C_i \subset F$. The fact that C is an ENR follows from two theorems. First, Dugundji has shown [7] that a closed, convex subset of a Banach space X is an ANR (see [1] for definitions). Since F is finite dimensional it can be taken to be a Banach space, so any closed, convex subset of F is an ENR. Second, a classical theorem states that if Y_1 and Y_2 are subsets of a metrizable space Y and Y_1, Y_2, and $Y_1 \cap Y_2$ are ANR's then $Y_1 \cup Y_2$ is an ANR (see [1] for a proof). In our case these results combine to show C is an ENR.

Now let Y be a compact Hausdorff ENR, G an open subset of Y, and $f: G \to Y$ a continuous map such that $\{x \in G: f(x) = x\}$ is compact. Then there is a unique integer valued function $i_Y(f, G)$ having the following properties. (O'Neill has shown uniqueness of the fixed point index for compact polytopes [17]. Since for any ENR E, there exists a polytope P, and continuous maps $j: E \to P$, $r: P \to E$ such that $r \circ j =$ identity on E, the methods of § 2 show uniqueness for ENR's).

1. (The additivity property). Let Y, f, and G be as above. If $S = \{x \in G: f(x) = x\}$, assume that $S \subset G_1 \cup G_2$, where G_1 and G_2 are disjoint open subsets of G. Then $i_Y(f, G) = i_Y(f, G_1) + i_Y(f, G_2)$. Further, if $i_Y(f, G) \neq 0$, f has a fixed point in G.

2. (The homotopy property). Let G be an open subset of a compact, Hausdorff ENR Y. Let $I = [0, 1] =$ the closed unit interval and let $F: G \times I \to Y$ be a continuous map. Assume that $S = \{(x, t) \in G \times I: F(x, t) = x\}$ is compact. Then if we define $F_t(x) = F(x, t)$, $i_Y(F_0, G) = i_Y(F_1, G)$.

3. (The normalization property). Let Y be a compact, Hausdorff ENR and let $f: Y \to Y$ be a continuous map. Then $i_Y(f, Y) = \Lambda(f)$,

where $\Lambda(f)$ is the Lefschetz number of f, using singular homology with rational coefficients.

4. (The commutativity property). Let Y_1 and Y_2 be compact, Hausdorff ENR's and let G_1 and G_2 be open subsets of Y_1 and Y_2 respectively. Let $f_1: G_1 \to Y_2$ and $f_2: G_2 \to Y_1$ be continuous maps. Setting $H_1 = f_1^{-1}(G_2)$ and $H_2 = f_2^{-1}(G_1)$, assume that $S_1 = \{x \in H_1: (f_2f_1)(x) = x\}$ is compact. Then $S_2 = \{x \in H_2: (f_1f_2)(x) = x\}$ is compact and $i_{Y_1}(f_2f_1, H_1) = i_{Y_2}(f_1f_2, H_2)$.

Notice that if Y, f, and G are as in 1 and $f(Y) \subset Y' \subset Y$, Y' an ENR, then $i_Y(f, G) = i_{Y'}(f, G \cap Y')$. This follows from 4 by using the inclusion $i: Y' \to Y$.

In order to generalize the above fixed point index to our context, we need some lemmas. First, we introduce some further notation. We shall denote subsets of $\{1, 2 \cdots, n\}$ by J, K, L, M and we define $|J|$ to be the number of elements in J. If C is a compact Hausdorff space, and $C = \bigcup_{i=1}^n C_i$, where C_i is a compact subset of C for $1 \leqq i \leqq n$, then for $L \subset \{1, 2 \cdots, n\}$, we shall write $C_L = \bigcap_{i \in L} C_i$.

With the aid of this notation we can state our first lemma, which is the basis of all our further work.

LEMMA 1. *Let C be a compact, Hausdorff space such that $C = \bigcup_{i=1}^n C_i$, C_i a compact subset of C. Let $\mathscr{U}$ be a finite open covering of C. Then there exists a finite open covering $\mathscr{V} = \{V_{J,i}: J \subset \{1, 2 \cdots, n\},$ $1 \leqq i \leqq k_J\}$* (i.e., indexed by ordered paris (J, i), J a subset of $\{1, 2, \cdots, n\}$, i an integer for which $1 \leqq i \leqq k_J$, k_J an integer depending on J) *such that* (1) *$\mathscr{V}$ is a refinement of $\mathscr{U}$* (2) *$V_{J,i}$ is empty if C_J is empty and $V_{J,i} \cap C_J$ is nonempty if $V_{J,i}$ is nonempty.* (3) *If $k \notin J$, $cl(V_{J,i}) \cap C_k$ is empty.* (4) *If $|L| \geqq |K|$ but $L \not\supset K$, $V_{L,i} \cap V_{K,j}$ is empty.*

Proof. We construct $\{V_{J,i}\}$ by induction on $|J|$, starting with $|J| = n$. The inductive assumption at the $(n\text{-}r)^{\text{th}}$ step, $1 < r \leqq n$, is that there exists a collection of open sets

$$\{V_{J,i}: J \subset \{1, 2, \cdots, n\}, |J| \geqq r, 1 \leqq i \leqq k_J\}$$

which satisfies 1–4 above and is such that $\bigcup_{|J| \geqq r} \bigcup_{i=1}^{k_J} V_{J,i} \supset \bigcup_{|J| \geqq r} C_J$. The object is to define open sets $V_{J,i}$ for $J \subset \{1, 2, \cdots, n\}$ for which $|J| = r - 1$ and $1 \leqq i \leqq k_J$ and such that $\{V_{J,i}: |J| \geqq r - 1, 1 \leqq i \leqq k_J\}$ satisfies 1–4 and gives an open covering of $\bigcup_{|J| \geqq r-1} C_J$.

Step 1. If $|J| = n$, let $\{V_{J,i}: 1 \leqq i \leqq k_J\}$ be the collection of $U \in \mathscr{U}$ which have nonempty intersection with C_J. This collection may be empty.

Step 2. Assume for some $r, 1 \leqq r \leqq n$, we have constructed $\{V_{J,i}: |J| \geqq r, 1 \leqq i \leqq k_J\}$ which satisfies the inductive hypothesis. If $r = 1$, we are done, so assume $r > 1$. For each $K \subset \{1, 2, \cdots, n\}$ with $|K| = r - 1$, let $A_K = C_K - \bigcup_{|L| \geqq r} \bigcup_{1 \leqq i \leqq k_L} V_{L,i}$ and notice that if $j \notin K$, A_K and C_j are disjoint compact sets (since, for $j \notin K$, $C_K \cap C_j = C_{K \cup \{j\}}$ and $|K \cup \{j\}| \geqq r$). It is also clear that $A_K \cap A_{K'}$ is empty for all K' such that $K' \neq K$, since $A_{K'} \subset C_j$ for $j \in K' - K$ and $A_K \cap C_j$ is empty. It is thus not hard to see that there exist open neighborhoods O_K of A_K for all $K \subset \{1, 2, \cdots, n\}$ with $|K| = r - 1$ such that $\mathrm{cl}(O_K) \cap C_j$ is empty for $j \notin K$ and $\mathrm{cl}(O_K) \cap \mathrm{cl}(O_{K'})$ is empty for all K, K' with $|K| = |K'| = r - 1$ but $K \neq K'$.

Next, for a given $K \subset \{1, 2, \cdots, n\}$ with $|K| = r - 1$, consider all $L \subset \{1, 2, \cdots, n\}$ such that $|L| > r - 1$ but $L \supset K$. For each such L, select $j \in K$ such that $j \notin L$. If (L, j) is such a pair, we know by inductive assumption that $\mathrm{cl}\,(\bigcup_{i=1}^{k_L} V_{L,i}) \cap C_j$ is empty so there exists an open neighborhood $W_{(L,j)}$ of $C_j \supset C_K$ such that $\mathrm{cl}\,(\bigcup_{i=1}^{k_L} V_{K,i}) \cap W_{(L,j)}$ is empty. We set $W_K = \bigcap_{(L,j)} W_{(L,j)}$, where the intersection is taken over all ordered pairs (L, j) as above. We define $T_K = O_K \cap W_K$, an open neighborhood of A_K.

Let $\{V_{K,i}: 1 \leqq i \leqq k_K\}$ be the collection of sets of the form $T_K \cap U$, $U \in \mathscr{U}$, such that $A_K \cap U$ is nonempty. This collection may be empty. Clearly $\{V_{K,i}: 1 \leqq i \leqq k_K\}$ is an open covering of A_K. Observe also that $\{V_{J,i}: |J| \geqq r - 1, 1 \leqq i \leqq k_J\}$ gives an open covering of $U_{|J| \geqq r-1} C_J$. By induction we certainly have an open covering of $U_{|J| \geqq r} C_J$. If $x \in C_K$ for $|K| = r - 1$ and $x \notin U_{|L| \geqq r} V_{L,i}$, then by definition $x \in A_K$ so that $x \in V_{K,i}$ for some i.

It remains to check conditions 1–4 for the new covering. Since $V_{K,i}$, $|K| = r - 1$, was selected so $V_{K,i} \subset U$ for some $U \in \mathscr{U}$, condition 1 holds. Condition 2 holds trivially: If C_K is empty, A_K is empty and $\{V_{K,i}: 1 \leqq i \leqq k_K\}$ is empty. If $V_{K,i}$ is nonempty, $V_{K,i} = U \cap T_K$ for some $U \in \mathscr{U}$ such that $U \cap A_K$ is nonempty. If $|L| \geqq |K|$ but $L \supset K$, we want to show $V_{L,i} \cap V_{K,j}$ is empty. If $|K| > r - 1$, the result is true by inductive hypothesis. If $|K| = r - 1$, and $|L| > r - 1$, select $k \in K$ such that $k \notin L$. By our construction we have $V_{K,j} \subset W_K \subset W_{(L,k)}$ and $W_{(L,k)} \cap \mathrm{cl}\,(U_{t=1}^{k_L} V_{L,t})$ is empty, so that in particular $V_{L,i} \cap V_{K,j}$ is empty. If $|K| = |L| = r - 1$, then again by our construction, $V_{K,j} \cap V_{L,i} \subset O_K \cap O_L$ is empty. In either event, 4 is satisfied. Finally, to show 3 it suffices to show that if $|K| = r - 1$ and $j \notin K$, $\mathrm{cl}\,(V_{K,i}) \cap C_j$ is empty. Since $V_{K,i} \subset O_K$, and $\mathrm{cl}\,(O_K) \cap C_j$ is empty, this follows immediately from our construction.

This completes the inductive step. After n repetitions, the desired covering is obtained.

Our next lemma provides the justification for proving Lemma 1.

Lemma 2 is the basic result upon which all work in this section depends.

LEMMA 2. *Let C be a compact subset of an lctvs X and suppose $C = \bigcup_{i=1}^{n} C_i$, C_i a compact, convex subset of X. Then given any continuous seminorm q and $\varepsilon > 0$, there exists a continuous, finite dimensional map $r: C \to C$ such that if $x \in C_i$ for $1 \leqq i \leqq n$, then $r(x) \in C_i$ also and $q(r(x) - x) \leqq \varepsilon$.* (By finite dimensional we mean that the range of r lies in some finite dimensional subspace of X).

Proof. For each $x \in C$, let $U_x = \{y \in C: q(y - x) < \varepsilon/2\}$. This gives an open covering of C, and by the compactness of C, there exists a finite subcovering $\mathscr{U} = \{U_{x_1}, U_{x_2}, \cdots, U_{x_t}\}$. Clearly, if $U \in \mathscr{U}$ and $x, y \in U$, then $q(x - y) < \varepsilon$. By Lemma 1, there exists a refinement $\mathscr{V} = \{V_{J,i}: J \subset \{1, 2, \cdots, n\}, 1 \leqq i \leqq k_J\}$ satisfying conditions 1-4 of Lemma 1. By condition 2, for each nonempty $V_{J,i}$ select $P_{J,i} \in V_{J,i} \cap C_J$; also define $P_{J,i} = 0$ if $V_{J,i}$ is empty. Let $\{\phi_{J,i}: J \subset \{1, 2, \cdots, n\}, 1 \leqq i \leqq k_J\}$ be a partition of unity subordinate to $V_{J,i}$ (with the convention that $\phi_{J,i} \equiv 0$ if $V_{J,i}$ is empty), so that supp $(\phi_{J,i}) \subset V_{J,i}$ and $\sum_{J,i} \phi_{J,i}(x) = 1$ for $x \in C$. Define $r(x) = \sum_{J,i} \phi_{J,i}(x) P_{J,i}$. It is clear that r is a continuous, finite dimensional map. Also, it is easy to show that $q(x - r(x)) \leqq \varepsilon$. For $q(x - r(x)) = q(\sum_{J,i} \phi_{J,i}(x)(x - P_{J,i})) \leqq \sum_{J,i} \phi_{J,i}(x) q(x - P_{J,i})$, and if $\phi_{J,i}(x) \neq 0$, $x \in V_{J,i}$, so that $q(x - P_{J,i}) \leqq \varepsilon$ and $\sum_{J,i} \phi_{J,i}(x) q(x - P_{J,i}) \leqq \sum_{J,i} \phi_{J,i}(x) \varepsilon = \varepsilon$.

The nontrivial statement (and the reason for introducing $\{V_{J,i}\}$) is that $r(x) \in C_i$ if $x \in C_i$. Thus suppose $x \in C_i$. If for $L \subset \{1, 2, \cdots, n\}$, $i \notin L$, we know by property 3 of the covering that $\mathrm{cl}\,(V_{L,j}) \cap C_i$ is empty, so that $\phi_{L,j}(x) = 0$ (since supp $\phi_{L,j} \subset V_{L,j}$). Thus we have $r(x) = \sum_{(L,j), i \in L} \phi_{L,j}(x) P_{L,j}$. Since $P_{L,j} \in C_L \subset C_i$, this is just a convex combination of points in C_i and hence lies in C_i.

Notice that the proof of Lemma 2 only uses properties 1-3 of the covering $\{V_{J,i}\}$.

Before proceeding with our main line of development, let us state a proposition which indicates again the usefulness of Lemma 1. The following proposition is standard if C is a compact metric ANR, and it plays a key role in some developments of the fixed point index for compact, metric ANR's. Since we shall not need this result, we shall not give a proof except to say that it follows straightforwardly from Lemma 1. We refer the reader to Hanner's article [10], where theorems along the the general lines of the following proposition are proved for metric ANR's.

PROPOSITION. *Let C be a compact subset of an lctvs X and suppose*

$C = \bigcup_{i=1}^{n} C_i$, C_i compact, convex. Let α be a finite open covering of C. Then there exists a finite open covering β of C which is refinement of α and has the following property: If N_β denotes the nerve of β, then there exist continuous maps $f_\beta: C \to N_\beta$ and $g_\beta: N_\beta \to C$ such that $g_\beta \, o f_\beta$ is homotopic to the identity I on C. Furthermore the homotopy $F_t: C \to C$, $0 \leqq t \leqq 1$, $F_0 = g_\beta o f_\beta$, $F_1 = I$, can be chosen so that for each $x \in C$, $\exists U \in \alpha$ for which $F_t(x) \in U$ for $0 \leqq t \leqq 1$.

With the aid of Lemma 2 we can define our fixed point index. Let C be a compact subset of an lctvs X and G an open subset of C. Suppose that $C = \bigcup_{i=1}^{m} C_i$, C_i compact and convex. Let $f: \mathrm{cl}\,(G) \to C$ be a continuous map such that $f(x) \neq x$ for $x \in \partial G = \mathrm{cl}\,(G) - G$. Since $\{x - f(x): x \in \partial G\}$ is a compact set not containing 0 and X is an lctvs, there exists a continuous seminorm q such that $q(f(x) - x) \geqq \varepsilon > 0$ for $x \in \partial G$. We shall say that a continuous map g: $\mathrm{cl}\,(G) \to C$ is an admissible approximation with respect to $< f, \{C_i\} >$ (C_i as above) if (1) There exists a continuous seminorm q such that $q(f(x) - x) \geqq \varepsilon > 0$ for $x \in \partial G$ and $q(f(x) - g(x)) < \varepsilon$ for $x \in \partial G$. (2) For all $x \in \mathrm{cl}\,(G)$, if $f(x) \in C_i$, then $g(x) \in C_i$ (3) g is finite dimensional map, i.e. the range of g lies in a finite dimensional subspace of X. With the notation above we take C_i', $1 \leqq i \leqq m$, to be any finite dimensional compact, convex subset of C_i such that $g(G) \subset \bigcup_{i=1}^{m} C_i' \equiv C'$. (If g is as above and V is any finite dimensional vector space containing the range of g, we can define $C_i' = C_i \cap V$). By our previous remarks C' is a compact ENR. Notice that $g: \mathrm{cl}\,(G) \to C'$, and by condition 1 on g, $g(x) \neq x$ for $x \in \partial G$. Thus we see that $i_{C'}(g, G \cap C')$ is defined. We shall show below that if we define $i_C(f, G) = i_{C'}(g, G \cap C')$, this gives a well defined definition.

THEOREM 1. *Let C be a compact subset of an lctvs X and assume $C = \bigcup_{i=1}^{t} E_i$, E_i a compact, convex set. Let G be an open subset of C and $f: \mathrm{cl}\,(G) \to C$ a continuous may such that $f(x) \neq x$ for $x \in \partial G$. Let q be any continouous seminorm and $\varepsilon > 0$. Then there exists an admissible approximation θ with respect to $< f, \{E_i\} >$ such that $q(\theta(x) - f(x)) < \varepsilon$ for all $x \in \mathrm{cl}\,(G)$. Furthermore, suppose $C = \bigcup_{i=1}^{m} C_i$ and $C = \bigcup_{j=1}^{n} D_j$, C_i and D_j compact and convex. If g is an admissible approximation with respect to $< f, \{C_i\} >$, h is an admissible approximation with respect to $< f, \{D_j\} >$, C_i', $1 \leqq i \leqq m$, is a finite dimensional compact, convex subset of C_i such that $g(G) \subset \bigcup_{i=1}^{m} C_i' \equiv C'$ and D_j', $1 \leqq j \leqq n$, is a finite dimensional compact, convex subset of D_j such that $h(G) \subset \bigcup_{j=1}^{n} D_j' \equiv D'$, then $i_{C'}(g, G \cap C') = i_{D'}(h, G \cap D')$.*

Proof. Let q be as above and let p be a continuous seminorm such that $p(f(x) - x) \geqq \delta > 0$ for $x \in \partial G$. Define $q'(x) = \max \{p(x), q\{(x)\}$

and $\varepsilon' = \min\{\varepsilon, \delta\}$; of course q' is a continuous seminorm. By Lemma 2, there exists a continuous map $r: C \to C$ such that

(1) $q'(r(x) - x) < \varepsilon'$ for all $x \in C$

(2) $r(x) \in E_i$ if $x \in E_i$ for all $x \in C$ and $1 \leqq i \leqq t$.

(3) r is a finite dimensional map.

We define $\theta(x) = r(f(x))$. It is immediate that θ is an admissible approximation with respect to $< f, \{E_i\} >$.

Now let g be an admissible approximation with respect to $< f, \{C_i\} >$ and h an admissible approximation with respect to $< f, \{D_j\} >$. Thus there exist continuous seminorms q_1 and q_2 such that $q_1(f(x) - x) \leqq \varepsilon_1$, $q_1(f(x) - g(x)) < \varepsilon_1$, $q_2(f(x) - x) \geqq \varepsilon_2$, and $q_2(f(x) - g(x)) < \varepsilon_2$ for all $x \in \partial G$. We define $\tilde{q}(x) = \max\{q_1(x), q_2(x)\}$, a continuous seminorm, and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Notice that $C = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} C_i \cap D_j$, $C_i \cap D_j$ a compact, convex subset of X. If we set $E_{i,j} = C_i \cap D_j$, by the first part of this theorem, there exists an admissible approximation θ with respect to $< f, \{E_{i,j}\} >$ such that $\tilde{q}(f(x) - \theta(x)) < \varepsilon$ for $x \in \mathrm{cl}\,(G)$.

Before proceeding further, let us recall the elementary theorem that if A and B are compact, convex subsets of a topological vector space X, then $\{sx + (1 - s)y: 0 \leqq s \leqq 1, x \in A, y \in B\}$ is a compact, convex subset of X. In particular, if A and B are also finite dimensional, this shows $\mathrm{cocl}\,(A \cup B)$ (cocl denotes convex closure) is compact, convex, and finite dimensional. In our case let V and W be finite dimensional subspaces of X such that range $(g) \subset V$ and range $(h) \subset W$ and define $C_i'' = C_i \cap V$ and $D_j'' = D_j \cap W$. Let U be a finite dimensional subspace of X such that range $(\theta) \subset U$ and define $E'_{i,j} = E_{i,j} \cap U$. Now define $F' = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} [\mathrm{cocl}\,(C_i' \cup C_i'' \cup E'_{i,j}) \cup \mathrm{cocl}\,(D_j' \cup D_j'' \cup E'_{i,j})]$. By our above remarks, F' is finite union of compact, convex, finite dimensional sets, and hence an ENR. It is also easy to see that $F' \subset C$, $C' \cup D' \subset F'$ and $\theta(G) \cup g(G) \cup h(G) \subset F'$. We thus see that $i_{F'}(g, G \cap F')$, $i_{F'}(\theta, G \cap F')$, and $i_{F'}(h, G \cap F')$ are defined. Because $g(G \cap F') \subset C'$ and $h(G \cap F') \subset D'$, it follows (by the commutativity property) that $i_{C'}(g, G \cap C') = i_{F'}(g, G \cap F')$ and similarly for h. Thus to show that $i_{C'}(g, G \cap C') = i_{D'}(h, G \cap D')$, it suffices to show $i_{F'}(g, G \cap F') = i_{F'}(\theta, G \cap F') = i_{F'}(h, G \cap F')$. We prove that $i_{F'}(g, G \cap F') = i_{F'}(\theta, G \cap F')$, the proof for h being the same. Consider the homotopy $sg(x) + (1 - s)\theta(x)$, $0 \leqq s \leqq 1$, $x \in \mathrm{cl}\,(G \cap F')$. For $x \in \mathrm{cl}\,(G \cap F')$, we know that $f(x) \in E_{i,j}$ for some i, j, so $\theta(x) \in E'_{i,j}$, $g(x) \in C_i''$, and $sg(x) + (1 - s)\theta(x) \in \mathrm{cocl}\,[E'_{i,j} \cup C_i''] \subset F'$. Also, since $q_1(f(x) - sg(x) - (1 - s)\theta(x)) \leqq sq_1(f(x) - g(x)) + (1 - s)q_1(f(x) - \theta(x)) < \varepsilon_1$ and since $q_1(f(x) - x) \geqq \varepsilon_1$ for $x \in \partial G$, $sg(x) + (1 - s)\theta(x) \neq x$ for $x \in \mathrm{cl}\,(G \cap F') - G \cap F'$. Thus the homotopy is permissible and $i_{F'}(g, G \cap F') = i_{F'}(\theta, G \cap F')$.

DEFINITION. Suppose that $C \in \mathscr{F}_0$, $C \subset X$ an lctvs, G is an open subset of C, and F: cl $(G) \to C$ is a continuous map such that $f(x) \neq x$ for $x \in \partial G$. Let $\{C_i: 1 \leqq i \leqq m\}$ be a covering of C by compact, convex sets $C_i \subset C$ and let g be an admissible approximation with respect to $< f, \{C_i\} >$. Let C'_i, $1 \leqq i \leqq n$, be any finite dimensional compact convex sets such that $C'_i \subset C_i$ and $g(G) \subset \bigcup_{i=1}^m C'_i \equiv C'$. Then we define $i_C(f, G) = i_{C'}(g, G \cap C')$.

Theorem 1 shows that g exists and that our definition does not depend on the particular admissible approximation g or the particular C'_i. Notice that if C happens to be an ENR, so that $i_C(f, G)$ is already defined, then our definition reduces to the usual. To see this, just consider the homotopy $F(x, s) = (1 - s)f(x) + sg(x)$, $x \in \mathrm{cl}\,(G)$, $0 \leqq s \leqq 1$. Since g is an admissible approximation, if $f(x) \in C_i$, then $g(x) \in C_i$, so that $F(x, s) \in C_i$. Furthermore, condition 1 on g guarantees that $F(x, s) \neq x$ for $x \in G$, $0 \leqq s \leqq 1$. It follows that $i_C(f, G) = i_C(g, G)$, and since $g(G) \subset C'$, the commutativity property implies $i_C(g, G) = i_{C'}(g, G \cap C')$.

It is now easy to show that the various theorems about the fixed point index for ENR's extend to our context.

THEOREM 2. *Suppose* $C \in \mathscr{F}_0$, G *is an open subset of* C, *and* $f: \mathrm{cl}\,(G) \to C$ *is a continuous map such that* $f(x) \neq x$ *for* $x \in \partial G$. *Then if* $i_C(f, G) \neq 0$, f *has a fixed point in* G. *If* $S = \{x \in G: f(x) = x\} \subset G_1 \cup G_2$, *where* G_1 *and* G_2 *are disjoint open subsets of* G, *then* $i_C(f, G) = i_C(f, G_1) + i_C(f, G_2)$.

Proof. Suppose $i_C(f, G) \neq 0$. Since $C \in \mathscr{F}_0$, there exist compact, convex sets C_i, $1 \leqq i \leqq n$, such that $C = \bigcup_{i=1}^n C_i$. By Theorem 1, for any continuous seminorm p and $\varepsilon > 0$, there exists an admissible approximation g with respect to $< f, \{C_i\} >$ such that $p(g(x) - f(x)) < \varepsilon$ for $x \in \mathrm{cl}\,(G)$; and furthemore $i_{C'}(g, G \cap C') = i_C(f, G) \neq 0$, where $C' \equiv \bigcup_{i=1}^n C'_i$ and C'_i are any compact, convex finite dimensional subsets of C_i, such that $g(G) \subset C'$. By the usual additivity property there exists $x \in G \cap C'$ such that $g(x) = x$, whence $p(f(x) - x) < \varepsilon$. It follows that if, for any continuous seminorm p and $\varepsilon > 0$, we define $C_{\varepsilon,p} = \{x \in \mathrm{cl}\,(G): p(f(x) - x) \leqq \varepsilon\}$, $C_{\varepsilon,p}$ is a nonempty, compact set. Also, the collection $\{C_{\varepsilon,p}: p$ a continuous, seminorm $\varepsilon > 0\}$ has the finite intersection property, since $\bigcap_{i=1}^n C_{\varepsilon_i,p_i} \subset C_{\varepsilon,q}$, where $\varepsilon = \min\{\varepsilon_i: 1 \leqq i \leqq n\}$ and $q(x) = \max\{p_i(x): 1 \leqq i \leqq n\}$. Thus there exists $x_0 \in \cap C_{\varepsilon,p}$, and since $p(f(x_0) - x_0) = 0$ for all continuous seminorms, $f(x_0) = x_0$.

To prove the second part of the theorem, note that $f(x) \neq x$ for $x \in \partial G \cup \partial G_1 \cup \partial G_2$, so that exists a continuous seminorm p and $\varepsilon > 0$ such that $p(f(x) - x) \geqq \varepsilon$ for $x \in \partial G \cup \partial G_1 \cup \partial G_2$. By Theorem 1, there exists an admissible approximation g with respect to $< f, \{C_i\} >$ such that $p(f(x) - x) < \varepsilon$ for $x \in \mathrm{cl}\,(G)$, and we have $i_C(f, G) = i_{C'}(g, G \cap C')$, $i_C(f, G_i) = i_{C'}(g, G_i \cap C')$, as usual, $i = 1, 2$. By the additivity property for ENR's, $i_{C'}(g, G \cap C') = i_{C'}(g, G_1 \cap C') + i_{C'}(g, G_2 \cap C')$.

With the aid of Theorem 2 a slight, but useful generalization of our previous definition can be given.

DEFINITION. Suppose that $C \in \mathscr{F}_0$, G is an open subset of C, and $f: G \to C$ is a continuous map such that $S = \{x \in G: f(x) = x\}$ is compact. Then we define $i_C(f, G) = i_C(f, V)$, where V is any open neighborhood of S such that $\mathrm{cl}\,(V) \subset G$.

This definition makes sense, for if V_1 and V_2 are two open neighborhoods of S above such that $\mathrm{cl}\,(V_i) \subset G$, define $V = V_1 \cap V_2$. Then if we set $U_i = V_i - \mathrm{cl}\,(V) i_C(f, V_i) - i_C(f, U_i) = i_C(f, V)$, by Theorem 2. Also we have $i_C(f, U_i) = 0$, since f has no fixed points in U_i and Theorem 2 would imply f had fixed points in U_i if $i_C(f, U_i) \neq 0$. This shows $i_C(f, V_1) = i_C(f, V_2)$.

Henceforth we shall use this generalized definition. It is clear that Theorem 2 immediately generalizes to this context, the only difference in hypotheses being that we only assume f is defined on G and $S = \{x \in G: f(x) = x\}$ is compact.

THEOREM 3. *Suppose that $C \in \mathscr{F}_0$, G is an open subset of C, $I =$ $[0, 1]$, the closed unit interval, and $F: G \times I \to C$ is a continuous map such that $S = \{(x, t) \in G \times I: F(x, t) = x\}$ is compact. Then if we define $F_t: G \to C$ by $F_t(x) = F(x, t)$, $i_C(F_0, G) = i_C(F_1, G)$.*

Proof. Define $\pi: G \times I \to G$ by $\pi(x, t) = x$. It is clear that π is a continuous map, so $\pi(S) = T$ is a compact subset of G. Let V be an open neighborhood of T such that $\mathrm{cl}\,(V) \subset G$. Since $\{F(x, t) - x: (x, t) \in \partial V \times I\}$ is a compact set not containing 0, there exists a continuous seminorm p and $\varepsilon > 0$ such that $p(F(x, t) - x) \geqq \varepsilon$ for $(x, t) \in \partial V \times I$. Suppose $C = \bigcup_{i=1}^n C_i$, C_i compact and convex. By Lemma 2 there exists a continuous, finite dimensional map $r: C \to C$ such that for all $x \in C$, $p(r(x) - x) < \varepsilon$ and for all $x \in C$ and $1 \leqq i \leqq n$ $r(x) \in C_i$ if $x \in C_i$. We set $C' = \bigcup_{i=1}^n \mathrm{cocl}\, r(C_i)$, an ENR, and we define $H(x, t) = r(F(x, t))$ for $x \in \mathrm{cl}\,(V)$, $t \in I$. By definition we have $i_C(F_t, G) = i_C(F_t, V)$ and $i_C(F_t, V) = i_{C'}(H_t, V \cap C')$, since H_t is an admissible approximation with respect to $< F_t, \{C_i\} >$ and $H_t(G) \subset C'$, where C' is of the required form. Now by the ordinary homotopy property for ENR's, $i_{C'}(H_0, V \cap C') = i_{C'}(H_1, V \cap C')$.

COROLLARY. *Suppose that $C \in \mathscr{F}_0$, G is an open subset of C, and $F: \mathrm{cl}\, G \times [0, 1]$ is a continuous map such that $F(x, t) \neq x$ for $x \in \partial G$,*

$0 \leqq t \leqq 1$. *Then* $i_C(F_1, G) = i_C(F_0, G)$.

THEOREM 4. *Suppose that* $C \in \mathscr{F}_0$ *and* $f: C \to C$ *is a continuous map. Then* $H_n(C)$ (singular homology with rational coefficients) *is a finite dimentional vector space for all* n *and* $H_n(C) = 0$ *except for finitely many* n. *Furthermore, if* $\Lambda(f) = \sum_{n \geq 0} (-1)^n tr(f_*, n)$, *the Lefschetz number of* f, *then* $\Lambda(f) = i_C(f, C)$.

Proof. Since $C \in \mathscr{F}_0$, suppose $C = \bigcup_{i=1}^n C_i$, C_i compact and convex. By Lemma 2, there exists a continuous finite dimensional map $r: C \to C$ such that $r(x) \in C_i$ if $x \in C_i$. As before we define $C' = \bigcup_{i=1}^n \operatorname{cocl} r(C_i)$, an ENR. Recall that for any ENR E (or in fact any compact, metric ANR [1]) $H_n(E)$ is a finite dimensional vector space and $H_n(E) = 0$ except for finitely many n. In our case let us view r as a map from C to C' and let $i: C' \to C$ be the inclusion map. Since ir is homotopic to I, the identity on C, by the homotopy $s(ir) + (1 - s)I$, $0 \leqq s \leqq 1$, $(ir)_{*,n} = i_{*,n} r_{*,n} = I_{*,n}$. It follows that $i_{*,n}: H_n(C') \to H_n(C)$ must be onto, so $H_n(C)$ is a finite dimensional vector space and 0 for almost all n.

To show the second part of the theorem, note that irf is an admissible approximation to $< f, \{C_i\} >$ so by definition we have $i_C(f, C) = i_{C'}(rfi, C') =$ (by the normalization property for ENR's) $\Lambda(rfi)$. Since irf is homotopic to f (by the homotopy $s(irf)(x) + (1 - s)f(x)$, $0 \leqq s \leqq 1$), $\Lambda(f) = \Lambda(irf)$. However, we have $tr(irf)_{*,n}) = tr(i_{*,n}(rf)_{*,n}) = tr((rf)_{*,n} i_{*,n}) = tr((rfi)_{*,n})$, by the commutativity property for the trace operator tr on linear operators between finite dimensional vector spaces. This shows $\Lambda(rfi) = \Lambda(irf) = \Lambda(f)$.

The proof of the commutativity property is a little more involved than that of Theorems 2–4. First, we need some simple lemmas.

LEMMA 3. *Let* K *be a compact subset of an lctvs* X *and* $f: K \to Y$ *a continuous map of* K *into an lctvs* Y. *Then given any continuous seminorm* q *on* Y *and* $\varepsilon > 0$, *there exists a continuous seminorm* p *on* X *and* $\delta > 0$ *such that for all* $x, y \in K$ *with* $p(x-y) < \delta$, $q(f(x) - f(y)) < \varepsilon$.

Proof. For each $x \in K$, there exists a seminorm p_x and $\delta > 0$ such that if $y \in K$ and $p_x(y - x) < \delta_x$, $q(f(y) - f(x)) < \varepsilon/2$. Let $N_x = \{y \in K: p_x(y - x) < \delta_x/2\}$ an open neighborhood of x in K. The open sets N_x give an open covering of K, and since K is compact there exists a finite subcovering $N_{x_1}, \cdots, N_{x_n}$. Let $p(x) = \max_{1 \leq i \leq n} \{p_{x_i}(x)\}$ and $2\delta = \min_{1 \leq i \leq n} \{\delta_{x_i}\}$; of course p is a continuous seminorm. For convenience let $N_{x_i} = N_i$, $p_{x_i} = p_i$, and $\delta_{x_i} = \delta_i$. If we take $x, y \in K$

with $p(x-y)<\delta$, we can assume $x \in N_i$ for some i. Thus $p_i(x-x_i)<\delta_i/2$ and $p_i(y-x_i) \leqq p_i(y-x)+p_i(x-x_i) \leqq p(y-x)+p_i(x-x_i)<\delta_i$. It follows that $q(f(x)-f(x_i))<\varepsilon/2$ and $q(f(y)-f(x_i))<\varepsilon/2$, so that $q(f(y)-f(x))>\varepsilon$.

LEMMA 4. *Let K be a compact subset of an lctvs X. Let G be an open subset of K. Then given any compact subset $V \subset G$, there exists a continuous seminorm p and $\varepsilon>0$ such that $\{y \in K: x \in V$ with $p(y-x)<\varepsilon\} \equiv N_{p,\varepsilon}(V) \cap K \subset G$.*

Proof. For each $x \in V$ there exists a continuous seminorm p_x and $\varepsilon_x>0$ such that $\{y \in K: p_x(y-x)<\varepsilon_x\} \subset G$. Let $N_x=\{y \in K: p_x(y-x)<\varepsilon_x/2\}$. Since K is compact there exists a finite subcovering $N_{x_1}, N_{x_2}, \cdots, N_{x_n}$ of the open covering $\{N_x\}$ of V. As before we define $p(x)=\max_{1 \leqq i \leqq n}\{p_{x_i}(x)\}$ and $2\varepsilon=\min_{1 \leqq i \leqq n}\{\varepsilon_{x_i}\}$. Then if $y \in N_{p,\varepsilon}(V) \cap K$, so that there exists $x \in V$ such that $p(y-x)<\varepsilon$, select i such that $x \in N_{x_i}$. Then $p(y-x)<\varepsilon$ implies $p_{x_i}(y-x)<\varepsilon_{x_i}/2$, and since $p(x-x_i)<\varepsilon_{x_i}/2$, $p_{x_i}(y-x_i)<\varepsilon_{xi}$, whence $y \in G$.

THEOREM 5. (Commutativity property). *Assume that $C_k \in \mathscr{F}_0$, $C_k \subset$ an lctvs X_k, $k=1,2$. Let $G_k \subset C_k$ be open subsets of C_k and let $f_1: G_1 \rightarrow C_2$ and $f_2: G_2 \rightarrow C_1$ be continuous maps. Define $H_1=f_1^{-1}(G_2)$ and $H_2=f_2^{-1}(G_1)$ and assume that $S_1=\{x \in H_1: (f_2 f_1)(x)=x\}$ is compact. Then $S_2=\{x \in H_2: (f_1 f_2)(x)=x\}$ is compact and $i_{C_1}(f_2 f_1, H_1)=i_{C_2}(f_1 f_2, H_2)$.*

proof. The fact that S_2 is compact is immediate, since $f_1: S_1 \rightarrow S_2$, $f_2: S_2 \rightarrow S_1$, $(f_2 f_1)(x)=x$ for $x \in S_1$, and $(f_1 f_2)(y)=y$ for $y \in S_2$.

Let $\widetilde{G}_1$ and $\widetilde{G}_2$ be open neighborhoods of S_1 and S_2 respectively such that $\mathrm{cl}(\widetilde{G}_i) \subset G_i$. Let $\widetilde{H}_1=\{x \in \widetilde{G}_1: f_1(x) \in \widetilde{G}_2\}$ and similarly for H_2. It is clear that $\widetilde{H}_i$ is an open neighborhood of S_i and $\mathrm{cl}\, \widetilde{H}_i \subset H_i$, so that $(f_2 f_1)(x) \neq x$ for $x \in \mathrm{cl}\, \widetilde{H}_1-S_1$ and similarly $(f_1 f_2)(x) \neq x$ for $x \in \mathrm{cl}\, \widetilde{H}_2-S_2$. By Theorem 2, $i_{C_1}(f_2 f_1, \widetilde{H}_1)=i_{C_1}(f_2 f_1, H_1)$ and similarly for $f_1 f_2$. Thus we may as well assume at the start that f_i is defined on $\mathrm{cl}(G_i)$, S_1 is a compact subset of $H_1=\{x \in G_1: f_1(x) \in G_2\}$, S_2 is a compact subset of H_2, $(f_2 f_1)(x) \neq x$, for $x \in \mathrm{cl}\, H_1-S_1$ and $(f_1 f_2)(x) \neq x$ for $x \in \mathrm{cl}\, H_2-S_2$. Let U_i be a compact neighborhood of S_i such that $U_i \subset H_i$ and let V_i be an neighborhood of S_i such that $\mathrm{cl}\, V_i \subset H_i$, $f_1(V_1) \subset U_2$ and $f_2(V_2) \subset U_1$. Theorem 2 implies $i_{C_1}(f_2 f_1, H_1)=i_{C_1}(f_2 f_1, V_1)$ and $i_{C_2}(f_1 f_2, H_2)=i_{C_2}(f_1 f_2, V_2)$, so it is enough to show that $i_{C_1}(f_2 f_1, V_1)=i_{C_2}(f_1 f_2, V_2)$.

Since $(f_2 f_1)(x)-x \neq 0$ for $x \in \mathrm{cl}(H_1)-V_1$ (a compact set), there exists a continuous seminorm p_1 on X_1 and $\varepsilon_1>0$ such that $p_1((f_2 f_1)(x)-x) \geqq \varepsilon_1$ for $x \in \mathrm{cl}(H_1)-V_1$. Similarly, there exists a continuous seminorm p_2 on X_2 and $\varepsilon_2 > 0$ such that $p_2((f_1f_2)(x) - x) > \varepsilon_2$ for $x \in \mathrm{cl}(H_2) - V_2$. By Lemmas 3 and 4 there exist a continuous seminorm q_1 on X_1 and $\delta_1 > 0$ such that $\{y \in C_1: x \in U_1 \text{ with } q_1(y - x) < \delta_1\} \equiv N_{q_1, \delta_1}(U_1) \cap C_1 \subset H_1$ and for all $y, z \in \mathrm{cl}(G_1)$ such that $q_1(y - z) < \delta_1$, $p_2(f_1(y) - f_1(z)) < \varepsilon_2/_2$. For the same reasons there exist a continuous seminorm q_2 on X_2 and $\delta_2 > 0$ such that $N_{q_2, \delta_2}(U_2) \cap C_2 \subset H_2$ and for all $y, z \in \mathrm{cl}(G_2)$ such that $q_2(y - z) < \delta_2$, $p_1(f_2(y) - f_2(z)) < \varepsilon_1/_2$. Because $C_1, C_2 \in \mathscr{F}_0$, $C_1 = \bigcup_{i=1}^m C_{1,i}$ and $C_2 = \bigcup_{j=1}^n C_{2,j}$, $C_{1,j}$ and $C_{2,j}$ compact, convex sets. By Lemma 2 there exists a continuous, finite dimensional map $r_1: C_1 \to C_1$ such that for all $x \in C_1$, $r_1(x) \in C_{1\,i}$ if $x \in C_{1,i}$, $p_1(r_1(x) - x) < \varepsilon_1/_2$, and $q_1(r_1(x) - x) < \delta_1$. For the same reason there exists a continuous finite dimensional map $r_2: C_2 \to C_2$ such that for all $x \in C_2$, $r_2(x) \in C_{2,j}$ if $x \in C_{2,j}$, $p_2(r_2(x) - x) < \varepsilon_2/_2$ and $q_2(r_2(x) - x) < \delta_2$. As usual we define $C_1' = \bigcup_{i=1}^m \mathrm{cocl}\, r_1(C_{1,i})$ and $C_2' = \bigcup_{j=1}^n \mathrm{cocl}\, r_2(C_{2,j})$.

We now define two supplementary functions, $g_1 = r_2f_1 | H_1$ and $g_2 = r_1f_2 | H_2$. Since $q_2(r_2f_1(x) - f_1(x)) < \delta_2$, $f_1(x) \in U_2$ for $x \in \mathrm{cl}\, V_1$, and $N_{q_2, \delta_2}(U_2) \cap C_2 \subset H_2$, we see that $g_1(\mathrm{cl}\, V_1) \subset H_2$. For the same reasons we observe that $g_2(\mathrm{cl}\, V_2) \subset H_1$. If we set $0_1 = g_1^{-1}(H_2)$ and $0_2 = g_2^{-1}(H_1)$, the above observations show that $\mathrm{cl}\, V_1 \subset 0_1$ and $\mathrm{cl}\, V_2 \subset 0_2$. We claim that $(g_2g_1)(x) \neq x$ for $x \in 0_1 - V_1$ and $(g_1g_2)(x) \neq x$ for $x \in 0_2 - V_2$. To prove this for g_2g_1, recall that for all $x \in 0_1$ (so $r_2f_1(x) \in H_2$ and $f_1(x) \in G_1$), $q_2(r_2f_1(x) - f_1(x)) < \delta_2$. By the assumption on δ_2 this implies $p_1(f_2r_2f_2(x) - f_2f_1(x)) < \varepsilon_1/_2$, while the assumption on r_1 guarantees that $p_1(r_1f_2r_2f_1(x) - f_2r_2f_1(x)) < \varepsilon_1/_2$. Thus we see that for $x \in 0_1$, $p_1(g_2g_1(x) - f_2f_1(x)) < \varepsilon_1$, and since $p_1(f_2f_1(x) - x) \geqq \varepsilon_1$ for $x \in \mathrm{cl}\, H_1 - V_1$, it follows that $g_2g_1(x) \neq x$ for $x \in 0_1 - V_1$. The proof for g_1g_2 is the same. This observation shows that $i_{C_1}(g_2g_1, 0_1)$ and $i_{C_1}(g_2g_1, V_1)$ are defined and equal and similarly for g_1g_2.

Our next claim is that $i_{C_1}(g_2g_1, V_1) = i_{C_1}(f_2f_1, V_1)$. To see this we consider the homotopy $F: \mathrm{cl}\, V_1 \times I \to C_1$ defined by $F(x, t) = (1 - t)r_1f_2((1 - t)r_2f_1(x) + tf_1(x)) + tf_2((1 - t)r_2f_1(x) + tf_1(x))$ and we apply Theorem 3. We have to show that this homotopy is permissible. First note that for all $x \in \mathrm{cl}(V_1)$, $f_1(x) \in U_2$ and $q_2(r_2f_1(x) - f_1(x)) < \delta_2$. It follows that $(1 - t)r_2f_1(x) + tf_1(x) \in N_{q_2, \delta_2}(U_2) \cap C_2 \subset H_2$. (Of course $(1 - t)r_2f_1(x) + tf_1(x) \in C_2$, since if $f_1(x) \in C_{2,j}$, $r_2f_1(x) \in C_{2,j}$ and hence $(1 - t)r_2f_1(x) + tf_1(x) \in C_{2.j}$ for $0 \leqq t \leqq 1$). This shows that $f_2((1 - t)r_2f_1(x) + tf_1(x))$ is defined for $x \in \mathrm{cl}\, V_1$ and applying the usual reasoning we see that $F(x, t) \in C_1$ for $x \in \mathrm{cl}\, V_1$, $0 \leqq t \leqq 1$. It remains to show that $F(x, t) \neq x$ for $x \in \partial V_1$, $0 \leqq t \leqq 1$. We have seen above that $q_2(r_2f_1(x) - f_1(x)) < \delta_2$ and it follows that $q_2((1 - t)r_2f_1(x) + tf_1(x)) - f_1(x)) < (1 - t)\delta_2 \leqq \delta_2$. It follows that $p_1(f_2((1 - t)r_2f_1(x) + tf_1(x)) - f_2f_1(x)) < \varepsilon_1/_2$ and since $p_1(r_1(y) - y) < \varepsilon_1/_2$ for $y \in C_1$ we conclude that $p_1(F(x, t) - f_2f_1(x)) < (1 - t)\varepsilon_1 + t\varepsilon_1/_2$ or $p_1(F(x, t) - f_2f_1(x)) < \varepsilon_1$ Since $p_1(f_2f_1(x) - x) \geqq \varepsilon$ for $x \in \partial V_1$, our homotopy is permissible and $i_{C_1}(f_2f_1, V_1) = i_{C_1}(g_2g_1, V_1)$. By the same reasoning we also find $i_{C_2}(f_1f_2, V_2) = i_{C_2}(g_1g_2, V_2)$.

To complete our proof it only remains to show that $i_{C_2}(g_2g_1, V_1) = i_{C_2}(g_1g_2, V_2)$. However notice that g_2g_1 is an admissible approximation with respect to $< g_2g_1, \{C_{1,i}\} >$ and $g_2g_1(V_1) \subset C_1'$, where C_1' is of the required form. By our definition it follows that $i_{C_1}(g_2g_1, V_1) = i_{C_1'}(g_2g_1, V_1 \cap C_1')$. By the same reasoning we also see that $i_{C_2}(g_1g_2, V_2) = i_{C_2'}(g_1g_2, V_2 \cap C_2')$. If we consider $h_1 = g_1 | C_1' \cap H_1$ and $h_2 = g_2 | C_2' \cap H_2$, it is easy to see that $h_1: C_1' \cap H_1 \to C_2'$, $h_2: C_2' \cap H_2 \to C_1'$, $h_1^{-1}(C_2' \cap H_2) = 0_1 \cap C_1'$ and $h_2^{-1}(C_1' \cap H_1) = 0_1 \cap C_2'$. Since we have already shown that $g_2g_1(x) \neq x$ for $x \in 0_1 - V_1$ and $\mathrm{cl}\, V_1 \subset 0_1$, we thus see that $i_{C_1'}(h_2h_1, h_1^{-1}(H_2 \cap C_2'))$ is defined and equals $i_{C_1'}(h_2h_1, V_1 \cap C_1')$. For the same reasons we see that $i_{C_2'}(h_1h_2, h_2^{-1}(H_1 \cap C_1')) = i_{C_2'}(h_1h_2, V_2 \cap C_2')$. Since the ordinary commutativity property implies $i_{C_1'}(h_2h_2, h_1^{-1}(H_2 \cap C_2')) = i_{C_2'}(h_1h_2, h_2^{-1}(H_1 \cap C_1'))$, we are done.

2. In this section we shall define a fixed point index for continuous maps defined in topological spaces which are homeomorphic to retracts of spaces $C \in \mathscr{F}_0$. The method we shall use is not new, and we include this treatment for the sake of completeness. The basic technique of this section seems first to have been explicitly stated in its essentials by A. Deleanu [5]. A number of other authors, among them Dold [6], Browder [4], Granas [9] and Nussbaum [16] have also used variants of the same idea.

We begin with some notation. If D is a compact, Hausdorff space, we write $D \in \mathscr{F}$ if there exists $C \in \mathscr{F}_0$, a continuous map $j: D \to C$, and a continuous map $r: C \to D$ such that $rj = I_D$, the identity on D. If G is an open subset of D and $f: G \to D$ is a continuous map such that $S = \{x \in G: f(x) = x\}$ is compact, it is clear that $T \equiv \{x \in r^{-1}(G): (jfr)(x) = x\} \subset r^{-1}(S)$, and since $r^{-1}(S)$ is a compact subset of $r^{-1}(G)$, it follows that T is a compact subset of $r^{-1}(G)$. Thus $i_C(jfr, r^{-1}(G))$ is defined. If we write $jfr = (jf)(r)$ and *formally* try to apply the commutativity property to $r: C \to D$ and $jf: G \to C$, we find that $i_C(jfr, r^{-1}(G)) = i_D(rjf, G) = i_D(f, G)$. Thus it is natural to try to define $i_D(f, G) = i_C(jfr, r^{-1}(G))$. Our first theorem shows that this definition is well-defined.

THEOREM 6. *Suppose that $D \in \mathscr{F}$ and for $k = 1, 2$ suppose that $C_k \in \mathscr{F}_0$ and $j_k: D \to C_k$ and $r_k: C_k \to D$ are continuous maps such that $r_kj_k = I_D$, the identity on D. Let G be an open subset of D and $f: G \to D$ a continuous map such that $S = \{x \in G: f(x) = x\}$ is compact. Then $i_{C_1}(j_1fr_1, r_1^{-1}(G)) = i_{C_2}(j_2fr_2, r_2^{-1}(G))$.*

Proof. Write $j_2fr_2 = (j_2r_1)(j_1fr_2)$ and define $h_1 = j_2r_1: C_1 \to C_2$ and

$h_2 = j_1 f r_2 \colon r_2^{-1}(G) \to C_1$. It is easy to check that $h_2^{-1}(C_1) = r_2^{-1}(G)$ and $h_1^{-1}(r_2^{-1}(G)) = r_1^{-1}(G)$, so it follows by Theorem 5 that $i_{C_1}(h_2 h_1, r_1^{-1}(G)) = i_{C_2}(h_1 h_2, r_2^{-1}(G))$. However $h_2 h_1 = j_1 f r_2 j_2 r_1 = j_1 f r_1$, so we have the desired result.

Thus we can define $i_D(f, G) = i_C(jfr, r^{-1}(G))$ (same notation as before). As immediate consequences of Theorems 2 and 3 we obtain the following theorems, whose proofs we omit.

THEOREM 7. *Suppose that $D \in \mathscr{F}$, G is an open subset of D, and $f\colon G \to D$ is a continuous map such that $S = \{x \in G\colon f(x) = x\}$ is compact. If $i_D(f, G) \neq 0$, then f has a fixed point in G. If $S \subset G_1 \cup G_2$, where G_1 and G_2 are disjoint open subsets of G, then $i_C(f, G) = i_C(f, G_1) + i_C(f, G_2)$*

THEOREM 8. *Suppose that $D \in \mathscr{F}$, G is an open subset of D, $I = [0, 1]$, and $F\colon G \times I \to D$ is a continuous map such that $S = \{(x, t) \in G \times I\colon F(x, t) = x\}$ is compact. Then $i_D(F_0, G) = i_D(F_1, G)$.*

THEOREM 9. *Suppose that $D \in \mathscr{F}$ and $f\colon D \to D$ is a continuous map. Then $\Lambda(f)$, the Lefschetz number of f, (singular homology with rational coefficients) is defined and $\Lambda(f) = i_D(f, D)$.*

Proof. By definition there exist $C \in \mathscr{F}_0$ and continuous maps $j\colon D \to C$ and $r\colon C \to D$ such that $rj = I_D$, the identity on D. Since $rj = I_D$, $r_{*,n}\colon H_n(C) \to H_n(D)$ is onto and $H_n(D)$ is a finite dimensional vector space and 0 for almost all n. Again by definition $i_D(f, D) = i_C(jfr, C)$; and since $i_C(jfr, C) = \Lambda(jfr)$, it suffices to show $\Lambda(f) = \Lambda(jfr)$. However we have $\Lambda(jfr) = \sum_{n \geq 0} (-1)^n tr((jf)_{*,n} r_{*,n}) =$ (by the properties of trace) $\sum_{n \geq 0} (-1)^n tr(r_{*,n}(jf)_{*,n}) = \sum_{n \geq 0} (-1)^n tr f_{*\,n} = \Lambda(f)$.

THEOREM 10. *Assume that $D_1, D_2 \in \mathscr{F}$, G_1 and G_2 are open subsets of D_1 and D_2 respectively, $f_1\colon G_1 \to D_2$ and $f_2\colon G_2 \to D_1$ are continuous maps. Let $H_1 = f_1^{-1}(G_2)$, $H_2 = f_2^{-1}(G_1)$, and assume that $S_1 = \{x \in H_1\colon (f_2 f_1)(x) = x\}$ is compact. Then $S_2 = \{x \in H_2\colon (f_1 f_2)(x) = x\}$ is compact and $i_{D_1}(f_2 f_1, H_1) = i_{D_2}(f_1 f_2, H_2)$.*

Proof. The same proof as before shows S_2 is compact. Since $D_k \in \mathscr{F}$, $k = 1, 2$, there exist $C_k \in \mathscr{F}_0$ and continuous maps $j_k\colon D_k \to C_k$ and $r_k\colon C_k \to {}_k$ such that $r_k j_k = I_{D_k}$, $k = 1, 2$. We have to show that $i_{C_1}(j_1 f_2 f_1 r_1, r_1^{-1}(H_1)) = i_{C_2}(j_2 f_1 f_2 r_2, r_2^{-1}(H_2))$. Define $g_1 = j_2 f_1 r_1\colon r_1^{-1}(G_1) \to C_2$ and $g_2 = j_1 f_2 r_2\colon r_2^{-1}(G_2) \to C_1$. It is easy to check that $g_1^{-1}(r_2^{-1}(G_2)) = r_1^{-1}(H_1)$ and $g_2^{-1}(r_1^{-1}(G_1)) = r_2^{-1}(H_2)$; also we see that $g_2 g_1 = j_1 f_2 f_1 r_1$ and

$g_1 g_2 = j_2 f_1 f_2 r_2$. It follows by Theorem 5 applied to g_1 and g_2 that $i_{C_1}(j_1 f_2 f_1 r_1, r_1^{-1}(H_1)) = i_{C_2}(j_2 f_1 f_2 r_2, r_2^{-1}(H_2))$.

REMARK. The method of proof we have used shows that there is a unique integer-valued function $i_D(f, G)$ (defined for $D \in \mathscr{F}$, G an open subset of D, and $f: G \rightarrow D$ a continuons map such that $\{x \in G: f(x) = x\}$ is compact) which satisfies Theorems 7–10. For as we have already remarked there is a unique such function defined for ENR's D. The methods of § 1, using Theorem 1, the homotopy property, and the commutativity property, show that the index function is determined by its value for ENR's when $D \in \mathscr{F}_0$. Finally, we saw in this section that the commutativity property completely determined our definition in terms of the index for $D \in \mathscr{F}_0$.

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