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ACTIONS OF TORUS T^n ON $(n+1)$-MANIFOLDS M^{n+1}

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Let ξ be a principal T^l-bundle over a lens space $L(p, q)$. It is shown here that the total space of ξ can be identified with $L(k, q) \times S_1^1 \times \cdots \times S_l^1$, for some $k \leqq p$. Let (T^n, M^{n+1}) be an effective torus action on an orientable $(n+1)$-dimensional manifold. An elementary examination of the parity of dimensions of the slice S_x at $x \in M$ and of the orbit $T^n(x)$, shows that the circle subgroups are the only possible stability groups on M^{n+1}. From these two results and the cross-sectioning theorem we can conclude that T^{n+1} and $L(k, q) \times T^{n-2}$ are the only possible types of compact closed orientable $(n+1)$-manifolds which allow T^n actions.

It is shown in [3] that T^4 and $L(p, q) \times S^1$ are the only compact closed orientable 4-manifolds which allow effective T^3 actions. The purpose of this note is to show, using an argument similar to that of [3], that T^{n+1} and $L(m, q) \times T^{n-2}$ are the only possible compact closed orientable $(n+1)$-manifolds which allow effective T^n actions for $n \geqq 3$. Here $L(m, q)$ includes the case of $S^2 \times S^1$ and S^3. The key lemma used in the proof of this theorem is that every principal T^l-bundle over the lens space $L(p, q)$ can be identified with $L(k, q) \times T^l$ for suitable $k \leqq p$. In later papers we intend to work on T^n actions on compact closed non-orientable $(n+1)$-manifolds M^{n+1} and $(n+2)$-manifolds M^{n+2}.

Let G, a compact Lie group, act on a space X. If $x \in X$, $G_x = \{g \in G \mid g(x) = x\}$ will denote the stability group, or isotropy group of G at $x \in X$. $G(x) = \{g(x) \mid g \in G\}$ will be called the orbit of $x \in X$. The orbit space, the set of all orbits, will be denoted by $X/G = X^*$ or $\bar{X}$ with the quotient topology, and the orbit map by $\Pi: X \to X^*$. For each $x \in X$, one can find a certain subset S_x called the slice at x [1, Chapter VIII], with the following properties:

(i) S_x is invariant under G_x.

(ii) If $g \in G$, $y, y' \in S_x$, and $g(y) = y'$, then $g \in G_x$.

(iii) There exists a "cell neighborhood" C of G/G_x such that $C \times S_x$ is homeomorphic to a neighborhood of x. That is, if $f: C \to G$ is a local cross-section in G/G_x then the map $F: C \times S_x \to X$ defined by $F(c, s) = f(c)s$ is a homeomorphism of $C \times S_x$ onto an open set containing S_x in X. The principal orbits are those for which the stability groups are identity. An action is effective if $g(x) = x$ for every $x \in X$ implies $g = e$. We shall assume that G is acting smoothly and effectively on a smooth orientable manifold. By the slice theorem, given in [1, Chapter VIII], it follows that if T^n acts effectively on a compact closed $(n+1)$-manifold M^{n+1}, then there exist principal T^n orbits and the orbit space $M/T^n = M^*$ is a compact 1-manifold which we denote by S^1 or $[0, 1]$.

LEMMA 1. *Let (T^n, M^{n+1}) be a transformation group. Then the circle subgroups of T^n are the only possible nontrivial stability groups on M^{n+1}.*

Proof. Let $T^i \times F$, $i = 1, \cdots, n$, be a subgroup of T^n, where T^i is i-dimensional torus subgroup of T^n and F is any finite subgroup of T^n complementary to T^i. We assume that if $i = 1$, then F is nontrivial.

First we show that no nontrivial finite subgroup F of T^n can be a stability group. If $M^* = S^1$ then every point in M^* corresponds to a principal orbit, so that we don't have a finite group as a stability group. In any case, if we have a finite stability group F at x, then x is isolated. The orbit is n-dimensional and the slice is a 1-dimensional interval. Thus F must be Z_2 which reverses the orientation (a contradiction, since M is orientable and T^n is connected).

Now cosider the case of $T^i \times F, i = 1, \cdots, n$. The orbit will be $(n-i)$-dimensional, and there is an $(n+1)-(n-i) = (i+1)$ dimensional disk slice on which $T^i \times F$ must act as a rotation. But $T^i \times F \not\subset SO(i+1)$ for $i = 1, \cdots n$. Thus there is no point $x \in M$ such that $T^n_x = T^i \times F$ for $i = 1, \cdots n$. This also implies that the fixed point set $F(T^n, M^{n+1}) = \emptyset$ for $n > 1$.

LEMMA 2. *Let (T^n, M^{n+1}) be a transformation group. Then the orbit map $\Pi\colon M^{n+1} \to M^*$ has a cross-section.*

Proof. If $M^* = S^1$, then the T^n-bundle is trivial. If $M^* = [0, 1]$, then the action corresponding over $(0, 1)$ is the trivial T^n-bundle, so that we have a cross-section over $(0, 1)$. Now we can extend this cross-section trivially to both ends.

LEMMA 3. *If M^{n+1} is a principal T^{n-2}-bundle over $L(p, q)$, $n \geqq 3$, then M^{n+1} can be written as $L(k, q) \times T^{n-2}$ for some integer $k \leqq p$.*

Proof. By taking a circle subgroup T^1_1 of T^{n-2} and the complementary subgroup T^{n-3} to T^1_1 in T^{n-2}, we can consider M/T^{n-3} as a principal T^1_1-bundle over $L(p, q)$. Without loss of generality we can take T^1_1 be the first factor of $T^{n-2} = T^1 \times \cdots \times T^1$. But, this bundle is classified by $[L(p, q), K(z, 2)] \cong Z_p$, and (see [5]) for any element $f_i \in [L(p, q), K(z, 2)]$, $i \in Z_p$, the total space of the principal T^1_1-bundle determined by f_i is $L(m, q) \times S^1$, where $m = \gcd(i, p)$. Take a circle subgroup T_2^1 in T^{n-3} as in the first case and denote the complementary subgroup by T^{n-4}. Then M/T^{n-4} is principal T_2^1-bundle over $L(m, q) \times S^1$. This bundle is also classified by

$$[L(m, q) \times S^1, K(Z, 2)] \cong H^2(L(m, q) \times S^1, Z) .$$

Let $\xi \in [L(m, q) \times S^1, K(Z, 2)]$ and denote its total space by E'. Consider the following diagram:

$$\begin{array}{ccc} E' & \xrightarrow{\Pi'} & L(m, q) \times S^1 \\ \downarrow & & \downarrow \Pi \\ E'' & \xrightarrow{\Pi''} & L(m, q) . \end{array}$$

Here E'' is the total space of ξ restricted to $L(m, q) \times t$, where t is any chosen point of S^1. Here Π' and Π'' are bundle maps and Π is the projection map onto the first coordinate $L(m, q)$. Now E' is the pull-back of E'' relative to the projection map Π, so that we have $E' = E'' \times S^1$. Since ξ restricted to $L(m, q) \times t$ is an element of $[L(m, q), K(Z, 2)] \cong Z_m$ we can consider $f_j \in [L(m, q), K(Z, 2)]$, for some $j \in Z_m$ as representing this bundle element whose total space is E''. But $E'' \cong L(d, q) \times S^1$ as before, where $d = \gcd(j, m)$. Hence $E' \cong L(d, q) \times S^1 \times S^1 \cong L(d, q) \times T^2$. Repeating this process a finite number of times we eventually get $M \cong L(k, q) \times T^{n-2}$ for some $k \leqq p$.

THEOREM. *If T^n acts effectively on a compact closed orientable $(n+1)$-manifold M^{n+1}, then M^{n+1} must be either T^{n+1} or $L(k, q) \times T^{n-2}$ for $n \geqq 3$.*

Proof. If $M^* = S^1$, then every point on S^1 corresponds to a principal orbit, and the total space is a T^n-bundle over S^1. But these bundles are classified by

$$[S^1, K(Z, 2) \times \cdots \times K(Z, 2)] = H^2(S^1, Z + \cdots + Z) = 0 ,$$

so that the bundle is trivial and $M = S^1 \times T^n = T^{n+1}$.

If $M^* = [0, 1]$, then by Lemma 1 there are only two circle subgroups of T^n corresponding to the stability groups at 0 and 1. Let T_0 be a subgroup generated by these two circle subgroups. Then any $(n-2)$-dimensional subgroup T^{n-2} of T^n which is complementary to T_0 acts freely on M. Then M/T^{n-2} is a 3-dimensional orientable manifold $\bar{M}$ and T_0 acts on it so that $\bar{M}\backslash T_0 \cong [0, 1]$. But T_0 actions on 3-manifolds whose orbit spaces are isomorphic to $[0, 1]$ are classified as lens spaces $L(p, q)$ in [2]. Now, since T^{n-2} acts freely on M, M is a principal T^{n-2}-bundle over $L(p, q)$. But these bundles can be written as $L(k, q) \times T^{n-2}$ by the Lemma 3.

REMARK. Since the maximal torus subgroup of $SO(m)$ is T^n where $m = 2n$ or $m = 2n + 1$, we see that (T^n, M^m) can have no fixed points unless $m > 2n$ or $m > 2n + 1$. Also we can see from the theorem that a compact simply- connected $(n + 1)$-manifold does not allow effective T^n actions for $n \geqq 3$. Thus extending a result of R. Richardson, Jr. [4] which says that T^3 cannot act effectively on the 4-dimensional sphere S^4.

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