PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 3, 1972

# COMPOSITION CONSTRUCTIONS ON DIFFEOMORPHISMS OF $S^p \times S^q$

REINHARD SCHULTZ

**It is shown that a map from $S^{p+r}$ to $S^p$ in the image of the $J$-homomorphism sends particular types of diffeomorphisms of $S^p \times S^q$ into diffeomorphisms of $S^{p+r} \times S^q$. This is applied to the problem of determining the diffeomorphism type of an exotic $(p+q+r+1)$-sphere obtained by attaching $D^{p+r+1} \times S^q$ to $S^{p+r} \times D^{q+1}$ via a diffeomorphism of $S^{p+r} \times S^q$.**

The first section of this paper deals with the generalities of the construction. In the next two sections we exploit these ideas in a study of the plumbing pairing (compare [4]), and we show that the pairing vanishes in an infinite class of cases (3.2) and some selected low dimensions of other types (3.4). The last section of this paper uses the basic construction to investigate the existence of smooth semifree circle actions on homotopy nine spheres [5]. In particular, it is shown that if $\Sigma^9$ does not bound a spin manifold, then it has no semifree circle action with 5-dimensional fixed point set. See [23] for further nonexistence theorems concerning semifree circle actions on homotopy spheres and some geometric applications.

1. **Constructions for compositions.** Let $\alpha \in \pi_{p+r}(S^p)$ and $\beta \in \pi_p(SO_{q+1})$ be given. Then it is well known that $\beta$ induces a diffeomorphism of $S^p \times S^q$ and the composition $\beta \cdot \alpha \in \pi_{p+r}(SO_{q+1})$ induces a diffeomorphism of $S^{p+r} \times S^q$. If $\alpha$ is in the image of $J: \pi_r(SO_p) \to \pi_{p+r}(S^p)$ we shall give a geometric procedure for passing from the diffeomorphism induced by $\beta$ to that induced by $\beta \cdot \alpha$. In all our applications $\alpha$ will be the nontrivial homotopy class in $\pi_{p+1}(S^p)$.

PROPOSITION 1.1. *Let $X$ be an $H$-space, let $\gamma \in \pi_r(SO_p)$, and let $\beta \in \pi_p(X)$, where $p \geqq 2$. If $h$ is the diffeomorphism of $S^r \times S^p$ induced by $\gamma$, the map $\pi: S^r \times S^p \to S^p$ is projection, and $q: S^r \times S^p \to S^{p+r}$ is the collapsing map, then the following formula holds*:

$$h^*\pi^*\beta = \pi^*\beta \cdot q^*(\beta \cdot J(\gamma)) .$$

The dot represents multiplication in $[S^p \times S^r, X]$. The above result was proved for $X$ a double loop space in [21, Appendix].

*Proof.* Without loss of generality the diffeomorphism $h$ maps $* \times D^{r+1}$ to itself by the identity ($*$ is the basepoint). Thus if

$$K = S^p \times S^r \cup * \times D^{r+1} ,$$

then $h$ extends to a homeomorphim of $K$. It is easy to see that $K$ has the homotopy type of $S^p \vee S^{p+r}$, since the map $(\pi, g)\colon K \to S^p \times S^{p+r}$ deforms to a map into $S^p \vee S^{p+r}$ which induces isomorphisms in homology (the maps $\pi$ and $q$ are extensions of the maps on $S^p \times S^r$ to $K$). An explicit homotopy inverse to the above map may be constructed as follows: Define $g\colon S^p \vee S^{p+r} \to K$ so that $g|S^r$ is inclusion into $* \times S^r$ and $g$ maps

$$S^{p+r} = D^p \times S^r \cup S^{p-1} \times D^{r+1}$$

by sending $(x, y) \in D^p \times D^{r+1}$ to $(\rho(x), y) \in K \subseteqq S^p \times D^{r+1}$, where $\rho$ maps $D^p$ to $S^p$ by collapsing the boundary. We then compute $g^*h^*\pi^*\beta$ from its restrictions to $S^{p+r}$ and $S^r$; an argument like that of [21, Appendix] shows that $h^*\pi^*\beta$ has the desired form in $[K, X]$. The proposition follows upon restriciting to $S^p \times S^r$.

COROLLARY 1.2. *Let $f$ be a diffeomorphism of $S^p \times S^r$ representing $\gamma$ and let $g$ be a diffeomorphism of $S^p \times S^q$ represented by $\beta \in \pi_p(SO_q)$ (notation as in* 1.1). *Then the commutator diffeomorphism* $[1_q \times f, g \times 1_r]$ *on $S^p \times S^q \times S^r$ represents the homotopy class $q^*\beta J(\gamma) \in [S^r \times S^p, SO_{q+1}]$.*

*Proof.* By a direct computation it follows that the commutator diffeomorphism sends $(x, y, z) \in S^p \times S^q \times S^r$ into

$$(x, [g_0(x)^{-1}g_0\pi f(x, y)]z, y) ,$$

where $g_0\colon S^p \to SO_{q+1}$ is such that $f(x, y) = (x, g_0(x)y)$ and the multiplication within the brackets comes from the group structure of $SO_{q+1}$. By 1.1 we know that the homotopy class of $g_0\pi f = f^*\pi^*\beta$ is the product of the class of $g_0\pi$, $\pi^*\beta$ with the class $q^*\beta J(\gamma)$, and the corollary follows from this.

We shall need generalizations of 1.2 in §4. Suppose that $h$ is a diffeomorphism of $S^p \times S^q$ which is the identity near $D^p_+ \times S^q$ (compare [17, 2.3]). Then $h$ is homotopic to a map sending $(x, y) \in S^p \times S^q$ to $(x, h'(x)y)$, where $h'$ is a continuous map from $S^p$ into the space $G_{q+1}$ of self-maps of $S^q$ with degree $+1$. Let $\beta$ be the homotopy class in $\pi_p(G_{q+1})$ so obtained; it is immediate $\beta$ is a pseudo-isotopy invariant of $h$ and that the formula $\beta(h_1h_2) = \beta(h_1) + \beta(h_2)$ holds. For any integer $s$ satisfying $1 \leqq s \leqq p-1$, the diffeomorphism $h$ induces a diffeomorphism $h^{[s]}$ of $S^{p-s} \times S^s \times S^q$ such that $h^{[s]}|D \times S^q = h|D^p \times S^q$ (for some coordinate disk $D$ in $S^{p-s} \times S^s$) and $h^{[s]} = 1$ off $D \times S$.

PROPOSITION 1.3. *Let $h\colon S^p \times S^q \to S^p \times S^q$ be a diffeomorphism* *which is the identity near $D^p_+ \times S^q$, and let $f$ be a diffeomorphism of $S^p \times S^r$ induced by $\gamma \in \pi_p(SO_r)$. If $\beta \in \pi_p(G_{q+1})$ is the class associated to $h$, then the commutator diffeomorphism $[1_q \times f, h \times 1_r]$ of $S^p \times S^q \times S^r$ is equal to a diffeomorphism of the form $k^{[p]}$ for some $k$ on $S^{p+r} \times S^q$ which is the identity off $D^{p+r}_+ \times S^q$. Furthermore, the homotopy class associated to $k$ is the composition $\beta \cdot J(\gamma) \in \pi_{p+r}(G_{q+1})$.*

*Proof.* Pick a representative $f$ of $\gamma$ which is the identity near $D^r_+ \times S^p$. Then the commutator above is the identity near the complement of $D^p \times D^r \times S^q$, and hence does have the form $k^{[p]}$. The homotopy class of $k$ may be computed from that of $k^{[p]}$ by a computation of the commutator similar to that of 1.2. Notice that $-\beta$ is the homotopy class associated to $h^{-1}$.

Suppose that $q = 3$ and the above diffeomorphism $h$ of $S^p \times S^3$ is equivariant with respect to the free action of $S^1$ on the second factor by scalar multiplication. Then a diffeomorphism $\bar{h}$ of $S^p \times S^2$ is induced on the quotient manifold, and a homotopy class $\bar{\beta} \in \pi_p(G_3)$ is obtained. Given $f$ and $\gamma$ as above, one may form the commutators $[1_3 \times f, h \times 1_r]$ and $[1_2 \times f, \bar{h} \times 1_r]$ and obtain diffeomorphisms $k$ of $S^{p+r} \times S^3$ and $l$ of $S^{p+r} \times S^2$; the diffeomorphism $k$ is equivariant with respect to the action of $S^1$ on $S^3$, and $l$ is the diffeomorphism which it induces on $S^{p+r} \times S^2$.

PROPOSITION 1.4. *In the above notation, the homotopy class in $\pi_{p+r}(G_3)$ determined by $l$ is $\bar{\beta} j(\gamma)$.*

The proof is straightforward.

**2. Plumbing formulas.** Throughout this paper $\mathscr{D}(M)$ will denote the group of pseudoisotopy classes of orientation-preserving diffeomorphisms of the smooth closed manifold $M^n$. Recall that there is a canonical map $\sigma: \Gamma_{n+1} \to \mathscr{D}(M)$ (compare [21, 1. 7]), whose image we shall call $\Delta$; this map is $1-1$ if $M$ is a product of spheres [21, 1. 7]. Furthermore, $\Delta$ is contained in the center of $\mathscr{D}(M)$ (compare [17, p. 529]).

The plumbing pairing

$$\sigma_{p,q}: \pi_p(SO_q) \times \pi_q(SO_p) \longrightarrow \Gamma_{p+q+1}$$

may be defined as follows. There are canonical maps of $\pi_p(SO_q)$ and $\pi_q(SO_p)$ into $\mathscr{D}(S^p \times S^q)$ as noted previously and we denote the homotopy and pseudo-isotopy classes by the same letter. Let $f$ represent $u \in \pi_p(SO_q)$ and $g$ represent $v \in \pi_q(SO_p)$; without loss of generality $f$ is the identity near $D^p_+ \times S^q$ and $g$ is the identity near $S^p \times D^q_+$. The commutator $[u, v]$ is thus represented by a diffeomorphism $[f, g]$ which is the identity off $D^p_- \times D^q_-$, and hence $[u, v] = \sigma(\gamma) \in \Delta$, for some $\gamma \in \Gamma_{p+q+1}$. Since $\sigma$ is $1-1$ for $M = S^p \times S^q$, $\gamma$ is uniquely determined, and we define $\sigma(u, v)$ to be this element $\gamma$. The bilinearity of $\sigma$ follows easily from the definition.

REMARK. Often the definition of the plumbing pairing $\sigma_{p,q}$ is extended to $\pi_p(SO_q) \times \pi_q(SO_{p+1})$. This is done by taking $\gamma \in \Gamma_{p+q+1}$ to be defined by the attaching construction

$$S^{p+q+1}_\gamma = D^{p+1} \times S^q \bigcup\nolimits_{[u,v]} S^p \times D^{q+1} .$$

(The diffeomorphism class of the above manifold is a pseudo-isotopy invariant; compare [17, §5].) The fact that this extends the previous definition follows from 2.1 below. If one of $v_1$ or $v_2$ is not in the image of $\pi_q(SO_p)$ in $\pi_q(SO_{p+1})$, the formula $\sigma(u, v_1 + v_2) = \sigma(u, v_1) + \sigma(u, v_2)$ may be false.

LEMMA 2.1. *Let $\sigma(\gamma) \in \Delta \subseteqq \mathscr{D}(M^n)$ be given, and suppose $Q^{n+1}$ is a closed smooth manifold which $M^n$ separates into two components $N$ and $P$. Then the identification manifold $N \bigcup_{\sigma(\gamma)} P$ is diffeomorphic to $Q \# S^{n+1}_\gamma$.*

The proof of this follows exactly the same pattern as the more specialized remark in [17, 2.3, p. 526].

We include another result on manifolds obtained by boundary identification which will be used in subsequent proofs. Let $M, N, P$ be compact connected smooth manifolds with boundary, and let $Q_1, Q_2$ be closed smooth manifolds such that $\partial M = Q_1$, $\partial N = Q_1 \cup Q_2$ (a disjoint union), $\partial P = Q_2$. If $h$ and $k$ are diffeomorphisms of $Q_1$ and $Q_2$ respectively, then one can form the smooth manifold

$$X(h, k) = M \bigcup\nolimits_h N \bigcup\nolimits_k Q .$$

LEMMA 2.2. *Suppose there is a diffeomorphism $\varphi$ of $N$ mapping $Q_1$ and $Q_2$ into themselves such that $\varphi | Q_1 = h'$ and $\varphi | Q_2 = k'$. Then $X(h, k)$ is diffeomorphic to $X(h'^{-1}h, kk')$.*

*Proof.* Define the diffeomorphism $\Phi$ piece by piece. Let $\Phi | M \cup P$ be the identity, and let $\Phi | N = \varphi^{-1}$. Then $\Phi$ is a diffeomorphism because it is consistent with identifications along the boundaries.

The pairing $\tau_{p,q}$ of Milnor-Munkres-Novikov [16, p. 583] has a similar description, although for computational purposes it is best described as the map $\tau_{p\,q}\colon \pi_p(SO_q) \times \pi_q(PD/O) \to \pi_{p+q}(PD/O)$ for which $\tau(\alpha, \beta) = \beta \cdot J(\alpha)$. For our purposes it will be convenient to interpret these pairings in terms of pseudo-isotopy classes of diffeomorphisms of a product of three spheres; other elaborations are obviously possible.

One may define pairings

$$\sigma' \colon \pi_p(SO_{r+1}) \times \pi_q(SO_p) \longrightarrow \Gamma_{p+q+1}$$
$$\tau' \colon \pi_p(SO_{r+1}) \times \Gamma_{q+1} \longrightarrow \Gamma_{p+q+1}$$

via induced diffeomorphisms of $S^p \times S^{q-r} \times S^r$ in much the same way that $\sigma$ and $\tau$ are defined. One uses the fact that there are embeddings

$$S^p \times S^{q-r} \times D^{r+1} \subseteqq S^p \times D^{q+1} \subseteqq S^{p+q+1}$$

and glues together $S^p \times S^{q-r} \times D^{r+1}$ and the closure of its complement in $S^{p+q+1}$ via the diffeomorphism of the product obtained. We remark that $\pi_q(SO_p)$ maps into $[S^{q-r} \times S^r, SO_p]$ via maps fixed off $D_+^{q-r} \times D_+^r$; likewise, $\Gamma_{q+1}$ acts via diffeomorphisms fixed off $D_+^{q-r} \times D_+^r$.

REMARK. Levine has defined a 4-linear map $\delta$ which generalizes $\sigma$ and $\tau$ to some extent (see [18, §7]; actually, $\delta$ is expressible in terms of $\sigma$ and $\tau$). The result stated below could also be formulated in terms of $\delta$ and a related pairing $\delta'$.

PROPOSITION 2.3. *Let $i \colon SO_{r+1} \to SO_{q+1}$ be the inclusion. Then $\sigma'(\alpha, \beta) = \sigma(\alpha, i_*\beta)$ and $\tau'(\beta, \alpha) = \tau(i_*\beta, \alpha)$.*

*Proof.* Let $Q_1 = S^p \times S^{q-r}$ and let $Q_2 = S^p \times S^q$. Let $N = S^p \times N'$ be the cobordism between $Q_1$ and $Q_2$ constructed from the above embeddings; i.e. $N' = D^{q+1} - \operatorname{Int} S^{q-r} \times D^{r+1}$. Finally, let $M = S^p \times S^{q-r} \times D^{r+1}$ and let $P = D^{p+1} \times S^q$. Then $S^{p+q+1} = M \bigcup_1 N \bigcup_1 D$ and $M \bigcup_1 N = S^p \times D^{q+1}$. If we can extend the diffeomorphisms induced by $\sigma$, and $\tau$ on $S^p \times S^q$ and by $\sigma', \tau'$ on $S^p \times S^{q-r} \times S^r$ to diffeomorphisms of $N$, then the result will follow by 2.2. But let $\psi \colon D^q \times I \to N'$ be a proper embedding such that

(i) $\psi^{-1}(S^q) = D^q \times 1$ and $\psi(D_+^q \times 1) \subseteqq \operatorname{Int} D_+^q$

(ii) $\psi^{-1}(S^{q-r} \times S^r) = D^q \times 0$ and $\psi(D_+^q \times 0) \subseteqq \operatorname{Int} D_+^{q-r} \times D_+^r$.

Then $\pi_q(SO_p)$ and $\Gamma_{q+1}$ act on $S^p \times N'$ and $N'$ via maps which are the identity outside the image of $\psi$ and only depend on the $D^q$ coordinate. The map $\alpha \in \Gamma_{q+1}$ extends to $S^q \times N'$ via natural product extensions. On the other hand, there is an explicit embedding of $N'$ in $R^{q+1} = R^{r+1} \times R^{q-r}$ given by the relation

$N' = \{(x, y) \mid |x|^2 + |y|^2 \leqq 1$ and $(2|x| - 1)^2/4 + |y|^2 \geqq 1/16\}$. (See the figure below.) This smooth manifold is invariant under the standard linear action of $SO_{r+1}$ on $R^{q+1}$, and the restricted action on the boundaries is again standard. In particular, the restriction to $S^r \times S^{q-r}$ is merely the usual action on $S^r$ crossed with the identity on

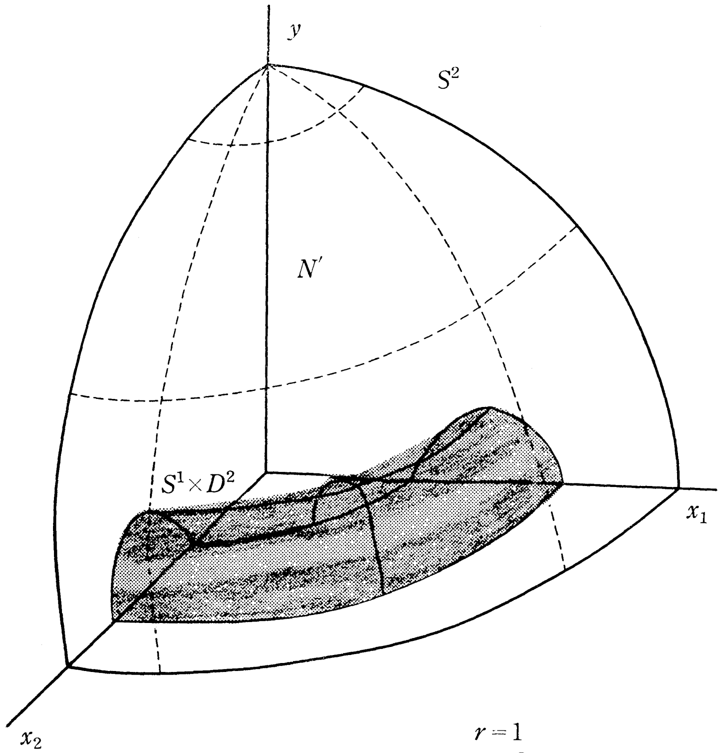


FIGURE 1

$S^{q-r}$. There are a homotopy class in $\pi_p(SO_{r+1})$ which induces a diffeomorphism of $S^p \times N'$ with the right restrictions to the boundaries, and this gives the desired extension of the map $\beta$. Since the pairings $\sigma$, $\tau$ and $\sigma'$, $\tau'$ are commutators in the restrictions of extendible maps, the diffeomorphisms associated to them are extendible, and the argument is completed.

The Milnor-Munkres-Novikov pairing may be generalized to products of three spheres in still another manner. Let $p$, $q$, $r$ be three positive integers, and let $\beta \in \pi_{p+q}(SO_r)$, $\alpha \in \Gamma_{p+q+r+1}$. Represent $\beta$ by a class in $[S^p \times S^q, SO_r]$ which is constant off $D^p_+ \times D^q_+$ and induces a diffeomorphism on $S^p \times S^r \times S^r$. Represent $\alpha$ by a diffeomorphism of $S^q \times S^r$ which is the identity off $D^q_- \times D^r_-$. Then the commutator $[\alpha, \beta]$ is the identity off $D^p_+ \times D^q_+ \times D^r_+$, and hence it determines an element $\tau_q(\beta, \alpha)$ of $\Gamma_{p+q+r+1}$.

THEOREM 2.4. *The pairing*

$$\tau_q: \pi_{p+q}(SO_r) \times \Gamma_{q+r+1} \longrightarrow \Gamma_{p+q+r+1}$$

*is trivial.*

*Proof.* Let $T^n$ be the $n$-dimensional torus. Then a pairing $\tau'_q$ may be formed using maps which are fixed off products involving $(D^1_+)^p$, $(D^1_+)^q$, and $D^r_+$, and an argument similar to 2.3 shows that $\tau'_q = \tau_q$.

If $\gamma$ is a pseudo-isotopy class of diffeomorphisms of $T^p \times T^q \times S^r$, let $X_\gamma$ be the closed smooth manifold formed by identifying two copies of $T^p \times T^q \times D^{r+1}$ (denoted by $+$ and $-$) along the boundary via $\gamma$. Let $\beta \in \pi_{p+q}(SO_r)$ and $\alpha \in \Gamma_{q+r+1}$ be as in the definition of $\tau_q$. By a standard argument [17, p. 540], $X_\alpha$ and $X_{\beta\alpha\beta-1}$ are diffeomorphic. However, the latter is merely $X_{\alpha\tau}$, which is $X_\alpha \# \Sigma_\tau$ by 1.1, and $X_\alpha$ is merely $T^p \times \{T^q \times S^{r+1} \# \Sigma_\alpha\}$. We have the standard inclusions of $T^p \times T^q \times D^{r+1}_\pm$ into $X_\gamma$, and the diffeomorphism $\beta'$ from $X_{\beta\alpha\beta-1}$ to $X_\alpha$ maps these to themselves by $-\beta$.

There are canonical homeomorphisms of $X_{\alpha_\tau}$ and $X_\alpha$ with $T^{p+q} \times S^{r+1}$, and we claim that under these homeomorphisms the diffeomorphism $\beta'$ corresponds to a map homotopic to the diffeomorphism of $T^{p+q} \times S^{r+1}$ induced by $-\beta \in \pi_{p+q}(SO_r)$. First notice that under the canonical homeomorphisms the standardly embedded $T^{p+q} \times D^{r+1}$ is mapped to itself by the identity; hence the remarks in the above paragraph imply $\beta'$ corresponds to $-\beta$ on this piece.

Next, consider the homeomorphism

$$F: T^{p+q} \times D^{r+1}_- \longrightarrow T^{p+q} \times D^{r+1}_-$$

corresponding to the diffeomorphism $\beta'$ on the included pieces $T^{p+q} \times D^{r+1}_-$, and let $H = F\beta$. Then $H$ maps the boundary $T^{p+q} \times S^r$ to itself by the identity by the previous remarks, and there is no cohomological obstruction to deforming $\pi_T H$ into $\pi_T$ rel the boundary (which is why we replaced $S^p$ and $S^p$ with $T^p$ and $T^q$). Thus $\beta'$ corresponds up to homotopy to a fiber homotopy equivalence over $T^{p+q}$ whose restriction to $T \times D_+$ agrees with the induced map of $-\beta$ and maps $T \times D_-$ into itself. An application of the Alexander trick implies that this fiber homotopy equivalence is fiber homotopic to $-\beta$, and hence that $\beta'$ is homotopic to the map induced by $-\beta$. But for any highly homotopy associative and commutative $H$-space $H$, the map $(-\beta)^*: [T^{p+q} \times S^{r+1}, H] \to [T^{p+q} \times S^{r+1}, H]$ applied to any element in the image of $\pi_{q+r+1}(H)$ is the identity [21, Appendix]. Combining this with a little smoothing theory (e.g., [21, 2.2]), we see that $\tau_q(\beta, \alpha)$ must vanish.

PROBLEM. Let $D_m$ be the topological group of diffeomorphisms of $D^m$ which are the identity near the boundary (in the $C^1$ topology). Then the map $\tau'_q$ factors through the group $\pi_p(D_{q+r})$ and the Novikov projection $\lambda: \pi_p(D_{q+r}) \to \Gamma_{p+q+r+1}$ [20, p. 227]. Is the associated map into $\pi_p(D_{q+r})$ trivial?

The above information has been assembled to prove the following factorization formula relating the plumbing pairing $\sigma$ and the Milnor-Munkres-Novikov pairing $\tau$.

THEOREM 2.5. *Let $\gamma \in \pi_r(SO_q)(r \leqq p-2)$, $\alpha \in \pi_p(SO_q)$, and $\beta \in \pi_q(SO_p)$. If $\beta$ is in the image of $\pi_p(SO_{p-r})$, then*

$$\tau_{r,p+q+1}(\gamma, \sigma_{p,q}(\alpha, \beta)) = \sigma_{p+r,q}(\alpha J(\gamma), \beta) .$$

Most of the interesting applications of the above formula occur when $r = 1$. The condition on $\beta$ may be weakened to assuming that the Samelson product of $\gamma$ and $\beta$ in $SO_{p+1}$ vanishes.

*Proof.* This basically reduces to an exercise in the algebra of the group $\mathscr{D} = \mathscr{D}(S^p \times S^q \times S^r)$. The homotopy groups involved in the statement 2.5 all map into $\mathscr{D}$ canonically, and we identify a given element with its image in $\mathscr{D}$. Also, there is the composite map

$$\rho: \Gamma_{p+q+1} \xrightarrow{\sigma} \mathscr{D}(S^p \times S^q) \xrightarrow{\times 1_r} \mathscr{D}(S^p \times S^q \times S^r) ,$$

the last map being cartesian product of a diffeomorphism with the identity on $S^r$.

By definition of the Milnor-Munkres-Novikov pairing, we have the equation:

$$\sigma(\alpha, \beta)\tau(\gamma, \sigma(\alpha, \beta)) = \gamma_\sigma(\alpha, \beta)\gamma^{-1} .$$

Now $\beta$ and $\gamma$ are both in the image of $[S^r \times S^q, SO_{p+1}]$ in $\mathscr{D}(S^p \times S^q \times S^r)$. Since $\beta$ comes from $SO_{p-r}$ and $\gamma$ comes from $SO_{r+1}$ (a pair of commuting subgroups up to homotopy), these elements commute in the above homotopy group. Hence we may continue to alter the last expression in the equation:

$$\gamma\alpha\beta\alpha^{-1}\beta^{-1}\gamma^{-1} = \gamma\alpha\beta\alpha^{-1}\gamma^{-1}\beta^{-1} = (\gamma\alpha\gamma^{-1})\beta(\gamma\alpha\gamma^{-1})^{-1}\beta^{-1} .$$

However, by 1.2, $\gamma\alpha\gamma^{-1} = \alpha\alpha_0 = \alpha_0\alpha$, where $\alpha_0$ is the image of the homotopy class of $\pi_{p+r}(SO_q)$ given by $\alpha J(\gamma)$. Continuing the derivation, we have

$$\begin{aligned}
&(\gamma\alpha\gamma^{-1})\beta(\gamma\alpha\gamma^{-1})\beta^{-1} = \alpha_0\alpha\beta\alpha^{-1}\alpha_0^{-1}\beta^{-1} \\
&= \alpha_0\sigma(\alpha, \beta)\beta\alpha_0^{-1}\beta^{-1} = \alpha_0\sigma(\alpha, \beta)\alpha_0^{-1}\sigma(\alpha_0, \beta) \\
&= \tau_p(\alpha_0, \sigma(\alpha, \beta))\sigma(\alpha, \beta)\sigma(\alpha_0, \beta) ,
\end{aligned}$$

where $\tau_p$ is as described preceding 2.4. Since $\tau_p$ vanishes by 2.4, we have shown that $\sigma(\alpha, \beta)\tau(\gamma, \sigma(\beta, \beta)) = \sigma(\alpha, \beta), \sigma(\alpha_0, \beta)$ and the theorem follows upon cancelling $\sigma(\alpha, \beta)$ from both sides of the equation.

**3. Special cases of Theorem 2.5.** If $n \equiv 0, 1,$ or $2 \bmod 8$, then $\pi_{n-1}(SO) = Z, Z_2,$ and $Z_2$ respectively, and composition with the nontrivial map $\eta: S^n \rightarrow S^{n-1}$ induces an epimorphism if $n \equiv 0$ and an isomorphism if $n \equiv 1$. Therefore it is reasonable to expect that Theorem 2.5 will have some nonempty applications; these are stated in terms of Wall's classification theory for highly connected almost closed manifolds [26, 27].

In the above papers suitable Grothendieck groups $\mathscr{G}_n^{2n}$ and $\mathscr{G}_n^{2n+1}$ were defined such that each almost closed $(n-1)$-connected $2n$- or $(2n+1)$-manifold $M$ determined a well-defined element $[M] \in \mathscr{G}_n$. There are also obstruction homomorphisms

$$\theta: \mathscr{G}_n^{2n} \longrightarrow \Gamma_{2n-1}(\theta: \mathscr{G}_n^{2n+1} \longrightarrow \Gamma_{2n})$$

such that $\theta(M)$ is the diffeomorphism class of the boundary of $M$. Now $\mathscr{G}_n^{2n}$ may be written $Z \oplus \mathscr{G}_n^{2n'}$ where $Z \subseteqq \operatorname{Ker} \theta$ [26, Th. 2, p. 176], and certain other elements are known to be either trivial or nontrivial. No relations are known in general for $\mathscr{G}_n^{2n+1}$, however; the obstruction homomorphism in the case $n = 4$ has been studied by D. Frank [8, p. 565]. Our techniques exhibit relationships between certain elements in the image of the obstruction homomorphisms if $n \equiv 0, 1, 2 \bmod 8$ and $n \geqq 16$. We refer the reader to Wall's papers for the computations of the Grothendieck groups and the manifold invariants which induce isomorphisms.

Let $n = 8k$, where $k \geqq 1$. In Wall's notation let

(i) $\mu \in \mathscr{G}_n^{2n}$ have invariants $\tau/8(\mu) = 0 \in Z$, $\chi^2/2(\mu) = 1 \in Z$

(ii) $\nu \in \mathscr{G}_n^{2n+1}$ have invariants $\varphi S\beta\hat{\alpha}(\nu) = 0 \in Z_2$, $\omega(\nu) = 1 \in Z_2$

(iii) $\xi \in \mathscr{G}_\pi^{2n+2}$ have invariants $\Phi(\xi) = 0 \in Z_2$, $\varphi\chi(\xi) = 1 \in Z_2$

(iv) $\kappa \in \mathscr{G}_{n+1}^{2n+3}$ have invariant $\omega(f)(\kappa) = 4 \in Z_8$

(v) $\lambda \in \mathscr{G}_{n+2}^{2n+4}$ have invariants $\tau/8(\lambda) = 0 \in Z$, $\chi^2/2(\lambda) = 1 \in Z_2$.

THEOREM 3.1. *If $n = 8k$ and $k \geqq 2$, the above elements are related by the compositions*

$$\theta(\lambda) = \theta(\kappa)\cdot\eta,\ \theta(\kappa) = \theta(\xi)\cdot\eta\ ,$$
$$\theta(\xi) = \theta(\nu)\cdot\eta,\ \theta(\nu) = \theta(\mu)\cdot\eta\ .$$

Note that $\eta$ always represents the nontrivial map from $S^{m+1}$ to $S^m$ in our discussions.

One consequence of 3.1 is that the homomorphism $\theta$ is completely determined for $\mathscr{G}_n^{2n'}$ if $n \equiv 2 \bmod 8$.

COROLLARY 3.2. *Suppose $n \equiv 2(8)$, $n > 10$. Then the obstruction homomorphism maps $\mathscr{G}_n^{2n'}$ onto $\partial P_{2n} \subseteqq \Gamma_{2n-1}$ sending the element with invariants $\tau/8 = s \in Z$ and $\chi^2/2 = t \in Z_2$ into $s$ times the generator of $\partial P_{2m}$.*

*Proof.* It is merely necessary to check the case $s = 0$, $t = 1$, i.e., the element $\lambda$ constructed above. But $\theta(\lambda) = \theta(\mu)\cdot\eta\cdot\eta\cdot\eta\cdot\eta$ by 3.1, and this element vanishes since $\eta^4 = 0$ in the stable homotopy groups of spheres.

We shall show later that the truth of 3.2 for $n = 10$ follows from results of Frank.

*Proof* of 3.1. Let $\alpha \in \pi_{n-1}(SO_n)$ go stably to the generator of $\pi_{n-1}(SO) = Z$ and have Hopf invariant zero; since $n \equiv 0 \bmod 8$ and $n \neq 8$ such a class exists. In fact, $\alpha$ is in the image of $\pi_{n-1}(SO_{n-3})$ under the canonical mapping by standard results of homotopy theory; thus 2.5 will apply to $\alpha$ or any composition $\alpha\cdot\zeta(\zeta \in \pi_h(S^{n-1}))$ provided $r = 1$ and $J(\gamma) = \eta$ in that theorem.

The obstruction homomorphism and the plumbing pairing are related as follows (compare [27, §12]):

$$\theta(\mu) = \sigma(\alpha, \alpha) \qquad \theta(\nu) = \sigma(\alpha\eta, i_*\alpha)$$
$$\theta(\xi) = \sigma(i_*\alpha\eta, i_*\alpha\eta) \quad \theta(\lambda) = \sigma(i_*\alpha\eta^2, i^*\alpha\eta^2)$$

($i_*$ denotes the homomorphism of $\pi_*(SO_{k-l})$ to $\pi_*(SO_k)$ induced by inclusion). The composition formulas now follow upon successive applications of 2.5.

REMARK. The element $\kappa \in \mathscr{G}_{n+1}^{2n+3}$ may be written $4\kappa'$ [27, §12] and hence by 3.1 we have $4\theta(\kappa') = \theta(\mu)\cdot\eta^3 = 4(3\theta(\mu)\cdot\rho)$, where $\rho \in \pi_3 = Z_{24}$ is the generator; it is obvious to ask whether $\theta(\kappa') = 3\theta(\mu)\cdot\rho$ holds in general.

For convenience we summarize some specific low-dimensional results on the obstruction homomorphism. All of these use the known multiplicative structure of the stable stems[1], and a few depend upon results of Brumfiel and Frank [6, 8].

Let $b\mathrm{Spin}_{8k+1} \subseteqq \Gamma_{8k+1}$ be the subgroup of homotopy spheres bounding spin manifolds. There are natural splitting mappings $s\colon \Gamma_{8k-1} \to \partial P_{8k}$ and $s'\colon b\mathrm{Spin}_{8k+2} \to \partial P_{8k+2}$ defined in [6, I] and [6, II] respectively. Finally notice that the image of the obstruction homomorphism on $\mathscr{G}_{8k+1}^{16k+2}$ is contained in $b\mathrm{Spin}_{16k+2}$ since the manifolds representing Grothendieck elements are highly connected.

[1] I wish to thank M. Mahowald and M. Tangora for discussing these results with me.

LEMMA 3.3. *Let $\xi$ and $\lambda$ be as in 3.1. Then $s'\theta(\xi) = 0$ and $s\theta(\lambda) = 0$.*

This may be directly verified from the construction of the mappings $s'$ and $s$. The latter statement has nonempty content only if $n = 10$ (otherwise $\theta(\lambda) = 0$ by 3.2).

THEOREM 3.4. (i) *The kernel of the obstruction homomorphism on $\mathscr{S}_n^{2n+1}$ contains an element of order 2 if $n = 8, 9, 16, 17, 25, 33$.*

(ii) *The kernel of the obstruction homomorphism $\mathscr{S}_n^{2n}$ consists of all elements with Arf invariant (i.e., $\Phi$ in Wall's notation) zero if $n = 9, 17, 25$.*

(iii) *The obstruction homomorphism maps $\mathscr{S}_{10}^{20}$ onto $\partial P_{20} \subseteqq \Gamma_{19}$, and the image of an element is completely determined by the index invariant $\tau/8$.*

Notice that (iii) settles the single case left unresolved in 3.2.

*Proof.* (i) If $n = 8$, we claim that the element $\theta(\nu)$ embeds in $R^{26}$ with trivial normal bundle; the triviality of $\theta$ then follows from [11]. Let $Q^{17}$ be the manifold formed by plumbing the disk bundles $\gamma$ (over $S^8$) and $\beta$ (over $S^9$), so that $\partial Q^{17} = S^{16}_{\sigma(\beta,\gamma)}$. According to an argument of Hirsch [10], $Q^{17}$ embeds in $R^{26}$ and the normal bundle of the boundary is explicitly computable. For let $h\colon Q^{17} \to S^8 \vee S^9$ be the standard homotopy equivalence and $j$ the inclusion of $S^{16}$ as the boundary of $Q^{17}$; then by a generalization of a theorem of James and Whitehead [12, 3.7, p. 206] the map $hj$ is homotopic to the sum

$$[i_8, i_9] + i_9J(\gamma) + i_8J(\beta) .$$

Thus the normal bundle of the embedded boundary of $Q^{17}$ is

$$i_*\{[\beta, \gamma] + \beta J(\gamma) + \gamma J(\beta)\} \in \pi_{16}(\beta SO_{10}) .$$

But the element $\theta(\nu)$ may be written in the form $\sigma(\tau, \eta, \alpha)$, where $\tau \in \pi_7(SO_8)$ classifies the tangent bundle and $\alpha$ generates $\pi_7(SO_9) = Z$ (compare [26, 27]). Since $i_*\tau = 0$, the normal bundle of this plumbing boundary is given by $-i_*\alpha J(\tau\eta) = -i_*\alpha J(\tau)\eta$. On the other hand, it is clear by a similar argument that $-i_*\alpha J(\tau)$ is the normal bundle for an embedding of the exotic sphere $\sigma(\tau, \alpha) \in \Gamma_{15}$ in $R^{25}$. But this is in the metastable range, and every exotic 15-sphere embeds in the metastable range with trivial normal bundle [11]. Thus we must also have $-i_*\alpha J(\tau\eta) = 0$, which proves our claim.

The case $n = 9$ is treated similarly. An argument like the previous one shows that $\theta(\kappa)$ embeds in $R^{27}$ with normal bundle $\zeta\eta$, where $\zeta$ is the normal bundle of the embedded homotopy 17-sphere $\sigma(\alpha\eta, \alpha\eta)$; but $\zeta$ is trivial by the argument in (ii) and [2, Table I], and hence $\theta(\kappa)$ embeds in $R^{27}$ with trivial normal budle. An extended investigation along the lines of [2] using results in [25, p. 164, line 7] and [25, p. 190] shows that the only exotic 18-sphere which embeds in $R^{27}$ with trivial normal bundle does not bound a spin manifold; hence the element $\theta(\kappa)$ vanishes since it corresponds to a sphere which does bound a spin manifold.

The cases $n = 16, 17$ follow from the relations given in 3.1 and the fact that the subgroup $\pi_{31}\cdot\eta \subseteqq \pi_{32}$ is contained in the image of $J$ [3, 18]. The case $n = 25$ follows from the fact that $\pi_{47}\cdot\eta^2 \subseteqq \pi_{49}$ is also contained in the image of $J$. The case $n = 33$ follows because $\pi_{63}\cdot\eta^3 = 0$ [24].

(ii) If $n = 9$, we first note that the element $\theta(\xi) \in \Gamma_{17}$ ($\xi \in \mathscr{G}_9^{18}$ as in 3.1) embeds in $R^{26}$ with trivial normal bundle. For if we repressent $\theta(\xi)$ as $\sigma(\alpha\eta, \alpha\eta)$, an argument like the preceding one shows that its normal bundle in $R^{26}$ is given by

$$i_*([\alpha\eta, \alpha\eta] + \alpha\eta J(\alpha\eta) + \alpha\eta J(\alpha\eta)) .$$

But $\alpha\eta J(\alpha\eta) = (\alpha\eta)_* J(\alpha\eta)$ has order two, and hence the normal bundle is merely $i_*\alpha_*[\eta, \eta]$. Using [9] and [25, Prop. 2.5, p. 22], we see that $[\eta, \eta] = [1_8, 1_8]\eta^2 = J(\tau_8)\eta^2$. Hence the normal bundle in $R^{26}$ is in fact $i_*\alpha J(\tau_8)\eta^2$; however, we have already noticed that $i_*\alpha J(\tau_8)$ is the normal bundle of an exotic 15-sphere embedded in $R^{24}$. But results of Kervaire [13], Levine [16], and Toda [25, pp. 104, 110] imply that a homotopy 15-sphere always embeds in $R^{24}$ with a trivial normal bundle and never embeds with a nontrivial normal bundle. Therefore we have shown that $i_*\alpha J(\tau_8)$ and $i_*\alpha J(\tau_8)\eta^2$ must vanish, and the claim concerning $\theta(\xi)$ follows immediately. Let $p(\theta(\xi)) \subseteqq \pi_{17}$ be the coset of the image of $J$ containing the Pontrjagin-Thom constructions on $\theta(\xi)$ [14]. Since $\theta(\xi) = \sigma(\alpha\eta, \tau)$ ($\tau \in \pi_8(SO_9)$ classifies the tangent bundle to $S^9$) and $J(\alpha\eta)$ factors in terms of $J(\alpha)$ and $\eta$, we may apply [8, Th. 2, p. 564]. In particular, $p(\theta(\xi))$ contains an element in the Toda bracket $B = \langle J(\alpha\eta), \beta, \eta\rangle$ for some $\beta \in \pi_7$. But clearly $J(\alpha)\langle\eta, \beta, \eta\rangle$ is contained in $B$, and the indeterminacy of $B$ is

$$J(\alpha\eta)\cdot\pi_9 + \pi_{16}\cdot\eta = \pi_{16}\cdot\eta .$$

On the other hand $\pi_7\cdot\pi_{10} \subseteqq \pi_{16}\cdot\eta$[25], and hence $p(\theta(\xi))$ contains an element of $\pi_{16}\cdot\eta$. Now the generators of $\pi_{16}\cdot\eta$ are $J(\alpha\eta^2)$ and $\eta^*\cdot\eta$. The latter cannot possibly be in $p(\theta(\xi))$ since it represents an exotic 17-sphere which does not embed in $R^{26}$ with trivial normal bundle [2] and hence $p(\theta(\xi)) =$ Image $J$. Hence $\theta(\xi) \in \partial P_{18}$, and the vanishing of $\theta(\xi)$ follows since $s'\theta(\xi) = 0$ by 3.3.

The cases $n = 17, 25$ follow from 3.1 and the previous remarks on $\pi_{31} \cdot \eta$ and $\pi_{47} \cdot \eta^2$.

(iii) By the theorem in [8] used above, we have that $p(\theta(\lambda))$ intersects the bracket $C = \langle J(\alpha\eta^2), \beta', \eta^2 \rangle$ for some $\beta' \in \pi_7$. As before we see that $J(\alpha)\langle \eta^2, \beta', \eta^2 \rangle \subseteqq C$ and the indeterminacy of $C$ is

$$J(\alpha\eta^2)\pi_{10} + \pi_{17}\eta^2 = \pi_{17}\eta^2 .$$

Since $\pi_{12} = 0$, we have $0 \in C$ and $C = \pi_{17}\eta^2$; but $\pi_{17}\eta^2$ is contained in the image of $J$, and hence $\theta(\lambda) \in \partial P_{20}$ as in the previous argument. The vanishing of $\theta(\lambda)$ now follows since $s\theta(\lambda) = 0$ by 3.3.

Of course, Theorem 3.1 reduces a great many computations of obstruction homomorphisms to the computation of the coset $p(\theta(\mu)) \subseteqq \pi_{2n-1}$; unfortunately, an effectively computable formula for this coset has not been discovered.

**4. Semifree circle actions on homotopy nine spheres.** It is not known whether every homotopy sphere admits a smooth circle action. However, every exotic 7-sphere has such an action which it inherits from its description as a Brieskorn-Hirzebruch variety. Furthermore, the Brieskorn-Hirzebruch description and a result of Bredon [4] give semifree circle actions on any exotic 9-sphere which lies in $b\mathrm{Spin}_{10}$. We shall consider those which do not lie in $b\mathrm{Spin}_{10}$. The proof of the following theorem uses some very explicit results and does not generalize to homotopy spheres not in $b\mathrm{Spin}_{8k+2}$ for arbitrary $k$.

THEOREM 4.1. *Let $\Sigma^9$ be a homotopy 9-sphere not bounding a spin manifold. Then $\Sigma^9$ has no semifree action whose fixed point set is 5-dimensional.*

If we combine this result with the proposition proved in [22, §4], we see that the fixed point set of a semifree circle action on $\Sigma^9 \notin b\mathrm{Spin}_{10}$ can only be 1-dimensional. The problem of determining whether actions with such fixed point sets exist may be reduced to homotopy theory by [5]; however, an actual homotopy-theoretic computation along the lines of [7] would be quite complicated and unrelated to the central ideas of this paper.

*Proof of* 4.1. We shall base our argument on the techniques developed in [5] and [23]. Suppose $\Sigma^9$ is a homotopy sphere having a smooth semifree circle action with $F^5$ as its fixed point set; thus $F^5$ is an integral homology sphere. Then $F^5$ bounds a contractible manifold $K^6$ by surgery theory, and the equivariant normal bundle of $F^5$ in $\Sigma^9$ is the trivial complex plane bundle (since it is fiber homotopically trivial [5]). Let $f: F \times D^4 \to \Sigma^9$ be an equivariant tubular neighborhood, let $V = \Sigma^9 - f(F \times \operatorname{Int} D^4)$, and let $W = V/S^1$. Then $\partial W = F \times S^2$, and as in [23, §2] there is a homotopy equivalence $h: (W, F \times S^2) \to (K \times S^2, F \times S^2)$. The restriction of $h$ to $F \times S^2$ is given up to homotopy by a class $R(\Phi, f) \in \pi_5(G_3) \cong \pi_5(F_2) \oplus \pi_5(SO_3)$. By surgery theory (compare [21, 2.2]) the restriction of $h$ to $F \times S^2$ is homotopic to a diffeomorphism $g$ fixed near the complement of $D^5 \times S^2$, where $D^5 \subsetneqq F$ is a closed disk.

We wish to determine the homotopy sphere $\Sigma^9$ (at least mod $b\mathrm{Spin}_{10}$) acted on by $\Phi$. This may be done since $h$ restricted to the boundary is a diffeomorphism, and hence we have a simply connected surgery problem. Actually, $h$ corresponds to a relative homotopy smoothing of the manifold pair $F \times S^2 \xrightarrow{i \cdot g} K \times S^2$, where $i: F \times S^2 \subsetneqq K \times S^2$ is inclusion and $g$ is the previously discussed diffeomorphism. We can compute the normal invariant of $h$ as usual, and the processes of passing to the total space of the circle bundle and equivariantly attaching $F \times D^4$ along $F \times S^3$ yield a homotopy smoothing

$$h': \Sigma^9 \longrightarrow F \times D^r \cup_{\bar{g}} S^9 - f(F \times \operatorname{Int} D^4)$$

whose normal invariant may be determined from that of $h$ by homotopy theory as in [23, §2] ($\bar{g}$ is an equivariant diffeomorphism of $F \times S^3$ covering $g$ on $F \times S^2$). It is immediate by direct computation of surgery obstructions that the normal invariant of $h$ in $[K \times S^2/F \times S^2, G/0]$ lies in the image of $[K \times S^2/F \times S^2, G]$. An elementary computation then shows that the normal invariant of $h'$ in $\pi_9(G/0)$ must come from an element in $\pi_9$ which bounds a spin manifold. Hence

$$\Sigma^9 \equiv F \times D^4 \cup_{\bar{g}} S^9 - f(F \times \operatorname{Int} D^4)$$

modulo $b\mathrm{Spin}_{10}$, and it is thus necessary to show that the manifold on the right does not bound a spin manifold.

Since the diffeomorphism $g$ of $F \times S^2$ is fixed off $\operatorname{Int} D^5 \times S^2$, we may assume that the equivariant covering $\bar{g}$ is fixed off $\operatorname{Int} D^5 \times S^3$ (it corresponds to a map $[F^5 - D^5] \times S^3 \to S^1$ on this complement). Therefore, by construction $g$ and $\bar{g}$ induce diffeomorphisms $\gamma$ and $\bar{\gamma}$ of $S^5 \times S^2$ and $S^5 \times S^3$ respectively, and an argument based upon Lemma 2.2 shows that

$$F \times D^4 \cup_{\bar{g}} S^9 - f(F \times \operatorname{Int} \mathrm{D}^4) \cong S^5 \times D^4 \cup_{\bar{\gamma}} D^6 \times S^3 .$$

If the diffeomorphism $\bar{\gamma}$ extends to $S^5 \times D^4$, then Lemma 2.2 implies that the homotopy sphere constructed is standard and hence the original $\Sigma^9$ bounds a spin manifold; we shall use Propositions 1.3 and 1.4 to determine whether this is the case.

Notice that composition with the nontrivial map $\eta: S^5 \to S^4$ induces an epimorphism from $\pi_4(G_3)$ to $\pi_5(G_3)$ by results of Toda [25, 5.1, 5.3, 5.8, 5.9, pp. 39, 40, 43, 44] and the natural splitting $\pi_i(G_3) \cong \pi_i(F_2) \oplus \pi_i(SO_3)$. Futhermore, since $\pi_4 = 0$ the fiber homotopy equivalence determined by any element of $\pi_4(G_3)$ is homotopic to a diffeomorphism $\gamma_1$ of $S^4 \times S^2$ which is fixed near $D^4 \times S^2$([21, 2.2] again), and hence $g$ is explicitly constructible from $g_1$ by Proposition 1.3. If $\bar{\gamma}_1$ is a diffeomorphism of $S^4 \times S^3$ fixed near $D^4_- \times S^3$ which covers $\gamma_1$, then Proposition 1.4 allows one to construct $\bar{\gamma}$ from $\bar{\gamma}_1$ in the same fashion. Let $h: S^1 \times S^4 \to S^1 \times S^4$ be a diffeomorphism induced by the nontrivial class in $\pi_1(SO_4) = Z_2$, so that $\bar{g}$ may be written as the commutator $[h \times 1_3, 1_1 \times \bar{\gamma}]$.

Since $\bar{\gamma}^1$ is fixed off $D^4_- \times S^3$, results of Levine [17, 2.3] imply that $\bar{\gamma}_1$ may be written as a product $uv$, where $u$ extends to $S^4 \times D^4$ and $v$ is fixed near the complement of $D^4_+ \times D^3_+$ (and hence corresponds to an element of $\Gamma_8$). According to Proposition 2.3, $[h \times 1, 1 \times v]$ is pseudo-isotopic to $\tau(\eta, v) \in \Gamma_9$. Since this element is central, it is easy to compute that

$$[h \times 1, 1 \times \bar{\gamma}] = [h \times 1, 1 \times u]\tau(\eta, v) .$$

But $[h \times 1, 1 \times \gamma]$ extends to $S^5 \times D^4$ because $u$ extends to $S^4 \times D^4$. Hence Lemma 2.2 implies that

$$S^5 \times D^4 \bigcup\nolimits_{\bar{\gamma}} D^6 \times S^3$$

is diffeomorphic to

$$S^5 \times D^4 \bigcup\nolimits_{\tau(\eta, v)} D^6 \times S^3 = S^9_{v\eta} .$$

But $S^9_{v\eta}$ bounds a spin manifold (e.g., see [1]); since we have shown that this sphere is the same as

$$F \times D^4 \bigcup\nolimits_{\bar{g}} S^9 - f(F \times \operatorname{Int} D^4)$$

and the latter is congruent to $\Sigma^9$ mod $b\mathrm{Spin}_{10}$, the fact that $\Sigma^9 \in b\,\mathrm{Spin}_{10}$ is immediate.

REMARK. Let $\bar{\gamma}_1$ be the map defined above with factorization $uv$. The factor $v$ may be shown to be nontrivial in certain cases if and only if the exotic 8-sphere has a semifree circle action with 4-dimensional fixed point set. Since such an action on the exotic 8-sphere may be explicitly constructed using [8, Th. 3], the factor $v$ can in fact be nontrivial. An explicit construction of the action will be given elsewhere.

## REFERENCES

1. D. W. Anderson, E. H. Brown and F. P. Peterson, *SU-cobordism, KO-characteristic numbers, and the Kervaire invariant*, Ann. of Math., **83** (1966), 54-67.

2. P. L. Antonelli, *On the stable diffeomorphism of homotopy spheres in the stable*

*range, $n < 2p$*, Bull. Amer. Math. Soc., **75** (1969), 343-346.
3. M. Barratt, M. Mahowald and M. Tangora, *Some differentials in the Adams spectral sequence II*, Topology, **9** (1970), 309-316.
4. G. Bredon, *A $\pi_*$-module structure for $\Theta_*$ and applications to transformation groups*, Ann. of Math., **86** (1967), 434-448.
5. W. Browder, *Surgery and the Theory of Differentiable Transformation Groups*, Proceedings of the Conference on Transformation Groups, New Orleans, 1967, 1-46. Springer-Verlag, New York, 1968.
6. G. Brumfiel, *On the homotopy groups of BPL and PL/O*, Ann. of Math., **88** (1968), 291-311; ibid, II, Topology, **8** (1969), 305-311.
7. ———, *Differentiable $S^1$ actions on homotopy spheres*, mimeographed.
8. D. Frank, *An invariant for almost closed manifolds*, Bull. Amer. Math. Soc., **74** (1968), 562-567.
9. P. J. Hilton and J. H. C. Whitehead, *Note on the Whitehead product*, Ann. of Math., **58** (1953), 429-442.
10. M. Hirsch, *Embeddings and compressions of polyhedra and smooth manifolds*, Topology, **4** (1966), 361-369.
11. W.-C. Hsiang, J. Levine and R. Szczarba, *On the normal bundle of a homotopy sphere in Euclidean space*, Topology, **3** (1964), 173-181.
12. I. M. James and J. H. C. Whitehead, *The homotopy theory of sphere bundles over spheres I*, Proc. London Math. Soc., (3) **4** (1954), 196-218.
13. M. Kervaire, *An interpretation of G. Whitehead's generalization of H. Hopf's invariant*, Ann. of Math., **69** (1959), 345-365.
14. M. Kervaire and J. Milnor, *Groups of homotopy spheres*, Ann. of Math., **78** (1963), 514-537.
15. R. Lashof (ed.), *Problems in differential and algebraic topology*, Seattle Conference, 1963, Ann. of Math., **81** (1965), 565-591.
16. J. Levine, *A classification of differentiable knots*, Ann. of Math., **82** (1965), 15-50.
17. ———, *Self-equivalences of $S^n \times S^k$*, Trans. Amer. Math. Soc., **143** (1969), 523-544.
18. ———, *Inertia groups of manifolds and diffeomorphisms of spheres*, Amer. J. Math., **92** (1970), 243-258.
19. M. Mahowald and M. Tangora, *Some differentials in the Adams spectral sequence*, Topology, **6** (1967), 349-369.
20. S. P. Novikov, *Differentiable sphere bundles*, Trans. Amer. Math. Soc., (2) **63** (1967), 217-243.
21. R. Schultz, *On the inertia group of a product of spheres*, Trans. Amer. Math. Soc., **156** (1971), 137-153.
22. ———, *The nonexistence of free $S^1$ actions on some homotopy spheres*, Proc. Amer. Math. Soc., **27** (1971), 595-597.
23. ———, *Semifree circle actions and the degree of symmetry of homotopy spheres*, Amer. J. Math., **93** (1971), 829-839.
24. M. Tangora, *verbal communication.*
25. H. Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Study No. 49. Princeton University Press, Princeton, 1962.
26. C. T. C. Wall, *Classification of $(n-1)$-connected $2n$-manifolds*, Ann. of Math., **75** (1962), 163-179.
27. ———, *Classification problems in differential topology VI. Classification of $(s-1)$-connected $(2s+1)$-manifolds*, Topology, **6** (1967), 273-296.

Received December 21, 1970 and revised form June 21, 1971. The author was partially supported by NSF Grant GP-19530.

PURDUE UNIVERSITY