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THE CLASS OF (p, q)-BIHARMONIC FUNCTIONS

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Our main interest in this paper is with the (p, q)-biharmonic boundary value problem, which takes the following form: Given continuous functions ϕ and Ψ on Wiener's or Royden's p-and q-harmonic boundaries α and β respectively, find a function u satisfying $(\Delta + q)(\Delta + p)u = 0$ and

$$u \mid \alpha = \varphi , \qquad u \mid \beta = \Psi .$$

We shall solve this problem by what we call the (p, q)-biharmonic projection.

In §1 we give some preliminary results. The (p, q)-biharmonic projection is introduced in §2 for various classes of functions, and in §4 for suitably restricted Riemannian manifolds. In §3 we characterize classes of manifolds with respect to significant subclasses of (p, q)-quasiharmonic functions by means of the p-harmonic Green's function and the q-elliptic measure on R. The (p, q)-quasiharmonic nondegeneracies of the manifold are the various conditions we impose on R in §4. Finally in §5 we give some explicit results concerning certain classes of density functions.

1. On a smooth noncompact Riemannian manifold R of dimension $m \geqq 2$ with a smooth metric tensor (g_{ij}), the Laplace-Beltrami operator is given by

$$\Delta \cdot = - \frac{1}{\sqrt{g}} \sum_{i=1}^{m} \frac{\partial}{\partial x^i} \sum_{j=1}^{m} \sqrt{g}\, g^{ij} \frac{\partial \cdot}{\partial x^j} ,$$

where $x = (x^1, \cdots, x^m)$ is a local coordinate system, $g = \det (g_{ij})$, and $(g^{ij}) = (g_{ij})^{-1}$. Let $p(x)$ be a density function, that is, a nonnegative C^2 function on R. A p-harmonic function is a C^2 solution of the equation $\Delta_p u = 0$ with

$$\Delta_p = \Delta + p .$$

We call a C^4 function (p, q)-*biharmonic* if it satisfies the equation

$$\Delta_q \Delta_p u = 0 ,$$

and we denote by $W_{pq} = W_{pq}(R)$ the family of (p, q)-biharmonic functions on R. An important subclass of W_{pq} is the class $Q_{pq} = Q_{pq}(R)$ of (p, q)-*quasiharmonic* functions, i.e., the C^2 solutions of $\Delta_p u = e_q$, where e_q is the q-elliptic measure on R (see No. 2).

Note that for $p \equiv q \equiv 0$, W_{pq} and Q_{pq} reduce to the classes W and Q of biharmonic and quasiharmonic functions respectively. For these classes our problem was solved in [5]-[7] and [11], which have greatly influenced our reasoning.

1. Auxiliary results.

2. Let Ω be a regular subregion of R, and $h_1^{p\Omega}$ the continuous function on R which is p-harmonic on Ω and 1 on $R - \Omega$. The limit e_p of the decreasing sequence $\{h_1^{p\Omega}\}$ as $\Omega \to R$ is called the *p-elliptic measure* of R. Clearly e_p is nonnegative and p-harmonic on R, with $0 \leqq e_p \leqq 1$. Explicitly, it is either identically zero or strictly positive. In particular, it is identically 1 if $p \equiv 0$. In the case $p \not\equiv 0$, we call a Riemannian manifold R *p-parabolic* if $e_p = 0$, and *p-hyperbolic* if $e_p > 0$. As in the case $p \equiv 0$, we shall follow the convention adopted by Royden [9] that R is called 0-parabolic if and only if R is parabolic.

3. The harmonic Green's function $g(x, y)$ on R exists only on a hyperbolic manifold. In contrast, the p-harmonic Green's function $g_p(x, y)$ for $p \not\equiv 0$ exists on every Riemannian manifold. Thus on an arbitrary Riemannian manifold R, hyperbolic if $p \equiv 0$, the operator G_p is well defined on the family of continuous functions by

$$G_p f = \int_R g_p(\cdot, y) f(y) dy ,$$

with dy the volume element of R. We are interested in the class

$$F_{p1} = \{f \mid G_p |f| < \infty\} .$$

LEMMA 1. *Let R be an arbitrary Riemannian manifold (hyperbolic if $p \equiv 0$). If $f \in C^\infty \cap F_{p1}$, then $\Delta_p G_p f = f$.*

Proof. For every $\varphi \in C_0^\infty$, we have

$$\int_R G_p f(x) \cdot \Delta_p \varphi(x) dx = \int_R G_p \Delta_p \varphi(y) \cdot f(y) dy$$
$$= \int_R \varphi(y) \cdot f(y) dy .$$

Therefore $\Delta_p G_p f = f$ in the sense of distributions, and the lemma follows by the hypoellipticity of Δ_p.

4. Let $M_{p1}(R)$ be the class of continuous p-harmonizable functions for which there is a continuous p-superharmonic function s_f with $s_f \geqq |f|$ on R, and $N_{p1}(R)$ the potential p-subalgebra of $M_{p1}(R)$, i.e.,

the family of functions in $M_{p1}(R)$ whose p-harmonic part $h_f^p = \lim_{\Omega\to R} h_f^{p\Omega}$ vanishes identically on R.

LEMMA 2. *Let R be an arbitrary Riemannian manifold (hyperbolic if $p \equiv 0$). If $f \in C^\infty \cap F_{p1}$, then $G_p f \in N_{p1}(R)$.*

Proof. Set $f = f^+ - f^-$ with $f^+ = f \cup 0$ and $f^- = -f \cup 0$. Clearly $G_p f^+$ and $G_p f^-$ are nonnegative and p-superharmonic on R. In view of $|G_p f| \leqq G_p f^+ + G_p f^-$, $G_p f \in M_{p1}(R)$.

It remains to show that $h^p_{G_p f} = 0$. Let Ω be a regular subregion of R and $g_{p\Omega}(x, y)$ the p-harmonic Green's function on Ω with value zero on $R - \Omega$. For a parametric disk $B_x \subset R$ about $x \in \Omega$ with radius ε, the Green's formula yields

$$\int_{\partial(\Omega - B_x)} \{[G_p f(y) - h^{p\Omega}_{G_p f}(y)] * dg_{p\Omega}(x, y) - g_{p\Omega}(x, y) * d[G_p f(y) - h^{p\Omega}_{G_p f}(y)]\}$$
$$= \int_{\Omega - B_x} g_{p\Omega}(x, y) \Delta_p G_p f(y) dy .$$

On letting $\varepsilon \to 0$ and then $\Omega \to R$ we obtain

$$G_p f(x) = h^{p\Omega}_{G_p f}(x) + \int_R g_{p\Omega}(x, y) \Delta_p G_p f(y) dy$$

and by Lemma 1,

$$G_p f = h^p_{G_p f} + G_p f .$$

Therefore $h^p_{G_p f} = 0$ and consequently $G_p f \in N_{p1}(R)$.

5. Denote by $H^p(R)$ the class of p-harmonic functions on R, and let $E(u)$ be the energy integral

$$E(u) = \int_R du \wedge * du + \int_R p(x) u^2(x) dx .$$

LEMMA 3. *The energy integral is lower semicontinuous:*

$$E(u_0) \leqq \varliminf_{n\to\infty} E(u_n)$$

for every sequence $\{u_n\}$ in $H^p(R)$ converging uniformly to u_0 on compacta of R.

Proof. For $x_0 \in R$ and a parametric ball $B \subset R$ about x_0,

$$u_n(x) = -\int_{\partial B} u_n(y) \frac{\partial g_{pB}(x, y)}{\partial n_y} dS_y , \qquad n = 0, 1, 2, \cdots ,$$

with $x \in B$, g_{pB} the p-Green's function on B, $\partial g_{pB}/\partial n$ the normal derivative of g_{pB}, and dS the surface element of ∂B. Then

$$\begin{aligned}\frac{\partial u_n(x)}{\partial x^i} &= -\int_{\partial B} u_n(y)\frac{\partial}{\partial x^i}\frac{\partial g_{pB}(x, y)}{\partial n_y}dS_y \\ &\longrightarrow -\int_{\partial B} u_0(y)\frac{\partial}{\partial x^i}\frac{\partial g_{pB}(x, y)}{\partial n_y}dS_y = \frac{\partial u_0(x)}{\partial x^i}\end{aligned}$$

as $n \to \infty$. Therefore $\partial u_n(x)/\partial x^i \to \partial u_0(x)/\partial x^i$ uniformly on every parametric ball as $n \to \infty$. The uniform convergence on compacta of R is a consequence of the fact that every compact set can be covered by a finite number of parametric balls. Clearly

$$F_\Omega(u_0) = \lim_{n\to\infty} E_\Omega(u_n) \leqq \varliminf_{n\to\infty} E(u_n)$$

for every relatively compact set Ω. The lemma follows as $\Omega \to R$.

6. Consider the real-valued linear operator $G_p(\cdot, \cdot)$ on $C \times C$ defined by

$$G_p(f, g) = \int_{R\times R} g_p(x, y)f(x)g(y)dxdy$$

for $f, g \in C$.

LEMMA 4. *If $f \in C^\infty$, then*

$$E(G_pf) = G_p(f, f)$$

whenever the right-hand side is finite.

Proof. Let

$$G_{p\Omega}f = \int_R g_{p\Omega}(\cdot, y)f(y)dy .$$

We have

$$E(G_{p\Omega}f) = \int_{R\times R} g_{p\Omega}(x, y)f(x)f(y)dxdy .$$

By the p-harmonicity of $G_pf - G_{p\Omega}f$ on Ω and the lower semicontinuity of E,

$$E(G_pf) \leqq \varliminf_{\Omega\to R} E(G_{p\Omega}f) \leqq G_p(|f|, |f|) < \infty .$$

In view of Lebesgue's convergence theorem,

$$E(G_pf) = \lim_{\Omega\to R} E(G_{p\Omega}f)$$

$$= \lim_{\Omega \to R} \int_{R \times R} g_{p\Omega}(x, y) f(x) f(y) dx dy$$
$$= \int_{R \times R} g_p(x, y) f(x) f(y) dx dy$$
$$= G_p(f, f) .$$

2. The (p, q)-biharmonic projection.

7. As a preparation for the (p, q)-biharmonic projection, we introduce a number of families of functions on R. Let $M_{p3}(R)$ be the class of continuous functions with finite energy integrals; $M_{p2}(R)$ and $M_{p4}(R)$ the Wiener and the Royden p-algebras on R; and $N_{pi}(R)$ the potential p-subalgebra of $M_{pi}(R)$ for $i = 2, 3, 4$, (cf. [3], [10], and [11]). We shall often omit R and write M_{pi} and N_{pi} instead of $M_{pi}(R)$ and $N_{pi}(R)$. For the sake of simplicity, we set $X_T = \{f \mid Tf \in X\}$ and $XY = X \cap Y$ for given classes of functions X, Y, and a given operator T. Furthermore, we write $M_{ij} = M_{pi}(M_{qj})_{\Delta_p}$ and $N_{ij} = N_{pi}(N_{qj})_{\Delta_p}$ for all i, j. Let P', B, and E be the classes of essentially positive functions, bounded functions, and functions with finite energy integrals respectively. Set $H_{p1} = H^p P'$, $H_{p2} = H^p B$, $H_{p3} = H^p E$, and $H_{p4} = H^p K$, where $K = BE$. It is known that the direct sum decompositions $M_{pi} = H_{pi} \oplus N_{pi}$ are valid for all i. The p-harmonic part of a function $f \in M_{pi}$ is called the p-harmonic projection of M_{pi} and denoted by $\pi_{pi} f$. It is also known that the decompositions are orthogonal in the sense that $E(f) = E(\pi_{pi} f) + E(f - \pi_{pi} f)$ for $f \in M_{pi}$ and $i = 3, 4$, (cf. e.g. [10]). Let

$$F_{p2} = \{f \in F_{p1} \mid \sup_R |G_p f| < \infty\} ,$$
$$F_{p3} = \{f \in F_{p1} \mid G_p(|f|, |f|) < \infty\} ,$$
$$F_{p4} = F_{p2} \cap F_{p3} ,$$

and

$$\Phi_{ij} = M_{ij}(F_{pi})_{\pi_{qj}\Delta_p} , \qquad i, j = 1, 2, 3, 4 .$$

THEOREM 1. *On an arbitrary Riemannian manifold R (hyperbolic if $p \equiv 0$), the functions in Φ_{ij} have a unique decomposition into (p, q)-biharmonic functions and (p, q)-potentials:*

$$\Phi_{ij} = W_{pq}\Phi_{ij} \oplus N_{ij}\Phi_{ij} .$$

Proof. Let $f \in \Phi_{ij}$. By the decomposition theorem of M_{pi} and M_{qj}, $f = \pi_{pi} f + h_i$ with $\pi_{pi} f \in H_{pi}$ and $h_i \in N_{pi}$, $\Delta_p f = \pi_{qj}\Delta_p f + k_j$ with $\pi_{qj}\Delta_p f \in H_{qj}$ and $k_j \in N_{qj}$. Since $\pi_{qj}\Delta_p f \in F_{pi}$ and $F_{pi} \subset F_{p1}$, the function $w_{ij} = \pi_{pi} f + G_p \pi_{qj} \Delta_p f$ is well defined. By Lemmas 1 and 2, we see that $w_{ij} \in W_{pq}$ for all i, j, and $w_{ij} \in \Phi_{ij}$ for $i = 1, 2$, and all j. In view of Lemma 4,

$$E(w_{ij}) \leqq E(\pi_{pi}f) + E(G_p\pi_{qj}\Delta_p f) = E(\pi_{pi}f) + G_p(\pi_{qj}\Delta_p f, \pi_{qj}\Delta_p f) < \infty$$

for $i = 3, 4$. Therefore $w_{ij} \in W_{pq}\Phi_{ij}$ for all i, j. It remains to show that $f - w_{ij} \in N_{ij}\Phi_{ij}$. Clearly $\Delta_p(f - w_{ij}) = k_j \in N_{qj}$ and $\pi_{qj}\Delta_p(f - w_{ij}) = 0$. By Lemma 2, $f - w_{ij} = h_i - G_p\pi_{qj}\Delta_p f \in N_{pi}$. Therefore $w_{ij} + (f - w_{ij})$ is the desired decomposition.

To prove the uniqueness, let $v \in W_{pq}\Phi_{ij} \cap N_{ij}\Phi_{ij}$. Since $\Delta_p v \in H_{qj} \cap N_{qj} = \{0\}$, $v \in H_{pi} \cap N_{pi}$, and consequently $v \equiv 0$ on R.

We call the function $w_{ij} \in W_{pq}\Phi_{ij}$ in Theorem 1 the *(p, q)-biharmonic projection* of $f \in \Phi_{ij}$. It is the solution of the (p, q)-biharmonic Dirichlet problem with

$$w_{ij} \mid \beta_i = f \mid \beta_i \quad \text{and} \quad \Delta_p w_{ij} \mid \beta_j = \Delta_p f \mid \beta_j \,,$$

where β_i and β_j are the p-and q-harmonic boundaries corresponding to M_{pi} and M_{qj} respectively. From the uniqueness of the decomposition, we see that the solution is unique except for the cases $i = 1$ or $j = 1$. In these cases there exist singular p-harmonic functions which vanish on the p-harmonic boundary.

3. (p, q)-quasiharmonic classification of Riemannian manifolds.

8. The (p, q)-biharmonic projection was obtained in Theorem 1 for certain restricted families of functions on arbitrary Riemannian manifold. In order to relax the conditions on the families, it is necessary to impose conditions on the manifold. We shall see that such conditions are intimately related to the (p, q)-quasiharmonic classification of manifolds.

Denote by O_X the class of Riemannian manifolds on which there exist no X-functions, and by P the class of positive functions. The various (p, q)-quasiharmonic null-manifolds are determined completely by the p-harmonic Green's function and the q-elliptic measure:

THEOREM 2. *On a q-hyperbolic Riemannian manifold R (hyperbolic if $p \equiv 0$),*

(i) $R \notin O_{Q_{pq}P}$ *if and only if* $G_p e_q < \infty$,
(ii) $R \notin O_{Q_{pq}B}$ *if and only if* $\sup_R G_p e_q < \infty$,
(iii) $R \notin O_{Q_{pq}E}$ *if and only if* $G_p(e_q, e_q) < \infty$,
(iv) $R \notin O_{Q_{pq}K}$ *if and only if* $\sup_R G_p e_q < \infty$ *and* $G_p(e_q, e_q) < \infty$.

Proof. For every $u \in Q_{pq}$ and every regular subregion $\Omega \subset R$,

$$u(x) = h_u^{p\Omega}(x) + \int_R g_{p\Omega}(x, y)e_q(y)dy \,.$$

Suppose $R \notin O_{Q_{pq}P}$, i.e., there exists a $v \in Q_{pq}P$. Clearly v is p-superharmonic and bounded from below on R. Therefore $h_u^p = \lim_{\Omega\to R} h_u^{p\Omega}$ exists. By the monotone convergence theorem,

$$G_p e_q = \lim_{\Omega\to R} \int_R g_{p\Omega}(\cdot, y)e_q(y)dy = u - h_u^p < \infty .$$

Conversely, $G_p e_q \in Q_{pq}P$, and (i) follows. Relation (ii) is established in a similar manner.

Suppose $R \notin O_{Q_{pq}E}$ and take a $v \in Q_{pq}E$. For every regular subregion $\Omega \subset R$,

$$v = h_v^{p\Omega} + G_{p\Omega}e_q$$

and

$$E(v) = E(h_v^{p\Omega}) + E(G_{p\Omega}e_q) .$$

As in the proof of Lemma 4,

$$E(G_{p\Omega}e_q) = G_{p\Omega}(e_q, e_q) .$$

The monotone convergence theorem yields

$$\lim_{\Omega\to R} G_{p\Omega}(e_q, e_q) = G_p(e_q, e_q) .$$

Since $G_p(e_q) - G_{p\Omega}(e_q)$ is p-harmonic on Ω, Lemma 2 implies

$$E(G_p e_q) \leqq \varliminf_{\Omega\to R} E(G_{p\Omega}e_q) \leqq E(v) < \infty .$$

By Lebesgue's convergence theorem,

$$G_p(e_q, e_q) = E(G_p e_q) < \infty .$$

Conversely, if $G_p(e_q, e_q) < \infty$, then $G_p e_q < \infty$ and $\Delta_p G_p e_q = e_q$. By virtue of

$$E(G_p e_q) \leqq \varliminf_{\Omega\to R} E(G_{p\Omega}e_q) \leqq G_p(e_q, e_q) < \infty ,$$

$G_p e_q \in Q_{pq}E$ and (iii) follows. The last assertion of Theorem 2 is an immediate consequence of (ii) and (iii).

9. An important bi-product of the proof of Theorem 1 is that the (p, q)-biharmonic functions restricted to the class Φ_{ij} can be uniquely decomposed into the p-harmonic part and the potential part:

THEOREM 3. *On an arbitrary Riemannian manifold R (hyperbolic if $p \equiv 0$), every function $w_{ij} \in W_{pq}\Phi_{ij}$ can be uniquely written as*

$$w_{ij} = u_i + G_p v_j ,$$

with $u_i \in H_{pi}$ *and* $v_j \in H_{qj}$ *for* $i, j = 1, 2, 3, 4$.

4. Nondegenerate manifolds.

10. We shall show that, by imposing a suitable condition on the manifold R, the restrictions we have set on the functions which have (p, q)-biharmonic projections can be relaxed.

We write $X_1 = P$, $X_2 = B$, $X_3 = E$, $X_4 = K$, and we let $W_{pq}X_1(X_j)_{\Delta_p}$ stand for $W_{pq}M_{p1}(X_j)_{\Delta_p}$.

THEOREM 4. *On a Riemannian manifold which carries* $Q_{pq}X_i$-*functions,*

$$M_{ij} = W_{pq}X_i(X_j)_{\Delta_p} \oplus N_{ij}$$

with $i = 1, 2, 3, 4$, *and* $j = 2, 4$.

Proof. It is sufficient to show that $f \in W_{pq}B_{\Delta_p}$ implies the p-harmonizability of f on $R \notin O_{Q_{pq}P}$.

For every regular subregion Ω of R, and every $f \in W_{pq}B_{\Delta_p}$,

$$f = h_f^{p\Omega} + \int_R g_{p\Omega}(\cdot, y)\Delta_p f(y)dy .$$

Since $R \notin O_{Q_{pq}P}$, $G_p e_q < \infty$ by Theorem 2. In view of $|g_{p\Omega} \cdot \Delta_p f| \leqq k \cdot g_p e_q$ for some constant k, the Lebesgue convergence theorem implies the existence of the limit of $\lim_{\Omega \to R}\int_R g_{p\Omega}(\cdot, y)\Delta_p f(y)dy$. Thus $h_f^{p\Omega}$ converges, and f is p-harmonizable.

11. With suitable conditions imposed on the manifold, we have the following direct sum decompositions of (p, q)-biharmonic functions:

THEOREM 5. *On a Riemannian manifold R which carries positive* Q_{pq}-*functions,*

$$W_{pq}X_i(X_j)_{\Delta_p} \subset H_{pi} \oplus G_p H_{qj}$$

with $i = 1, 2, 3, 4$, *and* $j = 2, 4$. *Moreover,*

$$W_{pq}X_i(X_j)_{\Delta_p} = H_{pi} \oplus G_p H_{qj}$$

if and only if $R \notin O_{Q_{pq}}X_i$.

The proof makes use of Theorems 2, 3, and 4.

On a manifold $R \notin O_{Q_{pq}X_i}$, let φ, ψ be continuous functions on the harmonic boundaries β_i and β_j corresponding to M_{pi} and M_{qj} respectively. The second assertion of Theorem 5 implies that if h_φ^p and h_ψ^q are solutions of the p-and q-harmonic boundary value problems with boundary values φ and ψ respectively, then our (p, q)-biharmonic Dirichlet problem has a solution which is in $W_{pq}X_i(X_j)_{\Delta_p}$ and takes the form $h_\varphi^p + G_p h_\psi^q$.

5. Special density functions.

12. In the case that the density function is bounded from below by a positive constant, we have more explicit results:

THEOREM 6. *If* $\inf_R p(x) > 0$ *on a q-hyperbolic Riemannian manifold, then*

$$M_{ij} = W_{pq}X_i(X_j)_{\Delta_p} \oplus N_{ij}$$

and

$$W_{pq}X_i(X_j)_{\Delta_p} = H_{pi} \oplus G_p H_{qj}$$

with $i = 1, 2,$ *and* $j = 2, 4$. *Furthermore, if* $\int_R p(x)dx < \infty$, *then the above assertion is true also for* $i = 3, 4$.

Proof. To prove the first assertion, it is sufficient to show that $R \notin O_{Q_{pqB}}$ for $\inf_R p(x) > 0$. On every regular subregion Ω, we have $1 = h_1^{p\Omega} + \int_R g_{p\Omega}(\cdot, y)p(y)dy$, and consequently $G_p p \leqq 1$ upon letting $\Omega \to R$. Therefore $G_p e_q \leqq G_p 1 \leqq 1/m$ with $m = \inf_R p$. By Theorem 2, $R \notin O_{Q_{pqB}}$. Suppose furthermore that $\int_R p(x)dx < \infty$. Then the volume of R is $V(R) = \int_R dx \leqq 1/m \int_R p(x)dx < \infty$ and

$$G_p(e_q, e_q) \leqq \frac{1}{m^2} V(R) < \infty .$$

The second assertion follows from Theorem 2.

13. By the fact that $g_p(x, y) \leqq g_r(x, y)$ for $p \geqq r$, and Theorem 2, we have the following:

PROPOSITION. *On a q-hyperbolic Riemannian manifold R (hyperbolic if* $p \equiv 0$*),*

(i) $O_{Q_{pqP}} \subset O_{Q_{pqB}} \subset O_{Q_{pqK}}$, *and* $O_{Q_{pqP}} \subset O_{Q_{pqE}} \subset O_{Q_{pqK}}$,
(ii) $O_{Q_{pqX}} \subset O_{Q_{rqX}}$ *for* $p \geqq r$,
(iii) $O_{Q_{pqX}} \subset O_{Q_{psX}}$ *for* $q \geqq s$,
(iv) $O_{Q_{pqX}} \subset O_{Q_{rsX}}$ *for* $p \geqq r$ *and* $q \geqq s$, *with* $X = P, B, E$ *or* K.

We note that if R is q-parabolic, $Q_{pq} = H^p$ and $\varnothing = O_{H^pP} \subset O_{H^pB} \subset O_{H^pE} = O_{H^pK}$, that is, (i) is still true. However, (ii)–(iv) are no longer valid, for $\varnothing = O_{H^pP} = O_{H^rP}$, $O_{H^rB} \subset O_{H^pB}$, and $O_{H^rE} = O_{H^rK} \subset O_{H^pK} = O_{H^pE}$ if $p \geqq r$.

From (iv) of the above proposition, we see that if the (r, s)-biharmonic Dirichlet problem is solvable by the decomposition method of Theorem 5, then the (p, q)-biharmonic boundary value problem has a solution for $p \geqq r$ and $q \geqq s$. In particular, the (p, q)-biharmonic Dirichlet problem is solvable if the biharmonic problem is.

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