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MATRIX INEQUALITIES AND KERNELS OF LINEAR TRANSFORMATIONS

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Let V be a finite dimensional unitary space and $\otimes^m V$ the unitary space of m-contravariant tensors based on V with the inner product induced from V. If T is a linear transformation on $\otimes^m V$ to itself and $X = (x_i, x_j)$ any positive semidefinite hermitian matrix define

$$d^T(X) = \| T(x_1 \otimes \cdots \otimes x_m \|^2 .$$

Let $\| \ \|_1$ be any norm on the space of $m \times m$ complex matrices, and $\mathscr{T} = \{x_1 \otimes \cdots \otimes x_m \colon x_i \in V\}$. The main result is that if T and S are any two linear transformations on $\otimes^m V$ to itself then the following are equivalent:
(a) $\ker(T) \cap \mathscr{T} \subseteq \ker(S) \cap \mathscr{T}$
(b) If X is positive semidefinite hermitian and $d^T(X) = 0$ then $d^S(X) = 0$.
(c) There exists a positive integer k and a constant $c > 0$ such that for all positive semidefinite hermitian matrices X

$$c \| X \|_1^{m(k-1)} d^T(X) \geqq (d^S(X))^k .$$

Some applications to inequalities for generalized matrix functions are given.

1. Introduction. Let V be a finite dimensional unitary space with inner product (,) and $\otimes^m V$ the space of m-contravariant tensors based on V. The inner product on V induces an inner product on $\otimes^m V$ as follows. If $x_1, \cdots, x_m; y_1, \cdots, y_m \in V$ define

$$(x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m) = \prod_{i=1}^m (x_i, y_i) .$$

Since the elements of the form $z_1 \otimes \cdots \otimes z_m$, $z_i \in V$, span $\otimes^m V$ we may extend the above to all of $\otimes^m V$ by conjugate bilinear extension and it is easy to check that this does define an inner product.

Let S_m be the symmetric group of degree m and suppose $\sigma \in S_m$. We define a linear map $P(\sigma) \colon \otimes^m V \to \otimes^m V$ by

$$P(\sigma)(x_1 \otimes \cdots \otimes x_m) = x_{\alpha(1)} \otimes \cdots \otimes x_{\alpha(m)} \ (\alpha = \sigma^{-1})$$

and linear extension. If G is a subgroup of S_m and χ an irreducible character on G let

$$T_\chi^G = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) P(g) . \qquad (|G| = \text{order of } G) .$$

It is not difficult to verify that T_χ^G is an idempotent hermitian operator and hence a projection onto its range. T_χ^G is called a symmetry operator. If $\underline{\bar{X}} = (x_{ij})$ is a m-square complex matrix then the generalized matrix function associated with G and χ, d_χ^G is defined by

$$d_\chi^G(\underline{\bar{X}}) = \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g)\prod_{i=1}^m x_{ig(i)} .$$

If $\underline{\bar{X}}$ is positive semidefinite hermitian then $\underline{\bar{X}}$ is a Gram matrix based on some set of vectors $\{x_1, \cdots, x_m\}$. That is, $x_{ij} = (x_i, x_j)$. A simple computation shows that

$$\| T_\chi^G(x_1 \otimes \cdots \otimes x_m)\|^2 = d_\chi^G(\underline{\bar{X}}) .$$

Hence the generalized matrix functions may be interpreted as norms of certain elements in a subspace of $\bigotimes^m V$.

In 1918 I. Schur [4] proved that G is any subgroup of S_m and χ is a character of degree r then

$$d_\chi^G(\underline{\bar{X}}) \geqq r \det(\underline{\bar{X}}) \text{ for all positive semidefinite } \underline{\bar{X}} .$$

It is easy to see that the determinant arises from the symmetry class associated with S_m and the alternating character. Recently Marcus [1, 3], Williamson [5] and others have discovered several inequalities of Schur type, that is, if G and H are subgroups of S_m and χ and ν are irreducible characters on G and H respectively then for certain choices of G, H, χ and ν there exists a constant $c > 0$ such that $c\, d_\chi^G(\underline{\bar{X}}) \geqq d_\nu^H(\underline{\bar{X}})$ for all positive semidefinite $\underline{\bar{X}}$. It is clear that if such an inequality exists then $d_\chi^G(\underline{\bar{X}}) = 0$ implies that $d_\nu^H(\underline{\bar{X}}) = 0$. It is the purpose of this note to give a type of converse to this result.

If T is a linear transformation on $\bigotimes^m V$ to itself and $\underline{\bar{X}} = (x_{ij})$ is a positive semidefinite hermitian matrix, choose $x_1, \cdots, x_m$ in V such that $x_{ij} = (x_i, x_j)$ and define

$$d^T(\underline{\bar{X}}) = \| T(x_1 \otimes \cdots \otimes x_m)\|^2 .$$

Further, let $\mathscr{T}$ be the set of all elements in $\bigotimes^m V$ of the form $x_1 \otimes \cdots \otimes x_m$ with x_i in V. The main result of this note is the following:

THEOREM. *Let $\|\ \|$ be any norm on the vector space of m-square complex matrices. If T and S are any two linear transformations on $\bigotimes^m V$ to itself then the following are equivalent*

(1) $\ker(T) \cap \mathscr{T} \subseteqq \ker(S) \cap \mathscr{T}$

(2) *If $\underline{\bar{X}}$ is positive semidefinite hermitian and $d^T(\underline{\bar{X}}) = 0$ then $d^S(\underline{\bar{X}}) = 0$*

(3) *There exists a positive integer k and a constant $c > 0$ such that for all positive semidefinite hermitian matrices $\underline{\bar{X}}$*

$$c\,||\underline{\bar{X}}||^{m(k-1)} d^T(\underline{\bar{X}}) \geqq (d^S(\underline{\bar{X}}))^k .$$

In the case that T and S are symmetry operators this result shows that a knowledge of $\ker(T) \cap \mathscr{T}$ and $\ker(S) \cap \mathscr{T}$ would allow us to decide whether an inequality of type (3) exists or not: For example, if χ is identically equal to one then one may show that $\ker\,(T_\chi^G) \cap \mathscr{T} = (0)$ hence there is always an inequality of type (3). Unfortunately the determination of $\ker(T_\chi^G)$ is a difficult problem and other than when the character is identically one or T_χ^G is the alternating operator little is known in this direction.

2. *Proof of Theorem.* Let $e_1, \cdots, e_n$ be an orthonormal basis for V and let $\Gamma = \{\omega = (\omega_1, \cdots, \omega_m): 1 \leqq \omega_i \leqq n,\ \omega_i$ and integer$\}$. It is known that the set $\{e_\omega = e_{\omega_1} \otimes \cdots \otimes e_{\omega_m}: \omega \in \Gamma\}$ is an orthonormal basis for $\bigotimes^m V$. If $x_j = \sum_{i=1}^n x_{ij} e_i$ then one computes, using the properties of the space of m-contravariant tensors, that

$$x_1 \otimes \cdots \otimes x_m = \sum_{\omega \in \Gamma} p_\omega e_\omega ;\ p_\omega = \prod_{i=1}^m x_{i\omega_i} .$$

If the x_i are considered as variable vectors then the p_ω are just polynomials in the mn unknowns x_{ij}.

Let $T(e_\omega) = \sum_{\tau \in \Gamma} t_{\tau\omega} e_\tau$ and $S(e_\omega) = \sum_{\tau \in \Gamma} s_{\tau\omega} e_\tau$ then

$$T(x_1 \otimes \cdots \otimes x_m) = \sum_{\tau \in \Gamma} \Big(\sum_{\omega \in \Gamma} t_{\tau\omega} p_\omega \Big) e_\tau$$

and

$$S(x_1 \otimes \cdots \otimes x_m) = \sum_{\tau \in \Gamma} \Big(\sum_{\omega \in \Gamma} s_{\tau\omega} p_\omega \Big) e_\tau .$$

Set $f_\tau = \sum_{\omega \in \Gamma} t_{\tau\omega} p_\omega$ and $g_\tau = \sum_{\omega \in \Gamma} s_{\tau\omega} p_\omega$, then

$$||\,T(x_1 \otimes \cdots \otimes x_m)\,||^2 = \sum_{\tau \in \Gamma} |f_\tau|^2$$

and

$$||\,S(x_1 \otimes \cdots \otimes x_m)\,||^2 = \sum_{\tau \in \Gamma} |g_\tau|^2 .$$

Hence, if (2) holds then $f_\tau = 0$ for all $\tau \in \Gamma$ implies that $g_\tau = 0$ for all $\tau \in \Gamma$.

Let J be the ideal in $\boldsymbol{C}[x_{11} \cdots x_{mn}]$ generated by the polynomials f_τ, $\tau \in T$ and $V = \{(a_{11} \cdots a_{mn}) \in \boldsymbol{C}^{mn}: f(a_{11} \cdots a_{mn}) = 0$ for all $f \in J\}$. Applying Hilbert's Nullstellensatz we conclude that

$$g_\tau \in \text{rad } J = \{h: h^k \in J \text{ for some positive integer } k\} .$$

Therefore there exists a positive integer k_τ such that $g^{k_\tau} \in J$. If k is the least common multiple of all the integers k_τ $(\tau \in \Gamma)$ then $g_\tau^k \in J$ for all τ in Γ so there exist $q_{\tau\omega} \in C[x_{11} \cdots x_{mn}]$ such that

$$g_\tau^k = \sum q_{\tau\omega} f_\omega .$$

Let $K = \{(a_{11} \cdots a_{mn}) \in C^{mn}$: if $A = (a_{ij})$ then A is positive semidefinite hermitian and $||A|| = 1\}$. Then K is a compact set in C^{mn} since the set of positive semidefinite hermitian matrices is closed.

Set $c_{\tau\omega} = \sup_{Z \in K} |q_{\tau\omega}| < \infty$ (since $q_{\tau\omega}$ is continuous) and $a = \max_{\omega,\tau \in \Gamma} c_{\tau\omega}^2$. Then on K

$$\begin{aligned} |g_\tau^k|^2 &= \Big|\sum_{\omega \in \Gamma} q_{\tau\omega} f_\omega\Big|^2 \\ &\leqq \sum_{\omega \in \Gamma} |q_{\tau\omega}|\,|f_\omega|^2 \\ &\leqq a \sum_{\omega \in \Gamma} |f_\omega|^2 . \end{aligned}$$

Now note that Γ contains n^m elements and apply inequality 2.4.6 in [2, p. 105] to obtain

$$\begin{aligned} \Big(\sum_{\tau \in \Gamma} |g_\tau|^2\Big)^k &\leqq n^{m(k-1)} \sum_{\tau \in \Gamma} |g_\tau^2|^k \\ &\leqq a\, n^m \sum_{\omega \in \Gamma} |f_\omega|^2 \text{ on } K . \end{aligned}$$

Letting $c = an^m$ the above becomes $(d_s(\underline{\bar{X}}))^k \leqq c\, d^T(\underline{\bar{X}})$ for all positive semidefinite hermitian matrices $\underline{\bar{X}}$ such that $||\underline{\bar{X}}|| = 1$. Now note that a simple calculation shows that $d^S(a\underline{\bar{X}}) = a^m d^S(\underline{\bar{X}})$ for $a \geqq 0$ and similarly for d^T, therefore if $\underline{\bar{X}} \neq 0$, $||1/||\underline{\bar{X}}||\, \underline{\bar{X}}|| = 1$, so

$$\begin{aligned} \frac{1}{||X||^{mk}} (d^S(\underline{\bar{X}}))^k &= \Big(d^S\Big(\frac{1}{||\underline{\bar{X}}||}\underline{\bar{X}}\Big)\Big)^k \\ &\leqq c\, d^T\Big(\frac{1}{||\underline{\bar{X}}||}\underline{\bar{X}}\Big) \\ &= \frac{1}{||\underline{\bar{X}}||^m} d^T(\underline{\bar{X}}) . \end{aligned}$$

Hence $c\, ||\underline{\bar{X}}||^{m(k-1)}\, d^T(\underline{\bar{X}}) \geqq (d^S(\underline{\bar{X}}))^k$ if $\underline{\bar{X}} \neq 0$. However, if $\underline{\bar{X}} = 0$ both sides are equal to zero so the result is trivial. This establishes that (2) implies (3).

The implications (1) if and only if (2) and (3) implies (2) are trivial.

3. Applications. Let $G < S_m$ be a finite group and

$$M\colon G \to GL(C, k)$$

an irreducable representation of G with character χ. If

$$Z(M) = \{g \in G\colon\ M(g)M(x) = M(x)M(g) \text{ for all } x \in G\}$$

then it is well known that $Z(M)$ consists of these elements of G whose image under the representation M are scalar matrices. If $g \in Z(M)$ and $x \in G$ then clearly $\chi(gx) = 1/k\,\chi(g)\chi(x)$. Suppose $H < Z(M)$, let $T_H^\chi = \chi(1)/|H| \sum_{h \in H} \chi(h^{-1})P(h)$ and $T_G^\chi = \chi(g)/|G| \sum_{h \in H} \chi(g^{-1})P(g)$ be the symmetry operators associated with G and H respectively. A simple computation, using the orthogonality relations on characters establishes that

$$T_G^\chi\ T_H^\chi = T_G^\chi\ .$$

Hence it follows that ker $T_G^\chi \supset k$er T_H^χ and we may conclude from the theorem that an inequality of type (3) exists.

In the case that the character χ is linear Williamson [5] showed that if $H < G < S_m$ then for all positive semidefinite hermitian matrices X there exists a constant $c > 0$ such that

$$c\, d_\chi^H(\bar{X}) \geqq d_\chi^G(\bar{\underline{X}})\ .$$

Further, Williamson gives a technique of computing the constant c. In a certain sense then, our results include Williamson's although they are purely of an existence type while his are computable.

In particular, if we choose H to be the group consisting of the identity alone then certainly $H < Z(M)$ and so there exists a constant $c > 0$ and a positive integer k such that

$$c\, \|\bar{\underline{X}}\|^{m(k-1)} \prod_{i=1}^{m} x_{ii} \geqq (d_\chi^G(\bar{\underline{X}}))^k$$

for any positive semidefinite hermitian matrix $\bar{\underline{X}}$.

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