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DISTRIBUTING TENSOR PRODUCT OVER DIRECT PRODUCT

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This paper is an investigation of conditions on a module A under which the natural map

$$A \otimes (\Pi C_\alpha) \longrightarrow \Pi(A \otimes C_\alpha)$$

is an injection. The investigation leads to a theorem that a commutative von Neumann regular ring is self-injective if and only if the natural map

$$(\Pi F_\alpha) \otimes (\Pi G_\beta) \longrightarrow \Pi(F_\alpha \otimes G_\beta)$$

is an injection for all collections $\{F_\alpha\}$ and $\{G_\beta\}$ of free modules. An example is constructed of a commutative ring R for which the natural map

$$R[[s]] \otimes R[[t]] \longrightarrow R[[s, t]]$$

is not an injection.

R denotes a ring with unit, and all R-modules are unital. All tensor products are taken over R.

We state for reference the following theorem of H. Lenzing [2, Satz 1 and Satz 2]:

THEOREM L. (a) *A right R-module A is finitely generated if and only if for any collection $\{C_\alpha\}$ of left R-modules, the natural map $A \otimes \Pi C_\alpha \to \Pi(A \otimes C_\alpha)$ is surjective.*

(b) *A right R-module A is finitely presented if and only if for any collection $\{C_\alpha\}$ of left R-modules, the natural map $A \otimes \Pi C_\alpha \to \Pi(A \otimes C_\alpha)$ is an isomorphism.*

THEOREM 1. *For any right R-module A, the following conditions are equivalent:*

(a) *If $\{C_\alpha\}$ is any collection of flat left R-modules, then the natural map $A \otimes \Pi C_\alpha \to \Pi(A \otimes C_\alpha)$ is an injection.*

(b) *There is a set X of cardinality at least card (R) such that the natural map $A \otimes R^X \to A^X$ is an injection.*

(c) *If B is any finitely generated submodule of A, then the inclusion $B \to A$ factors through a finitely presented module.*

Note that condition (c) always holds when R is right noetherian, for then all finitely generated submodules of A are finitely presented.

Proof. (b) $\Rightarrow$ (c): If R is finite, then it is right noetherian and

(c) holds. Thus we may assume that R is infinite.

Let $f\colon F \to A$ be an epimorphism with F_R free, and set $K = \ker f$. There is a finitely generated submodule G of F such that $fG = B$.

We have a commutative diagram with exact rows as follows (Diagram I):

$$\begin{tikzcd}
 & K \otimes R^X \arrow[r] \arrow[d, "\phi"] & F \otimes R^X \arrow[r] \arrow[d, "\phi'"] & A \otimes R^X \arrow[r] \arrow[d, "\phi''"] & 0 \\
0 \arrow[r] & K^X \arrow[r] & F^X \arrow[r] & A^X \arrow[r] & 0
\end{tikzcd}$$

DIAGRAM I

Since G is finitely generated, $G^X \leqq \phi'(F \otimes R^X)$. A short diagram chase (using the injectivity of ϕ'') shows that $(G \cap K)^X \leqq \phi(K \otimes R^X)$.

card $(G) \leqq$ card (R) because R is infinite, hence card $(G \cap K) \leqq$ card (X). Thus there is a surjection $\alpha \mapsto g_\alpha$ of X onto $G \cap K$. The element $g = \{g_\alpha\}$ in $(G \cap K)^X$ must be the image under ϕ of some element $h_1 \otimes r_1 + \cdots + h_n \otimes r_n$ in $K \otimes R^X$. It follows easily that $G \cap K$ is contained in the submodule H of K generated by $h_1, \cdots, h_n$. Note that $G \cap H = G \cap K$.

$G + H$ is contained in some finitely generated free submodule F_0 of F. The map f induces a monomorphism of $G/(G \cap H)$ into A, and this monomorphism factors through the finitely presented module F_0/H. Since $fG = B$, the inclusion $B \to A$ also factors through F_0/H.

(c) $\Rightarrow$ (a): Consider any x belonging to the kernel of the natural map $\phi\colon A \otimes \Pi C_\alpha \to \Pi(A \otimes C_\alpha)$. There is a finitely generated submodule

$$\begin{tikzcd}
B \otimes \Pi C_\alpha \arrow[rr] \arrow[dr] \arrow[ddd] & & A \otimes \Pi C_\alpha \arrow[ddd, "\phi"] \\
 & E \otimes \Pi C_\alpha \arrow[ur] \arrow[d, "\phi'"] & \\
 & \Pi(E \otimes C_\alpha) \arrow[dr] & \\
\Pi(B \otimes C_\alpha) \arrow[ur] \arrow[rr, "\psi"] & & \Pi(A \otimes C_\alpha)
\end{tikzcd}$$

DIAGRAM II

B of A such that x is in the image of the map $B \otimes \Pi C_\alpha \to A \otimes \Pi C_\alpha$. By (c), the inclusion $B \to A$ factors through some finitely presented module E.

We have a commutative diagram as follows (Diagram II):

ϕ' is an isomorphism by Theorem L, and ψ is a monomorphism because all the C_α's are flat. Another diagram chase now shows that $x = 0$.

COROLLARY. *Suppose that R is (von Neumann) regular. For any right R-module A, the following conditions are equivalent:*

(a) *If $\{C_\alpha\}$ is any collection of left R-modules, then the natural map $A \otimes \Pi C_\alpha \to \Pi(A \otimes C_\alpha)$ is an injection.*

(b) *There is a set X of cardinality at least card (R) such that the natural map $A \otimes R^X \to A^X$ is injective.*

(c) *All finitely generated submodules of A are projective.*

Proof. (b) $\Rightarrow$ (c): If B is a finitely generated submodule of A, then Theorem 1 says that the inclusion $B \to A$ factors through a finitely presented module E. E is flat (because R is regular) and hence is projective. Thus B can be embedded in a projective module. Since R is semihereditary, B must be projective.

(c) $\Rightarrow$ (a): All the C_α's are flat (since R is regular), and all finitely generated submodules of A are finitely presented, so this follows directly from Theorem 1.

THEOREM 2. *Assume that R is a commutative regular ring. Then the following conditions are equivalent:*

(a) *If $\{F_\alpha\}$ and $\{G_\beta\}$ are any collections of free R-modules, then the natural map $(\Pi F_\alpha) \otimes (\Pi G_\beta) \to \Pi(F_\alpha \otimes G_\beta)$ is an injection.*

(b) *There is a set X of cardinality at least card (R) such that the natural map $R^X \otimes R^X \to R^{X \times X}$ is an injection.*

(c) *R is injective as a module over itself.*

Proof. (b) $\Rightarrow$ (c): By [1, Theorem 2.1], it suffices to show that any finitely generated nonsingular R-module B is projective.

[1, Lemma 2.2] says that we can embed B in a finite direct sum $Q_1 \oplus \cdots \oplus Q_n$, where each Q_i is a copy of the maximal quotient ring Q of R. Then B can be embedded in a direct sum $B_1 \oplus \cdots \oplus B_n$, where B_i is a finitely generated R-submodule of Q_i. Since R is semihereditary, B will be projective provided each B_i is projective. Thus without loss of generality we may assume that B is an R-submodule of Q.

Let $b_1, \cdots, b_n$ generate B. Since R is an essential submodule of Q, there is an essential ideal I of R such that $b_i I \leqq R$ for all i.

Since R is commutative, the multiplications by the elements of I induce homomorphisms of B into R. Together, these homomorphisms induce a homomorphism $f: B \to R^I$. Q is a nonsingular R-module because it has the nonsingular R-module R as an essential submodule. Thus no nonzero element of B is annihilated by I; i.e., $f: B \to R^I$ is an injection. Since $\operatorname{card}(I) \leqq \operatorname{card}(R) \leqq \operatorname{card}(X)$, there must also be an embedding of B into R^X.

Since the natural map $R^X \otimes R^X \to (R^X)^X$ is injective by (b), the corollary to Theorem 1 says that all finitely generated submodules of R^X are projective. Thus B must be projective.

(c) $\Rightarrow$ (a): By [1, Theorem 2.1], all finitely generated nonsingular R-modules are projective. Since R_R is nonsingular, ΠF_α is nonsingular; thus all finitely generated submodules of ΠF_α are projective. By the corollary to Theorem 1, the natural map $(\Pi F_\alpha) \otimes (\Pi G_\beta) \to \Pi_\beta[(\Pi F_\alpha) \otimes G_\beta]$ is an injection. Likewise, each of the maps $(\Pi F_\alpha) \otimes G_\beta \to \Pi_\alpha(F_\alpha \otimes G_\beta)$ is injective. Thus the map $(\Pi F_\alpha) \otimes (\Pi G_\beta) \to \Pi(F_\alpha \otimes G_\beta)$ must be injective.

In particular, Theorem 2 asserts that if R is a countable commutative regular ring which is not self-injective, then the natural map $R^X \otimes R^X \to R^{X \times X}$ is not an injection for any infinite set X. For example, let $F_1, F_2, \cdots$ be a countable sequence of copies of some countable field F; let R be the subalgebra of ΠF_n generated by 1 and $\bigoplus F_n$. R is obviously a countable commutative regular ring. Since ΠF_n is a proper essential extension of R_R, R_R is not injective.

If N is the set of natural numbers, then the natural map $R^N \otimes R^N \to R^{N \times N}$ is not an injection. Thus the tensor product of two one-variable power series rings, $R[[s]] \otimes R[[t]]$, is not embedded in $R[[s, t]]$ by the natural map.

References

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