

ON A CONJECTURE OF L. B. PAGE

C. F. SCHUBERT

If \mathcal{H} is an Hilbert space and \mathcal{M} a subspace which is invariant under a unilateral shift S on \mathcal{H} one can ask when a bounded operator T on \mathcal{M} which commutes with S can be extended to a bounded operator on all of \mathcal{H} which also commutes with S . Here this problem is considered in the special case that \mathcal{H} is a Hardy space H^2 of functions analytic in the unit-disk with values in a finite dimensional Hilbert space. For this situation an easily derived necessary condition is shown to be sufficient. Further those \mathcal{M} for which the extension to \mathcal{H} is unique are characterized.

The problem posed above has recently been considered by Lavon B. Page [4], who conjectured that the condition (2) below was sufficient as well as necessary and who proved the validity of this conjecture in several cases.

Throughout this note we shall use facts, mainly elementary or standard, about unilateral shifts with few direct references given. These facts can all be found in [1.2.3]. A subspace will always be a closed linear manifold and all operators will be linear and continuous.

Let \mathcal{H}_n denote the Hardy space of H^2 functions in the unit disk $\Delta = \{z \mid |z| < 1\}$ with values in the complex Hilbert space C_n , ($n < +\infty$). When convenient $u \in \mathcal{H}_n$ will be considered as an $n \times 1$ column vector of elements in $\mathcal{H}_1 = H^2$, and we shall freely identify $\mathcal{H}_p \oplus \mathcal{H}_q$ with \mathcal{H}_{p+q} . Let S denote the unilateral shift on \mathcal{H}_n generated by multiplication by z , i.e., $(Su)(z) = zu(z)$ for all $z \in \Delta$ and $u \in \mathcal{H}_n$. Then $S^*S = I$ but $SS^* \neq I$. For each positive integer m put $P_m = I - S^m S^{*m}$. Then P_m is an orthogonal projection and for any operator T commuting with S , $P_m T = P_m T P_m$. Thus for all $u \in \mathcal{H}_n$ we have,

$$(1) \quad \|P_m T u\| = \|P_m T P_m u\| \leq \|T\| \|P_m u\|.$$

If \mathcal{M} is a subspace of \mathcal{H}_n which is invariant under S and T is an operator on \mathcal{M} which commutes with S and has a continuous extension to all of \mathcal{H}_n which also commutes with S , then clearly this extension must satisfy (1) on \mathcal{H}_n . Hence T itself must satisfy (1) on \mathcal{M} . This led Page to the following conjecture [4]:

If \mathcal{M} is a subspace of \mathcal{H}_n , invariant under S , and T an operator on \mathcal{M} which commutes with S and satisfies (2) $\|P_m T u\| \leq \alpha \|P_m u\|$ for all $u \in \mathcal{M}$ and m , then T has an extension to all of \mathcal{H}_n which commutes with S and has norm less than or equal to α .

We have taken the liberty here of reducing Page's conjecture

for general \mathcal{H} and S to our situation. The purpose of this note is to prove the validity of the above conjecture. As our methods are essentially finite dimensional matrix arguments we are unable to prove Page's original conjecture. However some slight extensions are possible (see Remark 2.1). We can also decide whether or not an extension is unique.

1. Existence. If \mathcal{M} is a subspace of \mathcal{H}_n , invariant under S , then \mathcal{M} may be represented as $A\mathcal{H}_p$, for some $p \leq n$, where A is an isometry and $AS = SA$. Further A can be represented as an $n \times p$ matrix $A(z)$ of elements in H^∞ with the further property that $A(e^{i\theta})$ is an isometry in C_p , i.e., $\bar{A}'(e^{i\theta})A(e^{i\theta}) = I_p$, a.e. $0 \leq \theta < 2\pi$. Here $\bar{A}'(e^{i\theta})$ is the complex conjugate transpose of the matrix $A(e^{i\theta})$. In this representation A acts on $u \in \mathcal{H}_p$ by matrix multiplication, $(Au)(z) = A(z)u(z)$ for all $z \in \Delta$. $A(z)$ is uniquely determined by \mathcal{M} to within multiplication on the right by a constant unitary matrix and $\text{rank } A(z) = p$ except at a discrete set of points in Δ .

If T is an operator on \mathcal{M} which commutes with S , then there exists an operator G on \mathcal{H}_p , with $\|G\| = \|T\|$, defined uniquely by the equation

$$(2) \quad TAu = AGu \text{ for all } u \in \mathcal{H}_p.$$

Moreover G commutes with S and so is representable as a $p \times p$ matrix of elements of H^∞ . We base our extension procedure on G .

In order to reduce the inequality (2) to a less intrinsic but more computible form, let \mathcal{L}_n denote the space of L^2 functions on the unit circle with values in C_n . \mathcal{H}_n is a closed subspace of \mathcal{L}_n . If P_0 denotes orthogonal projection of \mathcal{L}_n onto the orthogonal complement of \mathcal{H}_n , then $(P_0u)(e^{i\theta})$ is that part of the Fourier series for $u \in \mathcal{L}_n$ which contains only the negative powers of $e^{i\theta}$. We use U to denote the minimal unitary extension of S to \mathcal{L}_n , i.e. U is the bilateral shift $(Uu)(e^{i\theta}) = e^{i\theta}u(e^{i\theta})$. Put $\hat{\mathcal{M}} = \{u \in \mathcal{L}_n \mid u(e^{i\theta}) = A(e^{i\theta})v(e^{i\theta}), v \in \mathcal{L}_p\}$. Then $\hat{\mathcal{M}}$ is a bilaterally invariant subspace which contains \mathcal{M} . It is easily seen to be the closure in \mathcal{L}_n of $\{u \in \mathcal{L}_n \mid U^m u \in \mathcal{M} \text{ for some integer } m \geq 0\}$. T can also be extended to \hat{T} on all of $\hat{\mathcal{M}}$ so that $\hat{T}U = U\hat{T}$ as $(\hat{T}u)(e^{i\theta}) = A(e^{i\theta})G(e^{i\theta})v(e^{i\theta})$ for all $u = Av \in \hat{\mathcal{M}}$. The above extensions can of course be carried out more abstractly but there seems little point in doing so here.

LEMMA 2.1. *If T is defined on \mathcal{M} , commutes with S and satisfies (2), then*

$$(3) \quad \|P_0\hat{T}u\| \leq \alpha \|P_0u\| \text{ for all } u \in \hat{\mathcal{M}},$$

$$(4) \quad \|(TA)(z)X\| = \|A(z)G(z)X\| \leq \alpha \|A(z)X\| \text{ for all } z \in \Delta \text{ and } X \in C_p.$$

Proof. Suppose $u \in \hat{\mathcal{M}}$ is such that $U^m u \in \hat{\mathcal{M}}$ for some integer m . Then, since T satisfies (2),

$$\begin{aligned} \|P_0 \hat{T}u\| &= \|P_m U^m \hat{T}u\| = \|P_m T U^m u\| \\ &\leq \alpha \|P_m U^m u\| = \alpha \|P_0 u\|. \end{aligned}$$

Since such $u \in \hat{\mathcal{M}}$ are dense, (3) holds by continuity for all $u \in \hat{\mathcal{M}}$. In particular if $z \in \Delta$ and $X \in C_p$ are arbitrary, then $v = X[e^{i\theta} - z]^{-1} \in \mathcal{L}_p$ and $Av \in \hat{\mathcal{M}}$. A short computation gives $(P_0 Av)(e^{i\theta}) = A(z)v(e^{i\theta})$ and $(P_0 TAv)(e^{i\theta}) = (TA)(z)v(e^{i\theta}) = A(z)G(z)v(e^{i\theta})$. Substitution of these identities into (3) yields (4) immediately.

THEOREM 2.1. *Let \mathcal{M} be a subspace of \mathcal{H}_n which is invariant under S . Let T be an operator on \mathcal{M} which commutes with S and satisfies (2). Then T has an extension T_1 to all of \mathcal{H}_n which commutes with S and satisfies $\|T_1\| \leq \alpha$.*

Proof. Following Lemma 2.1, $\|A(z)G(z)X\| \leq \alpha \|A(z)X\|$ for all $z \in \Delta$ and $X \in C_p$. Since $A(z)$ has rank p except on a discrete subset of Δ , there exists a $p \times p$ submatrix of $A(z)$ whose determinant vanishes on a discrete subset D of Δ . Without loss of generality this matrix may be taken as the first p rows of $A(z)$. Extend $A(z)$, $G(z)$, to $n \times n$ matrices $A_1(z)$, $G_1(z)$ as below,

$$A_1(z) = \left[\begin{array}{c|c} & 0 \\ A(z) & \\ \hline & I_{n-p} \end{array} \right], \quad G_1(z) = \left[\begin{array}{c|c} G(z) & 0 \\ \hline 0 & 0 \end{array} \right].$$

With $T_1(z) = A_1(z)G_1(z)A_1^{-1}(z)$ for $z \in \Delta - D$, we have $\|T_1(z)X\| \leq \alpha \|X\|$ for all X in the range of $A_1(z) = C_n$ and all $z \in \Delta - D$. For $Y \in C_n$, $F(z) = \bar{Y}'T_1(z)X$ is analytic and satisfies $|F(z)| \leq \alpha \|X\| \|Y\|$ for $z \in \Delta - D$. But D is discrete and so $F(z)$ may be extended, by Riemann's Theorem on removable singularities, to all of Δ as an analytic function. Since X, Y are arbitrary we have finally that $T_1(z)$ can be extended analytically to all of Δ and satisfies there $\|T_1(z)X\| \leq \alpha \|X\|$ for all $X \in C_n$. Now $T_1(z)A_1(z)X = A_1(z)G_1(z)X = (TA)(z)X$ for all $X \in C_p$ and $z \in \Delta$. Thus T_1 on \mathcal{H}_n defined by $(T_1 u)(z) = T_1(z)u(z)$ is an extension of T which clearly commutes with S and $\|T_1\| \leq \alpha$.

REMARK 2.1. The above theorem covers the general case of unilateral shifts of finite multiplicity. By minor modifications the proof will also be valid for unilateral shifts S on \mathcal{H} of infinite multiplicity

provided $(S\mathcal{K})^+$ contains a finite dimensional subspace M such that $u(z) \in M$ for all $z \in \Delta$ and $u \in \mathcal{M}$.

3. Uniqueness. It follows immediately from (1) and (2) that the extension T_1 of Theorem 2.1 is an extension of minimal norm. However it need not be the only extension. Clearly an extension will be unique if and only if \mathcal{M} is not contained in the kernel of any non-trivial operator on \mathcal{K}_n which commutes with S . The following lemma provides the necessary characterization in terms of the subspace $\mathcal{M}_z = \{u(z) | u \in \mathcal{M}\}$ of C_n , for each $z \in \Delta$.

LEMMA 2.1. *If $\mathcal{M} \subset \mathcal{K}_n$ is invariant under S , then \mathcal{M} is contained in the kernel of a non-trivial operator B on \mathcal{K}_n which commutes with S if and only if $\dim \mathcal{M}_z < n$ for all $z \in \Delta$.*

Proof. Suppose there exists an operator $B \neq 0$ on \mathcal{K}_n such that $BS = SB$ and $B\mathcal{M} = 0$. The corresponding matrix $B(z)$ is not identically zero and so vanishes at most on a discrete subset of Δ . Since $B(z)\mathcal{M}_z = 0$ we must have $\dim \mathcal{M}_z < n$ except possibly on a discrete subset of Δ . However $\mathcal{M} = A\mathcal{K}_p$ and so $\dim \mathcal{M}_z \leq p$ with strict inequality only on a discrete subset of Δ . Thus $p < n$ and $\dim \mathcal{M}_z < n$ for all z .

Conversely, if $\dim \mathcal{M}_z < n$ for all $z \in \Delta$ then $p < n$ in the representation of \mathcal{M} . With $A_1(z)$ the extension of $A(z)$ used in Theorem 2.1, put $B_1(z) = \det A_1(z) A_1^{-1}(z)$, i.e., the matrix of co-factors of $A_1(z)$ in transposed position. Then the elements of $B_1(z)$ are in H^∞ and if $B(z)$ denotes the $n \times n$ matrix whose first p rows are zero and whose last $n - p$ rows are the last rows of $A_1(z)$, we have $B(z)A(z) \equiv 0$. The continuous operator B on \mathcal{K}_n defined by $B(z)$ then satisfies $B\mathcal{M} = 0$.

COROLLARY 3.1. *If T on \mathcal{M} has an extension to all of \mathcal{K}_n which commutes with S then the extension is unique if and only if $\dim \mathcal{M}_z = n$ for some $z \in \Delta$.*

Even if $\dim \mathcal{M}_z = n$ for all $z \in \Delta$, T need not have any extension to all of \mathcal{K}_n . Modifying Example 2.2 of [4] take $n = p = 2$, $A: (u, v) \rightarrow (u, \varphi v)$ where φ is a singular inner function and $G: (u, v) \rightarrow (v, 0)$. Then T defined on $\mathcal{M} = A\mathcal{K}_2$ by $TA = AG$ is a continuous map of \mathcal{M} into itself which commutes with S . If T_1 is any continuous extension of T to \mathcal{K}_2 which commutes with S then also $\varphi T_1 = T\varphi$. But if $w \neq 0$ is in H^2 and is orthogonal to φH^2 then $T_1\varphi(0, w) = (w, 0)$ which is orthogonal to $\varphi T_1(0, w)$. Thus no continuous extension exists even though $\dim \mathcal{M}_z = 2$ for all $z \in \Delta$.

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QUEEN'S UNIVERSITY, KINGSTON, ONTARIO

