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NECESSARY CONVEXITY CONDITIONS FOR THE HAHN-BANACH THEOREM IN METRIZABLE SPACES

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Local convexity appears—by the Hahn-Banach theorem—as a sufficient condition for the (topological) dual of a topological vector space to separate points from closed subspaces. The aim in the present article is to obtain *necessary* conditions, in terms of local convexity, for the latter statement to hold for a metrizable topological vector space. In particular, certain classes of such spaces are found, for which local convexity is, really, a necessary condition for the dual to separate points from closed subspaces. The course of proof goes via consideration of the more general question how two metrizable vector space topologies on a linear space must be related to each other, given that the class of linear subspaces which are closed in one of them is larger than the class of those closed in the other.

It was early recognized that a space with point-separating dual may fail to be locally convex [11]; for the sequence spaces l^p with $0 < p < 1$ constitute examples of this kind. So do the Hardy spaces H^p with $0 < p < 1$ [12] resp. [7]. However, even if the dual of a topological vector space separates points from each other, it need not separate points from closed subspaces. The first example of such a space was given by Klee [5]. Later, it has been shown that the l^p $(0 < p < 1)$-spaces [8], the H^p $(0 < p < 1)$-spaces [3], and, in fact, all complete metrizable nonconvex spaces with Schauder bases [10] are also such spaces.

So it is natural to ask: Are there also quite general conditions—similar to non local convexity—which imply that the dual of a given, not necessarily metrizable, topological vector space (over the real or complex numbers) does not separate points from closed subspaces? It does not seem so. For consider the remark of Alan Schuchat presented in [3]: let an uncountably dimensional linear space be endowed with the finest topology which is compatible with the linear structure of the space. Then any linear form on the space is continuous; so it follows from a purely algebraic argument that the dual of the space separates points from subspaces. On the other hand, the space is not locally convex; indeed, it is very far from that, since every uncountably dimensional subspace resp. quotient space fails to be locally convex. (For a different way of obtaining nonlocally convex spaces whose duals separate points from closed subspaces, see [4].)

Thus—as suggested in [3]—we naturally restrict our attention to some smaller class of spaces; for instance, the *metrizable* ones. In this article, we shall show that if a metrizable space satisfies a certain condition, which is similar to non local convexity, though stronger, then the dual of the space must fail to separate points from closed subspaces; see Corollary A below. This result is got as a special case of a more general assertion, contained in the Main theorem below. The mentioned results are followed by applications to more specific situations.

For instance, we are able to continue the work commenced by Shapiro [10] of finding certain classes of metrizable spaces, in which local convexity is, indeed, a necessary condition for the dual to separate points from closed subspaces; Corollaries E, G, and G′, and the example just before Corollary G below. So in these cases, there is a kind of converse of the "qualitative part" of the Hahn-Banach theorem. We see that local convexity is then characterized entirely in terms of the behaviour of the closed subspaces—thus in terms which do not explicitly involve convexity.—Of course, the last statements would not make any reasonable sense unless the space classes in question did contain spaces which are not locally convex as well as spaces which are. They do; for example, they contain the l^p and H^p (with $0 < p \leqq \infty$). This means that to re-establish the known facts that the duals of these spaces fail to separate points from closed subspaces when $p < 1$, one could use either Corollary E or Corollary G below.

It is natural to ask for more general such "non convex" characterizations of local convexity. In this connection, let us mention that a metrizable space is known to be locally convex as soon as every weakly bounded subset is bounded [6].

2. Let us now see how our problem can be recognized as a special case of a more general one.

Remark. Let L be a vector space with a vector space topology τ. If $\mathscr{N}$ is the 0-neighbourhood filter for τ, denote by τ_1 the vector space topology whose 0-neighbourhood filter is $\{\mathrm{co}\, V \mid V \in \mathscr{N}\}$; here and in the sequel, co denotes "convex hull of". Then we have:

1° τ is locally convex if and only if $\tau = \tau_1$;

2° the dual of (L, τ) separates points from τ-closed subspaces if and only if every τ-closed subspace is τ_1-closed.

(By (L, τ) we mean, of course, the topological vector space whose underlying linear space is L and whose topology is τ.) Namely, notice that τ_1 is the finest locally convex topology that is coarser than τ. From this, 1° is immediate. For 2°, observe that a linear form on L is continuous with respect to τ if and only if it is continuous with respect to τ_1 (i.e., if a linear form is continuous with respect to τ

and thus bounded on $V \in \mathscr{N}$, say, then it is bounded also on co V and hence τ_1-continuous; the converse implication is obvious). By an elementary corollary of Hahn-Banach's theorem, a linear subspace of L is then τ_1-closed if and only if it can be separated from any point outside it by some τ-continuous linear form. Thus, to say that *any* τ-closed subspace fulfils the latter condition amounts to the same as to say that any τ-closed subspace is τ_1-closed.

It is also clear that τ_1 is metrizable if τ is (since τ_1 must then have a countable 0-neighbourhood base).

From this remark it follows that our problem is a special case of a more general question, that is:

Suppose that L is a vector space endowed with two topologies τ and τ_1 which are compatible with the linear structure of the space and metrizable. Then, if every τ-closed linear subspace of L is τ_1-closed, how must τ be related to τ_1?—If τ and τ_1 are both *locally convex*, it follows from standard results in duality theory that τ_1 must be finer than τ (see, e.g., [9]). (In fact, for this conclusion, it suffices that τ is locally convex; cf. [6].) So the question is, essentially: Can anything in that direction be said also in the general case, with no assumptions of local convexity?

The following theorem is a partial answer. What it says is, briefly, that if τ satisfies a certain condition stronger than failure of being coarser than τ_1, then every τ-closed subspace cannot be τ_1-closed.

MAIN THEOREM. *If a vector space L has vector space topologies τ and τ_1 which are metrizable, and a non void class $\mathscr{C}$ of linear subspaces satisfying the conditions*

(C) *There is a 0-neighbourhood U for τ such that for every $K_1 \in \mathscr{C}$ and finite-dimensional subspace $F \subset L$, the set*

$$K_1 \cap \bigcap_{K \in \mathscr{C}} (K+F+U)$$

fails to be a 0-neighbourhood for the subspace topology which τ_1 induces on K_1,

(D) *The space $K_1 \cap K_2$ belongs to $\mathscr{C}$ whenever both K_1 and K_2 do, then L has a linear subspace which is τ-closed but not τ_1-closed. In fact, if z_0 is any point in $L \backslash \bigcap_{K \in \mathscr{C}} \bar{K}$ (closures with respect to τ), then one can find a τ-closed subspace N, say, so that z_0 is outside it but belongs to its τ_1-closure.*

Furthermore, it will be apparent from our construction of N that in the case when $\mathscr{C}$ is a decreasing sequence of subspaces $K_1 \supset K_2 \supset \cdots$, we get an N so that $K_j \cap N$ has finite codimension in N for each $j \geqq 1$.

Proof. For convenience, let τ be metrized by a metric $d(\cdot, \cdot)$ that is given by a *pseudo-norm* (in the sense of [9]), i.e., a nonnegative real-valued functional $|\cdot|$ on L satisfying

(a) $|x + y| \leqq |x| + |y|$, $x, y \in L$,

(b) $|\lambda x| \leqq |x|$, λ scalar with $|\lambda| \leqq 1$, $x \in L$,

(c) $|x| = 0$ implies $x = 0$ for $x \in L$, and

(d) $d(x, y) = |x - y|$, $x, y \in L$.

(This is certainly possible; see, e.g., [9].)

Let us write $S(\lambda) = \{x \in L \mid |x| < \lambda\}$ for $\lambda > 0$. Then (b) says precisely that each set $S(\lambda)$ is circled (i.e., balanced); and so every set $S(\lambda) + H$, where H is a subspace, is circled. Hence (b) may be replaced by the stronger statement

(b)′ the function $\lambda \rightarrow d(\lambda x, H)$, $\lambda > 0$, is monotone for every fixed $x \in L$ and linear subspace H; further, $d(\theta x, H) = d(x, H)$, θ scalar with $|\theta| = 1$.

By $d(y, H)$ we mean, as usual, $\inf\{|y - z| \mid z \in H\}$. Further, we shall write $S_1(\lambda) = \{x \in L \mid |x|_1 < \lambda\}$, $\lambda > 0$, where $|\cdot|_1$ is a pseudo-norm for τ_1.

Let the element z_0 not belonging to the τ-closures of all the spaces of $\mathscr{C}$ be given. (By (C), such elements must exist.) We intend to construct recursively subspaces $N_1 \subset N_2 \subset \cdots$, such that

(i) $z_0 + N_k$ meets $S_1(1/k)$ for every $k \geqq 1$ and

(ii) all the affine subspaces $z_0 + N_k$, $k \geqq 1$, stay away from $S(\rho_1)$ for some $\rho_1 > 0$.

Having done this, form the subspace $N = \overline{\bigcup_{k \geqq 1} N_k}$ (the closure being taken with respect to τ). Then, by (ii), the point z_0 will have "τ-distance" at least ρ_1 to N and will thus be outside N; but by (i), it will belong to the τ_1-closure of N. And so, the assertion of the theorem will follow.

So we only have to construct the finite-dimensional subspaces N_k. First notice that the τ-0-neighbourhood U in condition (C) may clearly be replaced by a smaller one, so we can assume that $U = S(\rho)$ for some $\rho > 0$. Take $H_1 \in \mathscr{C}$ so that $z_0 \notin \bar{H}_1$. Write $N_0 = \{0\}$ and set $\rho_1 = \min(\rho/2, d(z_0, H_1)/2)$. We will now employ a recursive procedure, which will give the N_k as well as spaces $H_k \in \mathscr{C}$, $k \geqq 2$, in such a way that, in the place of (ii), the even stronger statement

(ii)′ $d(z_0 + N_{k-1} + H_k, 0) > \rho_1(1 + 1/k)$ for $k \geqq 1$

is valid. It is for $k = 1$; so suppose that for some $k \geqq 2$, the N_{k-2} and H_{k-1} have been found while N_{k-1} and H_k are wanted. Then since the function

$$\lambda \rightarrow d(\lambda z_0 + N_{k-2} + H_{k-1}, 0)$$

is continuous (this is a simple consequence of the triangle inequality), there is a number θ', $0 < \theta' < 1$, such that

$$(*) \qquad d(\theta' z_0 + N_{k-2} + H_{k-1}, 0) > \rho_1\Big(1 + \frac{1}{k-1}\Big) .$$

We are now going to use condition (C); by this, the set

$$H_{k-1} \cap \bigcap_{K \in \mathscr{C}} \Big(K + \tilde{N}_{k-2} + \frac{1}{1-\theta'} S(\rho)\Big) ,$$

where $\tilde{N}_{k-2}$ denotes the linear hull of N_{k-0} and $\{z_0\}$, cannot be a 0-neighbourhood with respect to the topology that τ_1 induces on H_{k-1}. Since $S_1(1/k)$ is such a 0-neighbourhood, we can now take $H_k \in \mathscr{C}$ so that first, $H_k \subset H_{k-1}$ (possible in virtue of (D)), and second, the set

$$S_1\Big(\frac{1}{k}\Big) \cap H_{k-1} \Big\backslash \Big(\frac{1}{1-\theta'} S(\rho) + H_k + \tilde{N}_{k-2}\Big)$$

becomes nonempty. Take a point z_k in this set. By its definition, this z_k must satisfy

1° $z_k \in S_1(1/k)$ and

2° $d((1-\theta')z_k, H_k + \tilde{N}_{k-2}) > \rho$.

Now we define N_{k-1} as the linear hull of N_{k-2} and $\{z_k - z_0\}$. We must verify (i) and (ii)′. First, $z_k \in z_0 + N_{k-1}$, whence 1° gives (i). For (ii)′, notice that

$$\begin{aligned} d(z_0 + N_{k-1} + H_k, 0) &= d(z_0 + \operatorname{lin}\{z_k - z_0\} + N_{k-2} + H_k, 0) \\ &= d(z_0 + \operatorname{lin}\{z_k - z_0\}, N_{k-2} + H_k) \\ &= \inf\{d(\theta z_0 + (1-\theta)z_k, N_{k-2} + H_k) \mid \theta \text{ scalar}\} \end{aligned}$$

with lin meaning linear hull. So we get (ii)′ by proving that

$$d(\theta z_0 + (1-\theta)z_k, N_{k-2} + H_k) \geqq \rho_1\Big(1 + \frac{1}{k-1}\Big)$$

for every scalar θ. Any θ occurs in at least one of two cases:

I. $|\theta| \geqq \theta'$. Then, by property (b)′ of the metric and by (*),

$$\begin{aligned} &d(\theta z_0 + (1-\theta)z_k, N_{k-2} + H_k) \\ &\quad \geqq d(\theta z_0, N_{k-2} + H_k + \operatorname{lin}\{z_k\}) \\ &\quad \geqq d(\theta z_0, N_{k-2} + H_{k-1}) , \quad \text{since} \quad H_k \cup \{z_k\} \subset H_{k-1} , \\ &\quad \geqq d(\theta' z_0, N_{k-2} + H_{k-1}) > \rho_1\Big(1 + \frac{1}{k-1}\Big) . \end{aligned}$$

II. $|1-\theta| \geqq 1-\theta'$. Then, likewise and by 2°,

$$\begin{aligned} &d(\theta z_0 + (1-\theta)z_k, N_{k-2} + H_k) \\ &\quad \geqq d((1-\theta)z_k, \tilde{N}_{k-2} + H_k) \end{aligned}$$

$$\geqq d((1-\theta')z_k, \tilde{N}_{k-2} + H_k)$$
$$> \rho \geqq \rho_1\Big(1 + \frac{1}{k-1}\Big) .$$

Applied to our original problem, the Main theorem gives the following corollary. What the latter says is, briefly, that if a metrizable space satisfies a certain condition stronger than non local convexity, then its dual must fail to separate points from closed subspaces.

COROLLARY A. *If a metrizable topological vector space E has a non void class $\mathscr{C}$ of linear subspaces satisfying the conditions*

(CC) *There is a 0-neighbourhood U in E such that for every $K_1 \in \mathscr{C}$ and finite-dimensional subspace F in E, the set*

$$K_1 \cap \bigcap_{K \in \mathscr{C}} (K + F + U)$$

does not contain any intersection of a convex 0-neighbourhood with K_1,

(D) *The space $K_1 \cap K_2$ belongs to $\mathscr{C}$ whenever both K_1 and K_2 do, then the topological dual of E fails to separate points from closed subspaces.*

Proof. Apply the Main theorem with E in the place of (L, τ) and with τ_1 defined as in the remark preceding the theorem; the assertion then follows in virtue of that remark.

3. Before we consider more specific situations where Corollary A applies, let us return for a while to the more general problem on the comparison of topologies.

COROLLARY B. *Let L be a vector space endowed with vector space topologies τ, τ_1, and τ_b, such that τ and τ_1 are metrizable, τ_b is separated and locally bounded, and such that τ_1 and τ_b are coarser than τ. Further, let $L_1 \supset L_2 \supset \cdots$ be a decreasing sequence of linear subspaces and let τ^{ij}, τ_1^{ij}, and τ_b^{ij}, resp., denote the topologies defined by τ, τ_1, and τ_b, resp., on L_i/L_j for $1 \leqq i \leqq j$.* If *a linear subspace N, such that every $L_i \cap N$ has finite codimension in N, is always τ_1-closed if it is τ-closed, then there must be a space L_{i_0} for which the topologies of all its quotient spaces L_{i_0}/L_j, $j \geqq i_0$, satisfy*

$$\tau^{i_0 j} \supset \tau_1^{i_0 j} \supset \tau_b^{i_0 j} ,$$

where $\supset$ means "finer than".

Proof. Apply the Main theorem with $\mathscr{C} = \{L_1, L_2, \cdots\}$—taking notice of the special assertion for the case when $\mathscr{C}$ is a sequence. Namely, we now show that if the space L_{i_0} does *not* exist, this $\mathscr{C}$ must satisfy condition (C). For U, take any *bounded* 0-neighbourhood of τ_b. If L_{i_0} does not exist, then, for any given i, the relation $\tau_1^{ij} \supset \tau_b^{ij}$ must fail as soon as j is taken sufficiently large. First, let us see what this means. Suppose that F is a given finite-dimensional subspace of L. Consider the diagram

$$\begin{array}{ccccc} & & L_i + F & \xrightarrow{g_2} & (L_i + F)/(L_j + F) \\ & & \uparrow{\scriptstyle g_3} & & \updownarrow \text{ canonical isomorphism} \\ L_i/L_j & \xrightarrow{g_1} & (L_i + F)/L_j & \xrightarrow{g_4} & ((L_i + F)/L_j)/g_3(F_1) \ , \end{array}$$

where F_1 is the finite-dimensional space supplementary to L_j in $L_j + F$, where g_1 is the imbedding, and where g_2, g_3, and g_4 are the quotient mappings. Here, if L is topologized by any linear topology, we have (since $L_i \supset L_j$) the topological isomorphisms

$$(L_i + F)/L_j \cong (L_i \times F_2)/L_j \cong (L_i/L_j) \times F_2 \ ,$$

where F_2 is the subspace supplementary to L_i in $L_i + F$. But since F_2 is finite-dimensional and thus can have only one linear topology, this means that the relation $\tau_1^{ij} \supset \tau_b^{ij}$ holds if and only if the corresponding relation holds between the topologies defined by τ_1 resp. τ_b on $(L_i + F)/L_j$. Now, $(L_i + F)/L_j$ is also isomorphic to

$$(((L_i + F)/L_j)/g_3(F_1)) \times g_3(F_1) \ ,$$

where $g_3(F_1)$ is finite-dimensional; so we can use the same argument once more to show that $\tau_1^{ij} \supset \tau_b^{ij}$ if and only if $\tilde{\tau}_1^{ij} \supset \tilde{\tau}_b^{ij}$, where $\tilde{\tau}_1^{ij}$ and $\tilde{\tau}_b^{ij}$ are the topologies defined by τ_1 resp. τ_b on the space

$$\frac{\dfrac{L_i + F}{L_j}}{g_3(F_1)} \cong \frac{L_i + F}{L_j + F} \, .$$

Further, since U is bounded 0-neighbourhood with respect to τ_b, the relation $\tilde{\tau}_1^{ij} \supset \tilde{\tau}_b^{ij}$ means precisely that the set $g_2((L_i + F) \cap U)$ is a 0-neighbourhood with respect to $\tilde{\tau}_1^{ij}$. Thus, we find that if the relation $\tau_1^{ij} \supset \tau_b^{ij}$ fails, then $g_2((L_i + F) \cap U)$ must fail to be a 0-neighbourhood with respect to $\tilde{\tau}_b^{ij}$.

Now we can show that $\mathscr{C}$ satisfies condition (C) under the assumption that L_{i_0} does not exist. For, given $K_1 = L_i$ in $\mathscr{C}$ and finite-dimensional $F \subset L$, we notice that the relation $\tau_1^{ij} \supset \tau_b^{ij}$ must then fail for j large. As we have just seen, this means that for j large, the set $g_2((L_i + F) \cap U)$ must fail to be a 0-neighbourhood in $\tilde{\tau}_1^{ij}$; so then, the inverse image under g_2 of that set, viz.,

$$(L_i + F) \cap U + L_j + F = (L_i + F) \cap (L_j + F + U)$$

cannot be a 0-neighbourhood in the topology induced on $L_i + F$ by τ_1. Hence, the set

$$L_i \cap \bigcap_{j \geqq 1} (L_j + F + U)$$

certainly fails to be a 0-neighbourhood in the topology defined by τ_1 on L_i, as required.

Let us rephrase this corollary for the special case when $\tau = \tau_b$:

COROLLARY C. *Let L be a linear space with a locally bounded topology τ and a coarser metrizable topology τ_1. Further, let $L_1 \supset L_2 \supset \cdots$ be a decreasing sequence of subspaces.* If *a linear subspace N, such that every $L_i \cap N$ has finite codimension in N, is always τ_1-closed if it is τ-closed, then there must be a space L_{i_0} for which the topologies defined by τ and τ_1 coincide on all of its quotient spaces L_{i_0}/L_j, $j \geqq i_0$.*

4. The remainder of this article will be devoted entirely to our original problem, which was also dealt with in Corollary A.

Let us first see what Corollary C says on this problem for locally bounded spaces.

COROLLARY D. *If E is a locally bounded space whose dual separates points from closed subspaces, then any decreasing sequence $E_1 \supset E_2 \supset \cdots$ of closed subspaces has a space E_{i_0}, all of whose quotient spaces E_{i_0}/E_j are locally convex.*

(*In fact, for the latter statement to be true for a particular sequence $E_1 \supset E_2 \supset \cdots$ it suffices that the dual separates points from those closed subspaces N in which all spaces $E_i \cap N$ have finite codimension.*)

Proof. We may assume E to be separated; for, at any rate, the intersection of all 0-neighbourhoods in the *locally bounded* E must be a closed linear subspace G; so we may consider E/G, which is separated, in the place of E (with E_i/G in the role of E_i, $i \geqq 1$). Apply Corollary C with E in the place of (L, τ) and with τ_1 defined as in the remark preceding the Main theorem; the assertion then follows in virtue of that remark.

Specializing further, we get a converse of the "qualitative part" of Hahn-Banach's theorem in a specific situation.

COROLLARY E. *Let E be a locally bounded space which is isomor-*

phic to the product space $E \times E$. Then the space E must be locally convex if its dual separates points from closed subspaces.

Proof. Write successively

$$\begin{aligned} E &= K_0 \times K_1 \\ K_1 &= K_{10} \times K_{11} \\ K_{11} &= K_{110} \times K_{111} \\ \cdots & \qquad\qquad , \end{aligned}$$

all the spaces $K_0, K_1, K_{10}, \cdots$ being isomorphic to E. Considering the spaces $K_1, K_{11}, K_{111}, \cdots$ as subspaces of E, we see that these form a decreasing sequence such that each of the quotient spaces K_1/K_{11}, $K_{11}/K_{111}, \cdots$ is isomorphic to E. Thus, *a fortiori*, if E fails to be locally convex, all these quotient spaces do; and Corollary D shows that the dual of E fails to separate points from closed subspaces.

REMARK. As pointed out in the introduction, one can use this corollary to prove the known facts that the duals of l^p resp. H^p $(0 < p < 1)$ do not separate points from closed subspaces. The corollary is evidently applicable to l^p; so it is to H^p, for this space can easily be seen to be canonically isomorphic to the product of the subspaces

$$\{f(z^2) \,|\, f(z) \in H^p\}$$
$$\{zf(z^2) \,|\, f(z) \in H^p\} ,$$

each of which is isomorphic to H^p.

This remark says that there are, indeed, non locally convex (as well as locally convex) spaces among those locally bounded spaces E which are isomorphic to their own "squares" $E \times E$. Hence, it makes sense to call Corollary E "a converse of 'the qualitative part' of a special case of Hahn-Banach's theorem".

5. We are now going to cosider cases where the subspace class $\mathscr{C}$ (occurring in the Main theorem and in Corollary A) is of quite another type than in the applications till now (cf. the proof of Corollary B).

COROLLARY F. *The dual of a metrizable space E can separate points from closed subspaces only if E has the property*

(P) *For any 0-neighbourhood V in E, the weak closure of $V + F$ contains a convex 0-neighbourhood for some finite-dimensional subspace $F \subset E$.*

Thus, a metrizable space whose dual separates points from closed subspaces must refrain from possessing a *certain kind* of non local convexity, that is, failure of (P). Notice that this kind of non convexity also involves "smallness" of weak closures—in other words, plentitude of continuous linear forms. It might seem a bit odd that a statement of the latter sort is a part of a sufficient condition for the dual *not* to separate points from closed subspaces. But the question is whether a non locally convex metrizable space with "lack" (in some suitable sense) of continuous linear forms could have a general "lack" of closed linear subspaces, to the effect that the dual would be able to separate points from closed subspaces.

Of course, this corollary would not be very relevant to our problem unless there were spaces without property (P). However, we shall soon take advantage of the fact that for certain classes of spaces (with "plenty" of continuous linear forms, but not always being locally convex), property (P) is, indeed, equivalent to local convexity.

Proof of Corollary F. Suppose that (P) fails for E. We shall apply Corollary A with $\mathscr{C}$ equal to the class of all *closed subspaces with finite codimension*. To that end, condition (CC) must be verified, i.e., a 0-neighbourhood U such that

$$\operatorname{co} V \cap K_1 \setminus \bigcap_{K \in \mathscr{C}} (K + F + U) \neq \phi ,$$

for any 0-neighbourhood V, $K_1 \in \mathscr{C}$, and finite-dimensional F, must be found. Notice that $\bigcap_{K \in \mathscr{C}} (K + F + U)$ is contained in the weak closure of $F + U$; so it will do to take U according to the following sublemma.

SUBLEMMA. *If a topological vector space fails to have property* (P) [resp. *fails to be locally convex*], *it has a* 0*-neighbourhood* U *such that for any closed subspace* K_1 *of finite codimension, every set* $\operatorname{co} V \cap K_1$, *where* V *is an arbitrary* 0*-neighbourhood, reaches outside the weak closure of* $F + U$ *for any finite-dimensional subspace* F [resp. *reaches outside* U].

Proof of sublemma. By assumption, there is a 0-neighbourhood W so that every $\operatorname{co} V$ reaches outside the weak closure of every $F + W$ [resp. outside W]. We shall take the 0-neighbourhood U such that $U + U \subset W$. Then, if w denotes weak closure, we get

$$(\dagger) \qquad \mathrm{w}(F + U) + U \subset \mathrm{w}(F + W)$$

for any F. Let K_2 be a subspace supplementary to K_1—by assumption, K_2 is finite-dimensional—and let π_1 resp. π_2 be the (continuous) projections onto K_1 resp. K_2 vanishing on K_2 resp. K_1. Since K_2 is locally convex, π_2 is continuous also with respect to the topology with 0-neighbourhood filter $\{\mathrm{co}\, V\}$, where V runs through the 0-neighbourhood filter for the original topology. Thus, $\pi_2(\mathrm{co}\, V) \subset U$ for V small. Since $\pi_1(x) = x - \pi_2(x)$, and since $\mathrm{co}\, V$ has elements outside the weak closure of $F + W$ [resp. outside W] for any given F and arbitrarily small V, this means, by (†), that $\pi_1(\mathrm{co}\, V)$ must reach outside the weak closure of $F + U$ [resp. outside U], which certainly gives the assertion. (For, notice that by continuity of π_1, the class of sets $\{\pi_1(\mathrm{co}\, V)\}$ is the 0-neighbourhood filter for the topology induced on K_1 by the topology with 0-neighbourhood filter $\{\mathrm{co}\, V\}$).

6. The Corollary F will yield more significant consequences when applied to certain cases in which E can be embedded into a complete metrizable space that has a *Schauder basis*. Remember that a Schauder basis in a complete topological vector space E is a sequence $\{x_k\}$ of elements thereof such that, for every element x in E, there is one and only one scalar sequence $\{\xi^k\}$ which makes the sequence $\{\sum_{k=1}^n \xi^k x_k\}_n$ converge to x as $n \to \infty$. The scalars ξ^k are given by continuous linear forms $\xi^k(\cdot)$ on E—the *coordinate functionals*.

Suppose that the complete space E is metrizable and has a Schauder basis. Introduce the mappings

$$\pi_n(\cdot) = \sum_{k=1}^{n} \xi^k(\cdot) x_k \quad (n \geqq 1) ,$$

of E into itself (notations as above). The following statement is a consequence of the definition of Schauder basis.

(B) *The class $\{\pi_n\}_n$ is equicontinuous.*

This can be seen from the fact that if $|\cdot|$ is a pseudo-norm on E, then $|\cdot|_1 = \sup\{|\sum_{k=1}^n \xi^k(\cdot)x_k| \mid n \geqq 1\}$ is an equivalent pseudonorm; for a verification, which rests upon the open mapping theorem, of the latter assertion, see [1], proof of Th. 2.

Next, we give some conditions for property (P) of Corollary F to be equivalent to local convexity.

PROPOSITION. *Suppose that E is a topological vecter space for which one of the following statements* (α)—(γ) *is valid:*

(α) *The conditions*

(N) *There is a 0-neighbourhood base consisting of weakly closed sets in E, and*

(S) *There is a 0-neighbourhood U of E such that $\bigcap_{\lambda>0} \lambda U = \{0\}$, are both fulfilled;*

(β) *The space E is locally bounded and fulfils* (N); *or*

(γ) *The space E is the dense subspace which is the linear hull of the set of basis vectors in a complete metrizable space with Schauder basis.*

Then E has property (P) *if and only if E is locally convex.*

We need three lemmas.

FIRST LEMMA. *If V is a closed and circled subset, which contains no ray from the origin, of a separated topological vector space, then $V + F$ is closed for every finite-dimensional subspace F.*

SECOND LEMMA. *If V is a closed, circled subset of a separated topological vector space with a finite-dimensional subspace F such that $V + F$ is a 0-neighbourhood and $V \cap F$ is a compact 0-neighbourhood in F, then $V + V + V$ is a 0-neighbourhood.*

THIRD LEMMA. *Let E be a metrizable space whose completion $\hat{E}$ has a Schauder basis such that E contains all the basis vectors and thus also their linear hull G, say. For any 0-neighbourhood W in E, there is a 0-neighbourhood U in E such that for any finite-dimensional subspace F of G, the* weak *closure of $U + F$ is contained in the closure of $W + F$.*

Proof of first lemma. Let a be a cluster point of $V + F$, and let $\mathscr{A}$ be a filter on $V + F$, converging to a. It must be shown that a belongs to $V + F$. Write

$$\mathscr{A}_1 = \{\{x - y \mid x \in A,\, y \in V\} \cap F \mid A \in \mathscr{A}\} .$$

Clearly, $\mathscr{A}_1$ is filter base on F. If A_1 has a cluster point b, it is seen that $a - b$ must belong to the closed set V; thus, $a = a - b + b \in V + F$, as required.

Suppose that $\mathscr{A}_1$ does not have any cluster point; this will be shown to give a contradiction. Let $\mathscr{A}_2$ be the trace of $\mathscr{A}_1$ on $F\backslash\{0\}$. If $|\cdot|$ is an ordinary Euclidean norm on F, define the positive number $\lambda(x)$ for any $x \in F\backslash\{0\}$ so that $|\lambda(x)x| = 1$. By compactness, the image of $\mathscr{A}_2$ under $x \to \lambda(x)x$ has a cluster point z_0. In virtue of the assumption that $\mathscr{A}_1$ has no cluster point, the local compactness of F is also seen to be contradicted unless $\{\lambda(A) \mid A \in \mathscr{A}_2\}$ converges to zero. On account of the continuity of multiplication, the filter base

$$\mathscr{A}_3 = \{\{\lambda(x)y \mid x \in A_2,\, y \in A\} \mid A_2 \in \mathscr{A}_2,\, A \in \mathscr{A}\}$$

also converges to zero (since $\mathscr{A}$ was convergent). Now, let ρ be an arbitrary positive number; the definitions of z_0 and $\mathscr{A}_1$ show that ρz_0 is a cluster point to the filter base

$$\{\{\rho\lambda(x - y)(x - y) \mid x \in A,\ y \in [\{x\} - (F\backslash\{0\})] \cap V\} \mid A \in \mathscr{A}\} .$$

Since $\mathscr{A}_3$ converges to zero, this means that ρz_0 is also a cluster point to

$$(\ddagger) \qquad \{\{-\rho\lambda(x - y)y \mid x \in A,\ y \in [\{x\} - (F\backslash\{0\})] \cap V\} \mid A \in \mathscr{A}\} .$$

But $\{\lambda(A) \mid A \in \mathscr{A}_2\}$ converges to zero, so the coefficient $\rho\lambda(x - y)$ in ($\ddagger$) must be smaller than one for A small; and so, since V is circled, $\rho\lambda(x - y)y$ in ($\ddagger$) belongs to V for A small. We have thus seen that ρz_0 is a cluster point to a filter base on V. But V is closed, whence $\rho z_0 \in V$. The point z_0 being distinct from zero and ρ arbitrary positive, the assumption that V contains no ray from the origin is contradicted.

Proof of second lemma. The boundary Z, say, of $V \cap F + V \cap F$ in F is disjoint from $V \cap F$. Further, Z is compact. Hence, by a standard argument (cf. [2], II. 4.3), we find a circled 0-neighbourhood W so that $V + W$ is disjoint from Z. But $(V + W) \cap F$ is circled, and Z meets every ray from the origin; so $(V + W) \cap F \subset V \cap F + V \cap F$. The set $U = (V + F) \cap W$, which, by assumption, is a 0-neighbourhood, clearly fulfils $U \subset V + (V + U) \cap F$ (for, writing $u = v + f,\ v \in V,\ f \in F$, given $u \in U$, we get $f \in -V + U = V + U$). Thus,

$$\begin{aligned} U &\subset V + (V + U) \cap F \subset V + (V + W) \cap F \\ &\subset V + V \cap F + V \cap F\,, \quad \text{by the choice of } W\,, \\ &\subset V + V + V\,, \end{aligned}$$

whence the assertion.

Proof of third lemma. On account of property (B) of Schauder bases, we take the 0-neighbourhood U so that $\pi_n(U) \subset W$ for every $n \geqq 1$. To see that this suffices, let z be an arbitrary point outside the closure of $W + F$ (for any fixed finite-dimensional $F \subset G$). It is to be shown that z is also outside the weak closure of $U + F$. By the relation $F \subset G$, there is an $m \geqq 1$ such that $\xi^k(F) = \{0\}$ for $k \geqq m$ (with $\xi^k(\cdot)$ denoting the coordinate functionals as in the explanations before (B) above). Choose $n \geqq m$ so that $\pi_n(z) \notin \overline{W + F}$. Since $n \geqq m$, the space F is contained in the range P, say, of π_n, and we have $\pi_n(F) = F$. Of course, the subset $\overline{(W + F)} \cap P$ of the finite-dimensional space P is *weakly closed*, and thus, so is the set

$$\pi_n^{-1}(\overline{(W + F)} \cap P) = \overline{\pi_n^{-1}(W \cap P) + F}\,;$$

the equality follows from what was just said about F. This weakly closed set contains $U+F$ but not z, whence the assertion.

Proof of propositon. The "if"-part being trivial, we prove the "only if"-part for the different cases $(\alpha)-(\gamma)$.

Case (α): Suppose that E has property (P). First, we see that for every 0-neighbourhood V, there must be a finite-dimensional subspace F so that the set $V+F$ *itself* contains a convex 0-neighbourhood. For, by (N), we may assume that V is weakly closed (replacing it by a smaller 0-neighbourhood if necessary); likewise (S) permits us to assume that V contains no ray from the origin, i.e., in virtue of the first lemma just given (applied to the linear space E endowed with the weak topology), that $V+F$ is weakly closed for any F. Hence, (P) says that $V+F$ contains a convex 0-neighbourhood for suitable F.

In other words: the set $U+F$ must be a 0-neighbourhood for the topology τ_1 with 0-neighbourhood filter $\{\operatorname{co} V\}$, where V runs through the 0-neighbourhood filter of E, for some finite-dimensional F, whenever a 0-neighbourhood U is given. Once again taking account of (N) and (S), let U be an arbitrarily small 0-neighbourhood which is weakly closed (and thus τ_1-closed) and which fulfils $\bigcap_{\lambda>0}\lambda U=\{0\}$ (so that $U\cap F$ is compact for finite-dimensional F). Hence, our second lemma, applied to the space E with topology τ_1, shows that $U+U+U$ contains a convex 0-neighbourhood. The U being arbitrarily small, E must be locally convex.

Case (β): If E is separated, this is a special case of (α). The general case is easily reduced to the separated case (cf. proof of Corollary D above).

Case (γ): We use the notation introduced in connection to statement (B) above. Write

$$P_k=\operatorname{lin}\{x_1,\cdots,x_k\}\quad\text{and}\quad H_k=\operatorname{lin}\{x_{k+1},x_{k+2},\cdots\}\,.$$

Suppose that E is not locally convex; by the sublemma of the proof of Corollary F above, it follows that there is a 0-neighbourhood W_1 for which $\operatorname{co} V\cap H_k$ reaches outside $W_1\cap H_k$ for every $k\geqq 1$ and every 0-neighbourhood V. Now, one can easily see that property (B) of Schauder bases stated above would be violated unless there were a 0-neighbourhood W, such that for any $k\geqq 1$, an element in H_k and outside W_1 must also be outside $P_k+(W+W)$; and then, of course, outside $\overline{P_k+W}$. Summing up, we see that every set $\operatorname{co} V$ reaches outside $\overline{P_k+W}$. By the third lemma just given (applied with

$G = E$), there is a 0-neighbourhood U such that the weak closure of $P_k + U$ ($k \geqq 1$) does not contain any co V. Since every finite-dimensional F is in some P_k, the space E cannot have property (P).

By the preceding Proposition, we can use Corollary F to obtain converses to the "qualitative part" of Hahn-Banach's theorem for specific situations.

EXAMPLE. If we let E in Corollary F be the linear hull of the basis vectors in a complete space with Schauder basis and apply (γ) of the mentioned proposition, we re-obtain a result of Shapiro [10]: *A complete metrizable space with Schauder basis must be locally convex if the dual separates points from closed subspaces.*

COROLLARY G. *Let E be a metrizable space that is embeddable into a complete metrizable space with Schauder basis* (or *let E be a metrizable space that elsewise fulfils condition* (N) *of the Proposition*), *and that also satisfies*

(S) *There is a 0-neighbourhood U of E such that $\bigcap_{\lambda>0} \lambda U = \{0\}$.*

Then the space E must be locally convex if its dual separates points from closed subspaces.

Shapiro [10] proved this for the case of complete E, embeddable into a separated locally bounded space with Schauder basis.

(Notice that (S) must be satisfied e.g. if E can be endowed with a coarser topology, which is separated and locally bounded.)

Proof. To see that we can use Corollary F in combination with case (α) of the proposition just given, we must verify that condition (N) of this proposition is always fulfilled. This is done by the statement of the following two sublemmata. The first of these is a particular case of third lemma before the proof of proposition, i.e., the case with $E = \hat{E}$ and $F = \{0\}$. The second one is immediate.

FIRST SUBLEMMA. *A complete and metrizable space with Schauder basis must fulfil* (N).

SECOND SUBLEMMA. *The property* (N) *is inherited by all linear subspaces of a space with that property.*

In view of this corollary, the question of [3] and [10] whether for every (or for every complete) metrizable space, local convexity must be present if the dual's ability to separate points from closed subspaces is so, can be decomposed into two partial questions; namely the questions whether conditions (N) resp. (S) may be eliminated in this corollary. Concerning the second of these questions, it is easily seen that (S) may be replaced by some less restrictive condition, e.g., as in

COROLLARY G′. *Let E be a metrizable space that is embeddable into a complete metrizable space with Schauder basis* (or *let E be a metrizable space that elsewise fulfils condition* (N) *of the proposition*), *and that also satisfies*

(SA) *There is a 0-neighbourhood W of E such that $\bigcap_{\lambda>0} \lambda W$ is contained in a linear subspace S of E with infinite codimension.*

Then the space E must be locally convex if its dual separates points from closed subspaces.

Proof. Suppose that E is non locally convex. By this, there is a closed 0-neighbourhood U which contains no convex 0-neighbourhood, and which can play the role of W in (SA). We shall define recursively finite-dimensional subspaces $N_1 \subset N_2 \subset \cdots$. Let $\{V_k\}_k$ be a countable 0-neighbourhood base of open sets. Put $N_1 = \{0\}$; let us find N_k given N_{k-1} ($k \geqq 2$). By the assumptions for U, the set $\operatorname{co} V_k \backslash U$ is nonempty and open; so we can take z_k in this set and outside $S + N_{k-1}$ (for S has infinite codimension, so this is a proper linear subspace of E and thus is nowhere dense). However, (for the same last-mentioned reason) we have $\operatorname{co} V_k = \operatorname{co}(V_k \backslash (S + N_{k-1}))$, so there is a finite subset Z, say, of $V_k \backslash (S + N_{k-1})$ such that z_k is in $\operatorname{co} Z$. Then define N_k as the linear hull of N_{k-1} and Z.

By sublemmata of proof of Corollary G, the space $N = \bigcup_{k \geqq 1} N_k$ satisfies condition (N). Further, the construction of N shows that $S \cap N = \{0\}$, so by (SA), we get $\bigcap_{\lambda>0} \lambda U \cap N = \{0\}$, so N satisfies (S) of Corollary G. But by its construction, N is also non locally convex, whence this corollary (applied to N) says that in N, points are not separated from closed subspaces by the dual; and so, of course, this must be so also in E. The proof is complete.

7. Finally, we state a sufficient condition for a certain inverse limit space to be locally convex.

COROLLARY H. *Let $\{E_\alpha, f_{\alpha\beta}\}$ be an inverse system with countably many metrizable spaces E_α embeddable into complete metrizable spaces (or with countably many metrizable E_α that elsewise fulfil condition* (N) *of the Proposition preceding Corollary* G), *and with continuous linear mappings $f_{\alpha\beta}: E_\beta \to E_\alpha$. Further, assume that for some α, the space E_α fulfils condition* (SA) *of Corollary* G′, *and the $f_{\alpha\beta}(\beta \geqq \alpha)$ are surjective. If every closed subspace K of any E_α is the intersection of images $f_{\alpha\beta(\iota)}(H_\iota)$ of weakly closed subspaces H_ι of suitable spaces*

$E_{\beta(\iota)}$, *then the inverse limit space* $\varprojlim E_\alpha$ *is locally convex.*

Proof. We shall recognize this corollary as a special case of Corollary G′—with $\varprojlim E_\alpha$ in the place of the space E. To that end, two verifications must be made (except the simple proof of the fact that $\varprojlim E_\alpha$ is metrizable; it follows from the condition that the E_α are countably many; i.e., letting α run through the index set and letting V run through a countable 0-neighbourhood base of each E_α, we get $f_\alpha^{-1}(V)$ running through accountable 0-neighbourhood of $\varprojlim E_\alpha$; see [2], I.4.4.) First, $\varprojlim E_\alpha$ satifies conditions (N) and (SA). Denoting by $f_\alpha\colon \varprojlim E_\alpha \to E_\alpha$, any α, the projection, we get a 0-neighbourhood base in $\varprojlim E_\alpha$ consisting of all sets $f_\alpha^{-1}(U)$ where U is an arbitrary weakly closed 0-neighbourhood in E_α (remember that (N) holds for E_α, in view of sublemmata of proof of Corollary G) and where α is arbitrary; see [2], I.4.4. But of course, each such $f_\alpha^{-1}(U)$ is weakly closed in $\varprojlim E_\alpha$; so this space fulfils (N). Let E_α satisfy (SA) and f_α be surjective; if W and $S \subset E_\alpha$ are as in (SA), it is seen that $f_\alpha^{-1}(S) \subset \varprojlim E_\alpha$ are related to each other as W and S are in (SA). Thus, (SA) is fulfilled also for the limit space $\varprojlim E_\alpha$.

Second, suppose that every closed subspace in any E_α is the intersection of images of weakly closed subspaces in the way stated; we must then show that an arbitrary closed subspace K in $\varprojlim E_\alpha$ can separated from any element x outside it by a continuous linear form on $\varprojlim E_\alpha$. We can write $K = \bigcap_\alpha f_\alpha^{-1}(\overline{f_\alpha(K)})$ (see [2], I.4.4). So fix α with $f_\alpha(x) \notin \overline{f_\alpha(K)}$. By assumption, there is a space E_β with a weakly closed subspace H so that $f_\alpha(K) \subset f_{\alpha\beta}(H)$ and $f_\alpha(x) \notin f_{\alpha\beta}(H)$.

It follows that x is outside, while K is contained in, the weakly closed subspace $f_\beta^{-1}(H)$ of $\varprojlim E_\alpha$; this is what we need.

Notice the following special case of Corollary H: Let L be a linear space with finer and finer metrizable topologies $\tau_1 \subset \tau_2 \subset \tau_3 \subset \cdots$, each fulfilling conditions (N), and (SA) or (S). Here the corollary says that *if* a subspace closed in *some* topology τ_i can always be separated from a given point outside it by a suitable linear form, continuous in a *suitable* topology τ_j, *then* the supremum topology of all the τ_1, $\tau_2, \cdots$ must be locally convex. And the conclusion can be rephrased: any 0-neighbourhood of an arbitrary τ_i must contain a convex 0-neighbourhood of a suitable τ_j.

END REMARK. It may be noticed that for the methods and results presented above, it is never, really, essential whether the spaces in question are *complete* or not. When completeness occurs in our assumptions, it does so only indirectly, as in Corollary G. (However, the example before that corollary is an exception.) Of course, we could not possibly say whether this reflects that completeness is immaterial to the problem, or just that our methods are not general enough to deal with cases where one would have to assume completeness in order to get any results of the desired kind.

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