PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 3, 1972

A CLASS OF BILATERAL GENERATING FUNCTIONS FOR CERTAIN CLASSICAL POLYNOMIALS

J. P. SINGHAL AND H. M. SRIVASTAVA

In this paper the authors first prove a theorem on bilateral generating relations for a certain sequence of functions. It is then shown how the main result can be applied to derive a large variety of bilateral generating functions for the Bessel, Jacobi, Hermite, Laguerre and ultraspherical polynomials, as well as for their various generalizations. Some recent results given by W. A. Al-Salam [1], S. K. Chatterjea [2], M. K. Das [3], S. Saran [6] and the present authors [9] are thus observed to follow fairly easily as special cases of the theorem proved in this paper.

Let the sequence of functions $\{S_n(x) \mid n = 0, 1, 2, \cdots\}$ be generated by

$$\sum_{n=0}^{\infty} A_{m,n} S_{m+n}(x) t^n = \frac{f(x, t)}{[g(x, t)]^m} S_m(h(x, t)) , \tag{1}$$

where m is a nonnegative integer, the $A_{m,n}$ are arbitrary constants, and f, g, h are arbitrary functions of x and t.

In the present paper we first prove the following

THEOREM. *For the $S_n(x)$ generated by* (1), *let*

$$F[x, t] = \sum_{n=0}^{\infty} a_n S_n(x) t^n , \tag{2}$$

where the $a_n \neq 0$ are arbitrary constants.

Then

$$\begin{aligned} &f(x, t) F[h(x, t), yt/g(x, t)] \\ &\quad = \sum_{n=0}^{\infty} S_n(x) \sigma_n(y) t^n , \end{aligned} \tag{3}$$

where $\sigma_n(y)$ is a polynomial of degree n in y defined by

$$\sigma_n(y) = \sum_{k=0}^{n} a_k A_{k,n-k} y^k . \tag{4}$$

We also show how this theorem can be applied to derive a large number of bilateral generating functions for those classical polynomial systems that satisfy a relationship like (1). In particular, we discuss the cases of the Bessel, Jacobi, Hermite, Laguerre and ultraspherical polynomials.

2. Proof of the theorem. If we substitute for the coefficients $\sigma_n(y)$ from (4) on the right-hand side of (3), we shall get

$$\begin{aligned}\sum_{n=0}^{\infty} S_n(x)\sigma_n(y)t^n &= \sum_{n=0}^{\infty} S_n(x)t^n \sum_{k=0}^{n} a_k A_{k,n-k} y^k \\ &= \sum_{k=0}^{\infty} a_k (yt)^k \sum_{n=0}^{\infty} A_{k,n} S_{n+k}(x) t^n \\ &= f(x, t) \sum_{k=0}^{\infty} a_k S_k(h(x, t))\{yt/g(x, t)\}^k ,\end{aligned}$$

by using (1), and the theorem follows on interpreting this last expression by means of (2).

3. Applications. As a first instance of the applications of our theorem, we recall the following known generating function for the ultraspherical polynomials [5, p. 280]:

$$(5) \qquad \sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{\lambda}(x) t^n = \rho^{-m-2\lambda} P_m^{\lambda}\left(\frac{x-t}{\rho}\right) ,$$

where $\rho = (1 - 2xt + t^2)^{-1/2}$.

Formula (5) is of type (1) with $f = \rho^{-2\lambda}$, $g = \rho$, $h = (x - t)/\rho$, and $A_{m,n} = \binom{m+n}{n}$, and therefore, our theorem, when applied to the ultraspherical polynomials, gives us

COROLLARY 1. *If*

$$(6) \qquad F[x, t] = \sum_{n=0}^{\infty} a_n P_n^{\lambda}(x) t^n ,$$

then

$$(7) \qquad \rho^{-2\lambda} F\left[\frac{x-t}{\rho}, \frac{yt}{\rho}\right] = \sum_{n=0}^{\infty} P_n^{\lambda}(x) b_n(y) t^n ,$$

where, as well as in what follows,

$$(8) \qquad b_n(y) = \sum_{k=0}^{n} \binom{n}{k} a_k y^k .$$

Corollary 1 was proved recently by Chatterjea [2]. Note that in his long and involved derivation of Corollary 1, Chatterjea made use of the following formula of Tricomi:

$$(9) \qquad P_n^{\lambda}\left(\frac{x}{\sqrt{(x^2-1)}}\right) = \frac{(-1)^n (x^2-1)^{\lambda+(1/2)n}}{n!} \frac{d^n}{dx^n} \{(x^2-1)^{-\lambda}\} .$$

Evidently, in view of the known generating function (5), formula (7) would follow from (6) and (8) in a straightforward manner, without using (9).

Next we consider the Laguerre polynomials which satisfy the relationship [5, p. 211]

$$\sum_{n=0}^{\infty}\binom{m+n}{n}L_{m+n}^{(\lambda)}(x)t^n = (1-t)^{-1-\lambda-m}\exp\left(\frac{-xt}{1-t}\right)L_m^{(\lambda)}\left(\frac{x}{1-t}\right), \tag{10}$$

which is of type (1) with $f=(1-t)^{-1-\lambda}\exp\{-xt/(1-t)\}$, $g=(1-t)$, $h=x/(1-t)$, and $A_{m,n}=\binom{m+n}{n}$. Thus we arrive at the following special case of our theorem:

Corollary 2. *If*

$$F[x,t]=\sum_{n=0}^{\infty}a_nL_n^{(\lambda)}(x)t^n, \tag{11}$$

then

$$(1-t)^{-1-\lambda}\exp\left(\frac{-xt}{1-t}\right)F\left[\frac{x}{1-t},\frac{yt}{1-t}\right] = \sum_{n=0}^{\infty}L_n^{(\lambda)}(x)b_n(y)t^n. \tag{12}$$

Corollary 2 provides us with the corrected version of a result proved earlier by Al-Salam [1, p. 134].

On the other hand, if we consider the formula (see, for instance, [4], p. 58)

$$\sum_{n=0}^{\infty}\binom{m+n}{n}L_{m+n}^{(\lambda-m-n)}(x)t^n = (1+t)^{\lambda-m}e^{-xt}L_m^{(\lambda-m)}(x(1+t)), \tag{13}$$

we shall obtain the following particular case of our theorem:

Corollary 3. *If*

$$F[x,t]=\sum_{n=0}^{\infty}a_nL_n^{(\lambda-n)}(x)t^n, \tag{14}$$

then

$$(1+t)^{\lambda}e^{-xt}F[x(1+t),yt/(1+t)] = \sum_{n=0}^{\infty}L_n^{(\lambda-n)}(x)b_n(y)t^n. \tag{15}$$

For the simple Bessel polynomials defined by [5, p. 293]

$$y_n(x) = {}_2F_0\left[-n, n+1; -; -\frac{1}{2}x\right], \tag{16}$$

we have [3, p. 409]

$$\begin{aligned}&\sum_{n=0}^{\infty} y_{m+n}(x)\frac{t^n}{n!}\\ &\quad= (1-2xt)^{-(m+1)/2}\exp\left\{\frac{1-\sqrt{(1-2xt)}}{x}\right\}y_m\left(\frac{x}{\sqrt{(1-2xt)}}\right),\end{aligned} \tag{17}$$

and on comparing (17) with (1) we are led to the following result of Das [3, p. 410]:

COROLLARY 4. *If*

$$F[x, t] = \sum_{n=0}^{\infty} a_n y_n(x)\frac{t^n}{n!}, \tag{18}$$

then

$$\begin{aligned}&(1-2xt)^{-1/2}\exp\left\{\frac{1-\sqrt{(1-2xt)}}{x}\right\}F\left[\frac{x}{\sqrt{(1-2xt)}}, \frac{yt}{\sqrt{(1-2xt)}}\right]\\ &\quad= \sum_{n=0}^{\infty} y_n(x)b_n(y)\frac{t^n}{n!}.\end{aligned} \tag{19}$$

Similarly, if we compare (1) with the known formula [5, p. 197]

$$\sum_{n=0}^{\infty} H_{m+n}(x)\frac{t^n}{n!} = \exp(2xt - t^2)H_m(x-t), \tag{20}$$

where $H_n(x)$ denotes the Hermite polynomial of degree n in x, we shall obtain a class of bilateral generating functions for these polynomials, given by

COROLLARY 5. *If*

$$F[x, t] = \sum_{n=0}^{\infty} \frac{a_n}{n!}H_n(x)t^n, \tag{21}$$

then

$$\begin{aligned}&\exp(2xt - t^2)F[x-t, yt]\\ &\quad= \sum_{n=0}^{\infty} H_n(x)b_n(y)\frac{t^n}{n!}.\end{aligned} \tag{22}$$

For the Jacobi polynomials we first observe that the special case

$y = 1$ of the bilinear generating relation (21), p. 465 of Srivastava [8] leads us to the elegant formula

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{m+n}{n}&P_{m+n}^{(\alpha-m-n,\beta-m-n)}(x)t^n\\ &=\left\{1+\frac{1}{2}(x+1)t\right\}^{\alpha-m}\left\{1+\frac{1}{2}(x-1)t\right\}^{\beta-m}\\ &\quad\times P_m^{(\alpha-m,\beta-m)}\left(x+\frac{1}{2}(x^2-1)t\right).\end{aligned}\tag{23}$$

Note that the last formula (23) is a generalization of the well-known result

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)t^n = \left\{1+\frac{1}{2}(x+1)t\right\}^{\alpha}\left\{1+\frac{1}{2}(x-1)t\right\}^{\beta},\tag{24}$$

which follows at once from (23) when $m = 0$.

A comparison of (23) with (1) yields

COROLLARY 6. *If*

$$F[x, t] = \sum_{n=0}^{\infty} a_n P_n^{(\alpha-n,\beta-n)}(x)t^n,\tag{25}$$

then

$$\begin{aligned}&\left\{1+\frac{1}{2}(x+1)t\right\}^{\alpha}\left\{1+\frac{1}{2}(x-1)t\right\}^{\beta}\\ &\quad\times F\left[x+\frac{1}{2}(x^2-1)t,\ yt\Big/\left\{1+\frac{1}{2}(x+1)t\right\}\left\{1+\frac{1}{2}(x-1)t\right\}\right]\\ &=\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)b_n(y)t^n.\end{aligned}\tag{26}$$

Next we set $v = 0$ in the bilinear generating relation (18), p. 464 of Srivastava [8]. On replacing α, γ and λ by $1+\alpha+\beta, 1+\alpha$ and $1+\alpha+m$ respectively, it is easy to see that

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{m+n}{n}&P_{m+n}^{(\alpha,\beta-m-n)}(x)t^n\\ &=(1-t)^{\beta-m}\left\{1-\frac{1}{2}(x+1)t\right\}^{-\alpha-\beta-1}P_m^{(\alpha,\beta-m)}(X),\end{aligned}\tag{27}$$

where, for convenience,

$$X = \left\{x-\frac{1}{2}(x+1)t\right\}\left\{1-\frac{1}{2}(x+1)t\right\}^{-1}.\tag{28}$$

We thus obtain

COROLLARY 7. *If*

$$F[x, t] = \sum_{n=0}^{\infty} a_n P_n^{(\alpha, \beta-n)}(x) t^n , \tag{29}$$

then

$$\begin{aligned}(1-t)^{\beta}\left\{1-\frac{1}{2}(x+1)t\right\}^{-\alpha-\beta-1} & F[X, yt/(1-t)] \\ &= \sum_{n=0}^{\infty} P_n^{(\alpha, \beta-n)}(x) b_n(y) t^n ,\end{aligned} \tag{30}$$

where X is given by (28).

Lastly, we recall the following generating relation [9, p. 79, eq. (3.6)]

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{m+n}{n} & G_{m+n}^{(\lambda)}(x, r, p, \alpha) t^n \\ &= (1-\alpha t)^{-m-\lambda/\alpha} \exp\left[p x^r\{1-(1-\alpha t)^{-r/\alpha}\}\right] \\ &\quad \times G_m^{(\lambda)}(x(1-\alpha t)^{-1/\alpha}, r, p, \alpha) ,\end{aligned} \tag{31}$$

where the $G_n^{(\lambda)}(x, r, p, \alpha)$ are polynomials in x^r introduced by us [9] in an attempt to provide an elegant unification of the various recent extensions of the classical Hermite and Laguerre polynomials given, for instance, by Gould and Hopper [4] and others referred to in our earlier paper [9]. A comparison of (31) with (1) would yield the following result:

COROLLARY 8. *If*

$$F[x, t] = \sum_{n=0}^{\infty} a_n G_n^{(\lambda)}(x, r, p, \alpha) t^n , \tag{32}$$

then

$$\begin{aligned}(1-\alpha t)^{-\lambda/\alpha} \exp\left[p x^r\{1-(1-\alpha t)^{-r/\alpha}\}\right] & F[x/(1-\alpha t)^{1/\alpha}, yt/(1-\alpha t)] \\ &= \sum_{n=0}^{\infty} G_n^{(\lambda)}(x, r, p, \alpha) b_n(y) t^n .\end{aligned} \tag{33}$$

Corollary 8, which incorporates Corollaries 2 and 5 as its particular cases, was proved earlier by us [9, p. 82, § 6] by using an operational technique.

Now we recall the sequence of functions $\{f_n(x) \mid n = 0, 1, 2, \cdots\}$ defined by Rodrigues' formula

$$f_n(x) = \mu(n)\phi(x) \frac{d^n}{dx^n} \{\Psi(x)\} , \tag{34}$$

where $\phi(x)$ and $\Psi(x)$ are independent of n. By using Taylor's theorem it is readily seen that the $f_n(x)$ are generated by

$$\sum_{n=0}^{\infty} \frac{\mu(m)}{n!\mu(m+n)} f_{m+n}(x)t^n = \frac{\phi(x)}{\phi(x+t)} f_m(x+t) , \quad m = 0, 1, 2, \cdots, \tag{35}$$

which evidently is of type (1) with

$$A_{m,n} = \frac{\mu(m)}{n!\mu(m+n)}, f = \frac{\phi(x)}{\phi(x+t)}, g = 1, h = x + t . \tag{36}$$

Thus the sequence $\{f_n(x)\}$, considered recently by Saran [6], is merely a proper subset of $\{S_n(x)\}$ defined by the generating relation (1).

Consequently, as a very special case of our theorem we can obtain the following corollary which happens to be the main result of Saran's paper [6]:

COROLLARY 9. *For the $f_n(x)$ defined by* (34), *let*

$$F[x, t] = \sum_{n=0}^{\infty} a_n f_n(x) t^n , \tag{37}$$

where the $a_n \neq 0$ are arbitrary constants.

Then

$$\frac{\phi(x)F[x-t, yt]}{\phi(x-t)} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!\mu(n)} f_n(x)c_n(y) , \tag{38}$$

where

$$c_n(y) = \sum_{k=0}^{n} (-n)_k \mu(k) a_k y^k . \tag{39}$$

By comparing (34) with Tricomi's formula (9) it would seem obvious that Corollary 1, involving ultraspherical polynomials, is contained in Corollary 9. However, it may be pointed out that the scope of Corollary 9 is very limited, since Rodrigues formulas of most of the classical polynomials require that the function $\Psi(x)$, involved in (34), depend upon both n and x. Besides, the factor $\mu(n)$ on the right-hand side of (34) is superfluous. Indeed, in equations (34), (35), (37), (38) and (39) one can replace, without any loss of generality, $f_n(x)$ by $\mu(n)f_n(x)$ and a_n by $a_n/\mu(n)$, $n = 0, 1, 2, \cdots$.

In conclusion, we remark that by assigning special values to the arbitrary coefficients a_n it is easy to obtain, from Corollaries 1 to 9, a large variety of bilateral generating functions for the Bessel, Jacobi, Hermite, Laguerre and ultraspherical polynomials, and their generalizations studied earlier. For example, Corollary 2 would lead us fairly easily to a number of extensions of the well-known Hille-Hardy formula given, for instance, by Srivastava [7] and Weisner [10]. The details involved are quite straightforward and are, therefore, omitted.

RERERENCES

1. W. A. Al-Salam, *Operational representations for the Laguerre and other polynomials*, Duke Math. J., **31** (1964), 127-142.

2. S. K. Chatterjea, *A bilateral generating function for the ultraspherical polynomials*, Pacific J. Math., **29** (1969), 73-76.

3. M. K. Das, *Sur les polynômes de Bessel*, C. R. Acad. Sci. Paris. Sér. A **271** (1970), 408-411.

4. H. W. Gould and A. T. Hopper, *Operational formulas connected with two generalizations of Hermite polynomials*, Duke Math. J., **29** (1962), 51-63.

5. E. D. Rainville, *Special functions*, Macmillan Co., New York, 1960.

6. S. Saran, *A general theorem for bilinear generating functions*, Pacific J. Math., **35** (1970), 783-786.

7. H. M. Srivastava, *An extension of the Hille-Hardy formula*, Math. Comp., **23** (1969), 305-311.

8. ———, *Some bilinear generating functions*, Proc. Nat. Acad. Sci. U.S.A., **64** (1969), 462-465.

9. H. M. Srivastava and J. P. Singhal, *A class of polynomials defined by generalized Rodrigues' formula*, Ann. Mat. Pure Appl. Ser. IV, **90** (1971), 75-85. See also Abstract 71T-B16, Notices Amer. Math. Soc., **18** (1971), 252.

10. L. Weisner, *Group-theoretic origins of certain generating functions*, Pacific J. Math., **5** (1955), 1033-1039.

Received June 21, 1971. Supported in part by NRC grant A-7353. See Abstract 684-B14 in Notices Amer. Math. Soc., **18** (1971), p. 515.

UNIVERSITY OF JODHPUR
JODHPUR, INDIA
AND
UNIVERSITY OF VICTORIA
VICTORIA, BRITISH COLUMBIA
CANADA