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THE OPEN MAPPING THEOREM FOR SPACES WITH UNIQUE SEGMENTS

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If X and Y are spaces with unique segments, an affine map from X into Y is a map which takes segments into segments. The purpose of this paper is to give conditions on spaces X and Y such that we can prove the following versions of the Open Mapping and Closed Graph Theorems: (1) a continuous affine map of X onto Y is open, and (2) an affine map of X onto Y with closed graph is continuous.

Let X and Y be spaces with unique segments, with $\Phi(x, y, t)$ $(0 \leqq t \leqq 1)$ denoting the intrinsically parametrized segment from x to y. A map T between two such spaces is said to be affine if $T(\Phi(x, y, t)) = \Phi(Tx, Ty, t)$, and a subset A is convex if $x, y \in A$, $0 \leqq t \leqq 1 \Rightarrow \Phi(x, y, t) \in A$. The purpose of this paper is to prove versions of the Open Mapping and Closed Graph theorems for classes of spaces with unique segments.

Throughout this paper, all metric spaces will be spaces with unique segments (unique curves of minimal, realizing the distance, length between any two points). The open sphere with center x and radius ε will be denoted by $S(x, \varepsilon)$.

DEFINITION 1. (X, d) is said to be regular if it is complete, the closure of convex sets is convex, open spheres are convex, and $x_n \rightarrow x_0 \Rightarrow \Phi(z, x_n, \alpha) \rightarrow \Phi(z, x_0, \alpha)$ for $z \in X$, $\alpha \in [0, 1]$.

DEFINITION 2. A sphere $S(x_0, \varepsilon)$ is said to be thick if for any $y \in S(x_0, \varepsilon)$ and $x \in X$, $x \neq y$, there is a $z \in S(x_0, \varepsilon)$ and $\alpha \in (0, 1]$ such that $y = \Phi(z, x, \alpha)$.

It is always possible to extend geodesics into thick spheres.

DEFINITION 3. A sphere $S(x_0, \varepsilon)$ is said to be round if, given $x \in S(x_0, \varepsilon)$, $y \in S(x, \varepsilon - d(x_0, x))$, and λ such that $d(x, y) < \lambda < \varepsilon - d(x_0, x) \Rightarrow$ there is a $z \in S(x, \varepsilon - d(x_0, x))$ and $\alpha \in (0, 1)$ such that $d(x, z) = \lambda$, $y = \Phi(x, z, \alpha)$.

Given any sphere $S(x, \delta)$ contained in a round sphere $S(x_0, \varepsilon)$ and any $y \in S(x, \delta)$, y lies on a geodesic connecting x and a point in $S(x, \delta)$ whose distance from x is arbitrarily close to δ. It should be noted that both thickness and roundness are hereditary properties; that is, if $S(x_0, \varepsilon)$ is thick (round) and $S(x, \delta) \subseteqq S(x_0, \varepsilon)$ then $S(x, \delta)$ is thick (round).

We now prove an Open Mapping theorem.

THEOREM 1. *Let (X, d) and (X, d') be regular, and assume that each point in X is the center of a thick sphere, and each point in Y is the center of a round sphere. Assume further that, for each open $U \subseteqq Y$, $y \in Y$ and $\alpha \in (0, 1]$, the set $\{\Phi(y, u, \alpha) \mid u \in U\}$ is open. Let T be continuous affine map of X onto Y. Then T is open.*

Proof. Let U be an open subset of X, let $x_0 \in U$, and let $\varepsilon > 0$ such that $S(x_0, \varepsilon)$ is thick and a subset of U. Let $U_k = S(x_0, k)$, $k = 1, 2, \cdots$; since T is into, $Y = \bigcup_{k=1}^{\infty} \overline{T(U_k)}$. By the Baire Category Theorem, some $\overline{T(U_k)}$ contains an $S(Tx', \varepsilon')$. Since U_k is convex and T is affine, $T(U_k)$ is convex, and so $\overline{T(U_k)}$ is convex. Since thickness is hereditary, we can assume $\varepsilon < k$. Choose $x \in S(x_0, \varepsilon)$ and $\alpha \in (0, 1]$ with $x_0 = \Phi(x, x', \alpha)$; letting $z = Tx$ we have $Tx_0 = \Phi(z, Tx', \alpha)$, and $z \in \overline{T(U_k)}$. Now $Tx_0 \in \{\Phi(z, u, \alpha) \mid u \in S(Tx', \varepsilon')\}$, which is open by hypothesis and also a subset of $\overline{T(U_k)}$. We can therefore find a round sphere $S(Tx_0, \varepsilon'') \subset \overline{T(U_k)}$.

If $d'(y, Tx_0) < \varepsilon''$, let $\gamma = 1/2(\varepsilon'' + d'(y, Tx_0))$; since $S(Tx_0, \varepsilon'')$ is round and $\gamma < \varepsilon''$, there is a $z \in S(Tx_0, \varepsilon'')$ with $d'(z, Tx_0) = \gamma$ and $y = \Phi(Tx_0, z, \alpha)$, clearly $\alpha = \gamma^{-1} d'(y, Tx_0)$. Now $z \in \overline{T(U_k)}$, so choose $\{x_n \mid n = 1, 2, \cdots\}$ with $d(x_0, x_n) < k$ and $Tx_n \to z$; let $z_n = \Phi(x_0, x_n, \alpha)$. Now $Tz_n = T(\Phi(x_0, x_n, \alpha)) = \Phi(Tx_0, Tx_n, \alpha) \to \Phi(Tx_0, z, \alpha) = y$, and $d(x_0, z_n) = \alpha d(x_0, x_n) = \gamma^{-1} d'(y, Tx_0) d(x_0, x_n) = 2d'(y, Tx_0) d(x_0, x_n)/(\varepsilon'' + d'(y, Tx_0)) \leqq (2k/\varepsilon'') d'(y, Tx_0)$.

We now pause for a brief recapitulation. Given a point x_0, if there is an integer n and a $\lambda > 0$ such that $S(Tx_0, \lambda)$ is round and $S(Tx_0, \lambda) \subset \overline{T(S(x_0, n))}$, then given $z \in S(Tx_0, \lambda)$ and $\eta > 0$ there is an $x \in X$ with $d(x, x_0) \leqq (2n/\lambda) d'(z, Tx_0)$ and $d'(Tx, z) < \eta$. We now show that a similar approximation can be performed uniformly in a neighbourhood of x_0.

Since T is continuous at x_0, there is a $\delta > 0$ such that $d(x_0, y) < \delta \Rightarrow d'(Tx_0, Ty) < \lambda/2$. Let $y \in S(x_0, \min(\delta, n))$. If $z \in S(Ty, \lambda/2)$, then $d'(z, Tx_0) \leqq d'(z, Ty) + d'(Ty, Tx_0) < \lambda$, so

$$S(Ty, \lambda/2) \subset S(Tx_0, \lambda) \subset \overline{T(S(x_0, n))} .$$

Now $z \in S(x_0, n) \Rightarrow d(z, y) \leqq d(z, x_0) + d(x_0, y) < 2n$, so $S(x_0, n) \subset S(y, 2n)$, and so $S(Ty, \lambda/2) \subset \overline{T(S(y, 2n))}$. Since $S(Ty, \lambda/2) \subset S(Tx_0, \lambda)$, $S(Ty, \lambda/2)$ is round, and by repeating the previous computation, given $z \in S(Ty, \lambda/2)$ and $\eta > 0$, there is an $x \in X$ with

$$d(x, y) \leqq [2(2n)/(\lambda/2)] d'(z, Ty) = \left(\frac{8n}{\lambda}\right) d'(z, Ty)$$

and $d'(Tx, z) < \eta$.

Therefore, for any $x_0 \in X$, there are constants δ, ε and M such that $d(x_0, y) < \delta$, $d'(z, Ty) < \varepsilon$ and $\eta > 0 \Rightarrow$ there is an $x \in X$ such that $d(x, y) \leqq M d'(z, Ty)$ and $d'(Tx, z) < \eta$.

Let δ, ε, and M be defined as above. We now show T is open by showing $T(S(x_0, \rho))$ contains the sphere $S(Tx_0, \alpha)$, where $\alpha = 1/2 \min(\varepsilon, \delta/2M, \rho/2M)$. For $n = 1, 2, \cdots$ let

$$\eta_n = \frac{1}{2} \min(\varepsilon, \delta/2^{n+1}M, \rho/2^{n+1}M) .$$

Let $z \in S(Tx_0, \alpha)$.

Since $d'(z, Tx_0) < \varepsilon$, choose $x_1 \in X$ with

$$d(x_0, x_1) \leqq Md'(Tx_0, z),\ d'(Tx_1, z) < \eta_1 .$$

Now $d(x_0, x_1) \leqq Md'(Tx_0, z) < \alpha M < \delta$ and $d'(Tx_1, z) < \eta_1 < \varepsilon$, so we can choose $x_2 \in X$ with $d(x_1, x_2) \leqq Md'(z, Tx_1)$ and $d'(Tx_2, z) < \eta_2$ Inductively, assume we have chosen $x_2, \cdots, x_n$ such that, for $2 \leqq j \leqq n$, $d(x_{j-1}, x_j) \leqq Md'(z, Tx_{j-1})$ and $d'(Tx_j, z) < \eta_j$. Now $d(x_0, x_n) \leqq d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) \leqq Md'(Tx_0, z) + M\sum_{j=1}^{n-1} d'(z, Tx_j) < M\alpha + M\sum_{j=1}^{n-1} \eta_j < M(\delta/2M) + M\sum_{j=1}^{n-1} (\delta/2^{j+1} M) = \sum_{j=0}^{n-1} \delta/2^{j+1} < \delta$, and also $d'(Tx_n, z) < \eta_n < \varepsilon$, so we can choose $x_{n+1} \in X$ with $d(x_n, x_{n+1}) \leqq Md'(z, Tx_n)$ and $d'(Tx_{n+1}, z) < \eta_{n+1}$.

If $1 \leqq n < m$, then

$$\begin{aligned} d(x_n, x_m) &\leqq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\ &\leqq M\sum_{j=n}^{m-1} d'(z, Tx_j) < M\sum_{j=n}^{m-1} \eta_j < \delta/2^n ; \end{aligned}$$

by completeness there is an $x \in X$ such that $x_n \to x$. Since $\eta_n \to 0$, $Tx_n \to z$, by continuity $Tx_n \to Tx$, so $Tx = z$. We are done if we can show that $d(x_0, x) < \rho$. For $n \geqq 1$, $d(x_0, x) \leqq d(x_0, x_1) + \cdots + d(x_{n+1}, x_n) + d(x_n, x) \leqq Md'(Tx_0, z) + M\sum_{j=1}^{n-1} d'(z, Tx_j) + d(x_n, x) < M\alpha + M\sum_{j=1}^{n-1} \eta_j + d(x_n, x) < M(\rho/3M) + M\sum_{j=1}^{n-1} (\rho/3\cdot 2_j M) + d(x_n, x)$, so, since $x_n \to x$, we have $d(x_0, x) \leqq \rho/3 + \sum_{j=1}^{\infty} (\rho/3\cdot 2^j) = 2\rho/3 < \rho$, completing the proof.

If the hypotheses seem somewhat cumbersome, it should be rememebered that the Uniform Boundedness and Open Mapping Theorems in linear topological spaces also place heavy restrictions on the topologies involved.

We conclude by making some remaks relevant to the Closed Graph Theorem, one of the most useful consequences of the Open Mapping Theorem. Call a space (X, d) which satisfies all the hypothesis listed in Theorem 1 a normal space. Given two normal spaces (X, d) and (Y, d'), if we can show that the product $(X \times Y, d'')$ (where $d''((x_1, y_1), (x_2, y_2)) = d((x_1, x_2) + d'(y_1, y_2))$ is normal, standard techniques (for example, [1] p. 100 and p. 116) will prove that, if $T: X \to Y$ is affine with closed graph, then T is continuous.

The verification that the product $(X \times Y, d'')$ of normal spaces (X, d) and (Y, d') follows quickly from the following lemma.

LEMMA 2. *$(X \times Y, d'')$ is a space with unique segments.*

Proof. Given a path λ in X and a path γ in Y, define $\lambda \times \gamma(t) = (\lambda(t), \gamma(t))$, and if μ is a path in $X \times Y$, define $\pi_X(\mu)$ and $\pi_Y(\mu)$ by $\mu(t) = (\pi_X(\mu)(t), \pi_Y(\mu)(t))$. Given a path ρ of finite length, its length will be denoted by $l(\rho)$, and the variation of ρ over a partition P of $[0, 1]$ will be denoted by $l(\rho; P)$. We assert that, if $\pi_X(\mu)$ and $\pi_Y(\mu)$ have finite length, then $l(\mu) = l(\pi_X(\mu)) + l(\pi_Y(\mu))$. Since $l(\mu; P) = l(\pi_X(\mu); P) + l(\pi_Y(\mu); P)$, clearly $l(\mu) \leqq l(\pi_X(\mu)) + l(\pi_Y(\mu))$. The reverse inequality is proved by choosing $\varepsilon > 0$, finding partitions P_X and P_Y which approximate $l(\pi_X(\mu))$ and $l(\pi_Y(\mu))$ to within $\varepsilon/2$, letting P denote the common refinement of P_X and P_Y will yield $l(\mu) + \varepsilon > l(\pi_X(\mu)) + l(\pi_Y(\mu))$. If λ and γ are geodesics in X and Y respectively, then $l(\lambda \times \gamma) = l(\lambda) + l(\gamma)$. If μ is a path in $X \times Y$ with $l(\mu) < l(\lambda \times \gamma)$, then $l(\pi_X(\mu)) + l(\pi_Y(\mu)) < l(\lambda) + l(\gamma) \Rightarrow$ either $l(\pi_X(\mu)) < l(\lambda)$ or $l(\pi_Y(\mu)) > l(\gamma)$, a contradiction. Similarly, if $l(\mu) = l(\lambda \times \gamma)$, then $l(\pi_X(\mu)) = l(\lambda)$ and $l(\pi_Y(\mu)) = l(\gamma)$, which shows that $\pi_X(\mu) = \lambda$ and $\pi_Y(\mu) = \gamma$, so $\mu = \lambda \times \gamma$ and geodesics in $(X \times Y, d'')$ are unique.

In conclusion, it is interesting to note that the hypothesis that $S(x_0, \varepsilon)$ is convex is clearly analogous to the assumption that a linear topological space is locally convex. It is clear that many standard concepts in linear topogical space theory are geodesically defined (such as an absorbing neighbourhood), and consequently are easily translated into the theory of spaces with unique segments. In addition to the question of which theorems from normed linear spaces hold in suitably restricted spaces with unique segments, the following problem arises quite naturally: is there a category whose objects include both linear topological spaces and spaces with unique segments, and whose morphisms are the "affine" maps? A reasonable condidate would be topological spaces X with a geodesic-like structure such as a map $\Phi: X \times X \times I \rightarrow X$ (where $I = [0, 1]$) satisfying such conditions as (1) for fixed $x, y \in X$, $\Phi(x, y, t)$ is a continuous function of t, (2) $\Phi(x, x, t) = x$, and (3) $\Phi(\Phi(x, y, s), \Phi(x, y, t), u) = \Phi(x, y, (1 - u)s + ut)$. Some work has been done on spaces with convexity structures, but from a purely algebraic standpoint. It would be extremely interesting to find out to what extent such theorems as the Open Mapping Theorem and the Uniform Boundedness Theorem can be generalized.

REFERENCES

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