

ON THE RELATIONSHIP OF A RING AND THE SUBRING GENERATED BY ITS SYMMETRIC ELEMENTS

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Let R be a ring with involution, and \bar{S} the subring generated by the symmetric elements of R . By placing various conditions on the elements of \bar{S} , it is shown that the same conditions are forced on R . For example, if \bar{S} is nil or algebraic, then so is R . Also, if R is assumed to be simple, prime, or semi-prime, then \bar{S} satisfies the same property. Lastly, each of these three conditions on \bar{S} implies the same property for R , modulo a nilpotent ideal of R .

I. R will always denote an associative ring with involution, $*$. Thus, $*$ is an anti-automorphism of R of period 2. Let $S(R) = \{r \in R \mid r^* = r\}$ denote the set of symmetric elements of R . When there can be no confusion, we write S in place of $S(R)$. The symbol \bar{S} will represent the subring generated by S , and Z will represent the center of R . For $x, y \in R$, let $[x, y] = xy - yx$. An additive subgroup U of R is called a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$.

It will be necessary to refer frequently to some well-known results, most of which can be found in [3]. For the sake of completeness and convenience, we present these results in a form more useful for our purposes. The first two tell us when \bar{S} contains an ideal of R .

LEMMA 1.1. \bar{S} is a Lie ideal of R .

Proof. We use the proof of Lemma 1 of [5]. Suppose $s \in S$ and $r \in R$. Then $sr - rs = sr + r^*s - (r + r^*)s \in \bar{S}$. If we assume that for $s_1, \dots, s_{k-1} \in S$ and $r \in R$ that $[s_1s_2 \cdots s_{k-1}, r] \in \bar{S}$, then by induction we have $[s_1s_2 \cdots s_k, r] = s_1s_2 \cdots s_{k-1}[s_k, r] + [s_1s_2 \cdots s_{k-1}, r]s_k \in \bar{S}$ for all k . Since every element of \bar{S} is a sum of such terms, the lemma is proved.

LEMMA 1.2. If U is a noncommutative subring and Lie ideal of R , then U contains a nonzero ideal of R .

Proof. This is essentially Lemma 1.3 of [3]. Let $a, b \in U$ so that $[a, b] \neq 0$. Let $r \in R$. Then $[a, rb] = r[a, b] + [a, r]b$. Since $[a, r] \in U$ and U is a subring, we have $r[a, b] \in U$. For $t \in R$ $[r[a, b], t] \in U$. Thus $r[a, b]t - tr[a, b] \in U$, and so $r[a, b]t \in U$. Consequently, U contains the ideal generated by $[a, b] \neq 0$.

The next lemma, together with Lemma 1.1, gives us information when \bar{S} is commutative. The lemma is essentially the first half of Lemma 1.3 of [3].

LEMMA 1.3. *Suppose U is a Lie ideal of R with $[U, U] = 0$. If $2R = 0$ and $x \in U$ then $x^2 \in Z$. If R is semi-prime and 2-torsion-free, then $U \subset Z$.*

Proof. If $2R = 0$ then $x \in U$ says $[x, [x, r]] = x^2r + rx^2 = 0$ for all $r \in R$. Thus $x^2 \in Z$.

Suppose that R is semi-prime and 2-torsion free. For $u \in U$ and $x, y \in R$, we have $[u, xy] \in U$, so $0 = [u, [u, xy]] = [u, x[u, y]] + [u, [u, x]y]$. Thus $2[u, x][u, y] = 0$, and so $[u, x][u, y] = 0$. Letting $y = rx$ for $r \in R$ gives $[u, x]R[u, x] = 0$. Since R is semi-prime, $[u, x] = 0$, so $u \in Z$.

In the study of rings with involution, often one can prove certain facts if either $2R = 0$, or if R is 2-torsion free. This situation arises in considering those properties in which we are interested, namely, the properties of being nil, locally nilpotent, algebraic, and locally algebraic. To enable us to reduce the general case to one of the two special situations of $2R = 0$ or R 2-torsion free, we introduce the following formalization:

Given a ring A , let $F(A)$ denote the set of finite subsets of A . Let P be a property of the elements of $F(A)$, satisfying:

(1) If $\{A_i\}$ is a collection of rings and the elements of $F(A_i)$ satisfy P for each i , then the elements of $F(\bigoplus A_i)$ satisfy P , where $\bigoplus A_i$ is the direct sum of the A_i .

(2) If B is an ideal of A and the elements of both $F(B)$ and $F(A/B)$ satisfy P , then so do the elements of $F(A)$.

Note that the four properties mentioned above are properties of finite subsets, and satisfy the two conditions stated.

Now we can state the lemma which will enable us to reduce the general case to the special cases cited above.

LEMMA 1.4. *For all rings B with either $2B = 0$, or B 2-torsion free, let P be a property of the elements of $F(B)$ satisfying the two conditions above. Then P is a property of the elements of $F(A)$ for any ring A .*

Proof. Let A be a ring, $T(A)$ the torsion ideal of A , and $T_p(A)$ the p -primary component of $T(A)$. Since $A/T_2(A)$ is 2-torsion free, P is a property of the elements of $F(A/T_2(A))$. Thus, it suffices to show

that P is a property of the elements of $F(T_2(A))$. Let $B = T_2(A)$, and $B_k = \{x \in B \mid 2^k x = 0\}$. Now $2B_1 = 0$ and $2(B_2/B_1) = 0$, so P is a property of the elements of $F(B_2)$. By induction, P is a property of the elements of $F(B_k)$ for any k . Hence, P is a property of the elements of $F(B)$, for $F(B)$ is the union of the $F(B_k)$ for all k .

II. In this section we consider the situation when \bar{S} is nil, locally nilpotent, algebraic, locally finite, and nilpotent. Clearly, if R satisfies any of these conditions, then so does \bar{S} . The converse holds, except for \bar{S} nilpotent and $2R = 0$.

One technique that we shall use often in this section involves considering certain homomorphic images of R . In general, a homomorphic image of a ring with involution need not have an involution itself. However, if A is an ideal of R and $A^* = A$, then the quotient R/A inherits an involution from R by defining $(r + A)^* = r^* + A$. It is this involution on R/A to which we shall refer in this section.

We begin by eliminating the trivial case.

LEMMA 2.1. *If $S = 0$ then $R^3 = 0$.*

Proof. Let $I = \{r \in R \mid 2r = 0\}$. I is an ideal of R and $I^* = I$. If $x \in I$ then $x + x^* \in S$, so $x + x^* = 0$. But $2x = 0$, so $x = x^*$. Thus $I = 0$ and R is 2-torsion free. For $x \in R$ $x + x^*$, $xx^* \in S$ so $0 = xx^* = -x^2$. But if R is 2-torsion free and nil of index 2, we must have $R^3 = 0$.

Our first main result is a complete answer when \bar{S} is nil.

THEOREM 2.2. *If \bar{S} is nil then R is nil.*

Proof. By Lemma 1.4, it suffices to assume that either R is 2-torsion free or $2R = 0$. In either case, let N be the sum of all nil ideals of R . Clearly $N^* = N$, so R/N has an induced involution, and further, contains no nonzero nil ideals. Also, if R is 2-torsion free then so is R/N , for $T = \{x \in R \mid 2x \in N\}$ is a nil ideal of R , so is contained in N . If $\rho: R \rightarrow R/N$ is the natural homomorphism, then $\rho(\bar{S})$ is a subring and Lie ideal of R/N . Suppose that $\rho(\bar{S})$ is not commutative. By Lemma 1.2 $\rho(\bar{S})$ would contain a nonzero ideal of R/N , which is impossible since $\rho(\bar{S})$ is nil. Thus $\rho(\bar{S})$ is commutative. Whether R is 2-torsion free or $2R = 0$, we can conclude, using Lemma 1.3, that $\rho(x)^2 \in Z(R/N)$ for $x \in \bar{S}$. But a nil element in the center of any ring generates a nilpotent ideal. Hence $\rho(x)^2 = 0$. Thus $[\rho(x), \rho(r)]^2 \rho(x) = 0$ for all $r \in R$ and any $x \in \bar{S}$. This implies that

$\rho(xR)$ is nil of index 3. Since R/N is semi-prime we must have $\rho(x) = 0$, using Levitzki's theorem [3, Lemma 1.1]. But then if $r \in R$ we have $0 = \rho(rr^*) = \rho(r)\rho(r^*)$ and $0 = \rho(r + r^*) = \rho(r) + \rho(r^*)$. Thus R/N is nil of index 2, so R must be nil.

Using the same technique but assuming a bit more gives

THEOREM 2.3. *If \bar{S} is locally nilpotent, so is R .*

Proof. Proceed exactly as in Theorem 2.2 using $\mathcal{L}(R)$, the sum of all locally nilpotent ideals [4, p. 197], in place of N .

As we have mentioned, \bar{S} nilpotent does not in general imply that R is nilpotent. However, when R is 2-torsion free it is nilpotent, and we can obtain a bound on the index of nilpotence.

THEOREM 2.4. *If R is 2-torsion free and $\bar{S}^n = 0$, then $R^{3n} = 0$.*

Proof. By Lemma 1.1 \bar{S} is a Lie ideal of R . Let $t_m = s_1 s_2 \cdots s_m$ for $s_i \in S$. Then for $r \in R$

$$[t_m, r] = \sum_{i=1}^m s_1 \cdots s_{i-1} [s_i, r] s_{i+1} \cdots s_m \in S^m + S^{m+1}$$

since $[s_i, r] = s_i r + r^* s_i - (r + r^*) s_i \in S + S^2$. Thus $[t_m, r] t_{n-m} = t_m r t_{n-m} = 0$. Repeating the argument on t_m and t_{n-m} gives $s_1 r_1 s_2 \cdots r_{n-1} s_n = 0$ for $s_j \in S$ and $r_i \in R$ or r_i formally equal to 1. Therefore, the ideal I generated by S in nilpotent of index n . If $r \in R$ then $rr^*, r + r^* \in S \subset I$. This gives $2R^3 \subset I$ or $2^n R^{3n} = 0$. Since R is 2-torsion free, $R^{3n} = 0$.

The proof of Theorem 2.4 fails in the case where $2R = 0$ at the point where we obtain R/I is a nil ring of index 2. Of course, one can have such a non-nilpotent ring in the characteristic 2 case. An example would be a polynomial ring in infinitely many indeterminates over $GF(2)$, modulo all squares. We present such an example with involution which will show that Theorem 2.4 is false if $2R = 0$. Thanks are due to Dennis Estes for conversations regarding the existence of an example and particularly for his suggestion to view the involution as we will, which makes verification of the example relatively simple.

Let $X = \{x_i\}$ and $Y = \{y_i\}$ be countable sets of indeterminates indexed by positive integers, and set T equal to the ring of polynomials in X and Y with coefficients in $GF(2)$ whose constant terms are zero. Let I be the ideal of T generated by all x_i^2 and $y_i y_j$, and let $R = T/I$. Denote the image of x_i by \bar{x}_i and of y_i by \bar{y}_i . Lastly,

denote by D_i the formal partial derivative on R with respect to \bar{x}_i . R has an involution defined by $\bar{x}_i^* = \bar{x}_i + \bar{y}_i$ and $\bar{y}_i^* = \bar{y}_i$. If $f(\bar{Y})$ is a polynomial in the $\{\bar{y}_i\}$ alone, then clearly $f(\bar{Y})^* = f(\bar{Y})$. If $f(\bar{X})$ is a polynomial in the $\{\bar{x}_i\}$ then since $\bar{y}_i\bar{y}_j = 0$, we have $f(\bar{X})^* = f(\bar{X}) + \sum \bar{y}_i D_i f(\bar{X})$. Thus $(f(\bar{X})\bar{y}_i)^* = (f(\bar{X}) + \sum \bar{y}_i D_i f(\bar{X}))\bar{y}_i = f(\bar{X})\bar{y}_i$, and so S contains the ideal generated by all \bar{y}_i . Suppose that $f(\bar{X}) \in S$. Then by above we must have $\sum \bar{y}_i D_i f(\bar{X}) = 0$, and so, if we take a polynomial $f(X) \in T$ which maps to our element $f(\bar{X}) \in R$ we have $\sum y_i D_i f(X) = \sum p_i(X, Y)x_i^2 + \sum q_{ij}(X, Y)y_i y_j$ where we may assume that $D_i f(X)$ has no monomials containing any x_j^2 , and where p_i and q_{ij} may have a nonzero constant term. By looking at degrees of elements of T in each y_i , and then in the other y_j , we obtain $y_i D_i f(X) = \sum p_i(X, Y)x_i^2$. Now consider the degrees of both sides with respect to the x_j . The result is that $y_i D_i f(X) = 0$. Since T is a domain we must have $D_i f(X) = 0$. But then no x_i appears in any monomial of $f(X)$; that is, $f(X) = 0$. This shows that in $R, S = (\bar{y}_i)$, the ideal generated by the $\{\bar{y}_i\}$. Clearly $S^2 = 0$ but R is not nilpotent, since $R/S \cong GF(2)[X](x_i^2)$, where $GF(2)[X]$ are the polynomials in X without constant terms.

To obtain a noncommutative counterexample to Theorem 2.4 take R_n , the complete $n \times n$ matrix ring over R , the commutative example above, and consider the subring $W = \{(a_{ij}) \in R_n \mid a_{ij} \in S \text{ for } i \neq j\}$. Define $(a_{ij})^* = (a_{ji}^*)$. Thus, the involution on W takes transposes and applies $*$ from R . Again, W is not nilpotent but since $S(W) \subset (S(R))_n$ we have $S(W)^2 = 0$.

Assume for the remainder of this section that R is an algebra over a field F , and that $*$ is compatible with the action of F in that $(ar)^* = ar^*$ for $a \in F$.

THEOREM 2.5. *If \bar{S} is algebraic over F , so is R .*

Proof. Let A be the algebraic kernel of R ; that is, the sum of all algebraic ideals of R . Clearly $A^* = A$, so R/A inherits the involution from R and contains no algebraic ideals. If $\rho: R \rightarrow R/A$ is the natural homomorphism, we have, as in Theorem 2.2, that $\rho(\bar{S})$ is a Lie ideal and subring of R/A . If $\rho(\bar{S})$ is not commutative it must contain a nonzero ideal of R/A by Lemma 1.2. But this ideal would be algebraic over F , which is impossible. Hence $\rho(\bar{S})$ is commutative. Given any $x \in R$ we have $\rho(x)^2 - \rho(x + x^*)\rho(x) + \rho(x^*x) = 0$, where $\rho(x + x^*)$ and $\rho(x^*x)$ are algebraic over F . Thus to show that R/A , and so R is algebraic over F , it is enough to do so for an algebra B with the property that for each $x \in B, x^2 + bx + c = 0$ for some $b, c \in B$ with b and c commuting with each other and each algebraic over F . Let $F[b, c] = C$ and $[C: F] = n$. Since $x^2 = -bx - c$ we have

$x^3 = -bx^2 - cx = -b(-bx - c) - cx$, and so $x^3 = b_2x + c_2$ for $b_2, c_2 \in C$. Similarly one obtains $x^k = b_{k-1}x + c_{k-1}$ for all $k \geq 2$. Now the collection $\{b_i\}$ is linearly dependent over F . Let $\sum_{i=1}^m f_i b_i = 0$ be a nontrivial dependence relation, and set $g(Y) = \sum_{i=1}^m f_i Y^{i+1}$. Then $g(x) = \sum f_i c_i = d \in C$. Let $h(Y)$ be a nontrivial polynomial satisfied by d . Then $0 = h(d) = h(g(x))$, and so x is algebraic over F .

In the same way that we went from nil to locally nilpotent, we may now consider the property of local finiteness, although here we need the result in Theorem 2.5 as well as the method of proof.

THEOREM 2.6. *If \bar{S} is locally finite, then so is R .*

Proof. The argument is the same as in Theorem 2.5 using L , the sum of all the locally finite ideals in place of the sum of all algebraic ideals [2]. Again we may assume that $\rho(\bar{S})$ is commutative and obtain that R/L is algebraic. Since $\rho(\bar{S})$ is commutative, the set of elements of the form $\rho(x) + \rho(x)^*$ satisfy a polynomial identity over F [2]. By a theorem of Amitsur [1], R/L itself satisfies a polynomial identity. Since R/L is algebraic, it must be locally finite by Theorem 6.4.3 of [2]. Hence $R = L$, so is locally finite.

We mention two examples relevant to the theorems proved in this section. First, if B is any ring and B^{op} its opposite ring, then $B \oplus B^{op}$ has an involution defined by $(b_1, b_2)^* = (b_2, b_1)$. Clearly, $S(B \oplus B^{op}) = \{(b, b) | b \in B\}$. This example shows that if \bar{S} satisfies one of the properties we have considered, one cannot hope to force R to satisfy a stronger property, in the sense, for example, that nilpotence is a stronger property than local nilpotence.

The second example concerns the following situation. If R is 2-torsion free and $k = \{r \in R | r^* = -r\}$ then $2R \subseteq S + K$ and $S \cap K = 0$. One might ask whether imposing conditions on \bar{K} as we did on \bar{S} will also force these conditions on R . In this case the answer is no. Let A be a commutative ring and N an ideal of A . If A_n is the complete $n \times n$ matrix ring over A , consider $R = \{(a_{ij}) \in A_n | a_{ij} \in N \text{ for } i \neq j\}$. Let $(a_{ij})^* = (a_{ji})$. If A is 2-torsion free then $\bar{K} \subset N_n$. In particular, N may be nil, nilpotent, or algebraic, and so \bar{K} has the same property, but R may fail to. Specifically, let $A = F[x, y]/(x^t)$ and $N = (x)$. Then \bar{K} is nilpotent but R is not even nil, or algebraic. One would like to have such examples which do not satisfy some polynomial identity. This can be accomplished by considering matrix rings described above and then taking direct sums of such rings with the size of the matrices becoming arbitrarily large.

III. In this section we consider the situation when either R or

\bar{S} is simple, prime, or semi-prime. The first two lemmas give us information on the "semi-primeness" of S .

LEMMA 3.1. *If R is semi-prime and $x \in S$ with $xSx = 0$, then $x = 0$.*

Proof. For $r \in R$ we have $x(r + r^*)x = 0$, so $xrx = -xr^*x$. Hence $xrxrx = -xrxr^*x = -x(rx^*r)x = 0$, since $xrx^* \in S$. Thus, xR is a nil right ideal of R of index 3. By Levitzki's theorem [3, Lemma 1.1] we must have $x = 0$.

LEMMA 3.2. *If R is semi-prime and $xSx = 0$, then $x + x^* = 0$.*

Proof. For $r \in R$ and $s \in S$, $rx^*s + srx^* \in S$. Hence

$$0 = x(rx^*s + srx^*)x = xrx^*sx.$$

Since R is semi-prime, $x^*Sx = 0$. Similarly, $0 = x(sx^*r + r^*xs)x$ implies $xSx^* = 0$. As $x^*Sx^* = 0$, we have, $(x + x^*)S(x + x^*) = 0$. By Lemma 3.1, $x + x^* = 0$.

The next theorem, that R simple implies that \bar{S} is simple, is an easy consequence of any one of several theorems in [3], together with an examination of the case when R is 4-dimensional over its center and $\text{char } R = 2$. We prefer to give a proof which depends only on the results presented thus far.

THEOREM 3.3. *If R is simple, then either $\bar{S} = R$ or \bar{S} is a field.*

Proof. If \bar{S} is not commutative, then by Lemma 1.1 and Lemma 1.2, \bar{S} must contain a nonzero ideal of R . Hence $\bar{S} = R$. If \bar{S} is commutative then $\bar{S} = S$ and for $x \in S$, $x^2 \in Z$ by Lemma 1.3. If $x \in S$ but $x^2 = 0$, then $xSx = 0$ so $x = 0$ by Lemma 3.1. Now $S \neq 0$ by Lemma 2.1, so Z is a field. Since every element of S is invertible, S is a field.

The situation when R is prime follows just as easily.

THEOREM 3.4. *If R is prime so is \bar{S} .*

Proof. If \bar{S} is commutative, proceed as in Theorem 3.3 to obtain $x^2 \in Z - 0$ for all $x \in S - 0$. Since R is prime, the nonzero elements of Z are not zero divisors in R . Thus S is a domain, so is prime. If \bar{S} is not commutative then \bar{S} contains $I \neq 0$ an ideal of R . If A and B are ideals of \bar{S} and $AB = 0$, then $AIB = 0$. Since R is prime,

either $A = 0$ or $B = 0$, proving that \bar{S} is prime.

The semi-prime case is a bit more complicated. We need first the following definition.

DEFINITION. If I is an ideal of the semi-prime ring T , then $\text{Ann } I = \{x \in T \mid Ix = 0\}$. An ideal J which has the form $J = \text{Ann } I$ is called an annihilator ideal of T . Note that since R is semi-prime we have $I(\text{Ann } I) = (\text{Ann } I)I = I \cap \text{Ann } I = 0$.

THEOREM 3.5. *If R is semi-prime then so is \bar{S} .*

Proof. If \bar{S} is commutative then it can have no nonzero nilpotent elements by Lemma 3.1, so is semi-prime. If \bar{S} is not commutative then by Lemma 1.2 \bar{S} contains the ideal generated by $[a, b]$ for $a, b \in \bar{S}$. Let I be the sum of all such ideals, and note that $I = I^*$. If $a, b \in S$ then $ab + I = ba + I$ in R/I , so if $x \in \bar{S}$ then $x + I$ is symmetric in R/I . Thus $x - x^* \in I$, for any $x \in \bar{S}$.

Let N be an ideal of \bar{S} with $N^2 = 0$. Now $NSN \subset N^2 = 0$, so $x + x^* = 0$ for all $x \in N$ by Lemma 3.2. Further, since $N\bar{S}N = 0$ we have $(NI)^2 = 0$. Because R is semi-prime, $NI = 0$. If $x \in N$, then we know that $x + x^* = 0$ and $x - x^* \in I$, so $2x \in I \cap N \subset I \cap \text{Ann } I = 0$. But $2x = 0$ and $x + x^* = 0$ imply $x = x^* \in S \cap N$. Since $NSN = 0$, Lemma 3.1 implies that $N = 0$, and so, \bar{S} must be semi-prime.

Although the converses to the last theorems are false, we can obtain reasonable partial converses modulo the existence of a unique maximal nilpotent ideal guaranteed by the next Lemma.

LEMMA 3.6. *If \bar{S} is semi-prime then R contains an ideal N with*

- (1) N is 2-torsion free
- (2) N is nil of index 2
- (3) $N^3 = 0$
- (4) $x \in N$ implies $x + x^* = 0$
- (5) N contains all nilpotent ideals of R .

Proof. Suppose that m is a nilpotent ideal of R . Since \bar{S} is semi-prime $m \cap S = 0$. Also, since $m + m^*$ is nilpotent we have $(m + m^*) \cap S = 0$. For $x \in m$, $x + x^* \in (m + m^*) \cap S = 0$ and $xx^* \in m \cap S = 0$. Now if $2x = 0$ then since $x + x^* = 0$, $x = x^* \in S \cap m = 0$. Hence m is 2-torsion free, and since $xx^* = x + x^* = 0$, we have $x^2 = 0$ for all $x \in m$. Thus $m^3 = 0$. Let N be the sum of all the nilpotent ideals of R . N is easily seen to satisfy the conditions of the Lemma.

Let us note that if $2R = 0$ and \bar{S} is semi-prime, then the Lemma implies that R is semi-prime. Also, the same proof can be used to show more. For example, if we assume that S intersects nil ideals, or locally nilpotent ideals, only in zero, then N still exists as indicated and contains all ideals of the type being considered, so they would be nilpotent as well.

Using Lemma 3.6 we can reduce to the situation in which R is semi-prime. To do this consider R/N which is semi-prime and has an induced involution, since $N^* = N$. Now by Theorem 3.5 we know that $\overline{S(R/N)}$ is still semi-prime. Unfortunately, even when we can assume that R is semi-prime, \bar{S} prime does not force R to be prime. However, we do have the following theorem.

THEOREM 3.7. *If \bar{S} is prime and R is semi-prime, then either R is prime, or R is a subring of $A \oplus A^{op}$, where $A = R/P$ is a prime ring, A^{op} is the opposite ring of A , $P \cup P^*$ contains every annihilator ideal of R , and $*$ in R is induced by interchanging co-ordinates in $A \oplus A^{op}$.*

Proof. Suppose that for every ideal $I \neq 0$ in R we have $I \cap \bar{S} \neq 0$. If $BC = 0$ for B and C ideals of R , then $(B \cap \bar{S})(C \cap \bar{S}) = 0$, forcing either $B \cap \bar{S} = 0$ or $C \cap \bar{S} = 0$. Thus either $B = 0$ or $C = 0$, so R is prime.

Assume now that R is not prime. Let P be an ideal of R maximal with respect to the property that $P \cap \bar{S} = 0$. Since \bar{S} is prime, we have by the usual arguments that P is a prime ideal, and so, $P \neq 0$. If $x \in P \cap P^*$ then xx^* and $x + x^*$ are in $P \cap S = 0$. Now if $2x = 0$, then $x = x^* \in P \cap S = 0$. Hence $P \cap P^*$ is 2-torsion free. But then $x^2 = 0$ for every $x \in P \cap P^*$, so $(P \cap P^*)^3 = 0$. Since R is semi-prime, $P \cap P^* = 0$. Thus R is a subdirect sum of R/P and R/P^* . Further R/P^* is naturally isomorphic to $(R/P)^{op}$ via $r + P^* \mapsto r^* + P$. We can embed R into $R/P \oplus (R/P)^{op}$ by the map $r \mapsto (r + P, r^* + P)$. Clearly, interchanging co-ordinates in $R/P \oplus (R/P)^{op}$ is an involution which restricts to $*$ on R . Lastly, suppose $BC = 0$ for ideals B and C of R . If $B \not\subset P \cup P^*$ then since P and P^* are prime ideals, $C \subset P \cap P^* = 0$. Thus every annihilator ideal of R is contained in $P \cup P^*$.

As an illustration of Theorem 3.7, consider the ring $A = F[x, y]/(xy)$ where $F[x, y]$ is the full polynomial ring in x and y over an arbitrary field F . If $A\{w_1, w_2, w_3\}$ is the free algebra with 1 over A and I is the ideal in this ring generated by w_1^2, w_2^2 , and w_3^2 , let $B = A\{w_1, w_2, w_3\}/I$. We extend the involution from A to B in the following way. If $at_1t_2 \cdots t_n$ is a monomial in B with $a \in A$ and each t_i some w_j , let $(at_1t_2 \cdots t_n)^* = a^*t_n t_{n-1} \cdots t_1$. One can show that B is a semi-prime

ring, basically because $bw_i b \neq 0$ for each i and $b \neq 0$, but B is not prime since $(x)(y) = 0$. Further, since each w_i is in S and x and y must both appear in an element of S if either one does, it can be shown that \bar{S} is prime but not a domain. Let $P = (x)$. Then B is a sub-direct sum of B/P and $(B/P)^{op}$, but not a direct sum since $(x) + (y) \neq B$. Also B/P is neither commutative nor a domain.

For an example which is not semi-prime, but where \bar{S} is still prime, let $\text{char } F \neq 2$, and let H be the free semi-group with unit generated by $\{z_i\}$ containing more than one indeterminate. Consider the semigroup ring $B[H]$, which can be viewed as the ring of non-commutative polynomials in the $\{z_i\}$ with coefficients in B . If J is the ideal of $B[H]$ generated by $\{z_i^2, z_i z_j + z_j z_i, w_i z_i, x z_i, y z_i\}$, let $C = B[H]/J$. Our involution on B can be extended to C by defining $(f z_i)^* = -f z_i$ for $f \in F$. Now since each z_i annihilates "almost everything" in C , fails to be symmetric, and since $(f z_i z_j)^* = f z_j z_i = -f z_i z_j$, we have $\overline{S(C)} \cong \overline{S(B)}$. Thus $\overline{S(C)}$ is prime, but the ideal generated by $\{z_i\}$ is the nonzero ideal N of Lemma 3.6.

Lastly, we examine the situation when \bar{S} is simple. It will be convenient to consider separately the cases where \bar{S} is commutative or not. Since the second case is more straightforward, we consider it first.

THEOREM 3.8. *If \bar{S} is simple and not commutative, and R is semi-prime, then $R = \bar{S}$ or R is prime with \bar{S} is unique minimal ideal, $r^2 \in \bar{S}$ for all $r \in R$ and $Z(R) = 0$.*

Proof. By Lemma 1.2, \bar{S} contains a nonzero ideal of R . Since \bar{S} is simple, it is an ideal of R . Consequently, $r^2 = (r + r^*)r - r^*r \in \bar{S}$ for all $r \in R$. Suppose $I \cap \bar{S} = 0$ for an ideal I of R . Since $x^2 \in \bar{S}$ for $x \in I$, I is nil of index 2, contradicting R semi-prime, unless $I = 0$. Hence \bar{S} is the intersection of all nonzero ideals of R , so is the unique minimal ideal. It follows that R is prime. Lastly, let $x \in Z(R)$. Then $x^2 \in Z(\bar{S})$, so \bar{S} has an identity if $x \neq 0$. But the identity of \bar{S} is a central idempotent in the prime ring R . Thus $1 \in R$, so $\bar{S} = R$ unless $Z(R) = 0$.

To see that the second possibility of the theorem can occur, let $B = Q\langle x, y \rangle / (xy - yx - 1)$, where $Q\langle x, y \rangle$ is the free algebra with identity over the rational numbers. B is well known to be a simple domain. Let $R = Qx + xBx$. Since B is simple, xBx is simple and an ideal in R . Consequently, since R is a domain, xBx is the unique minimal nonzero ideal of R . Define $*$ by setting $x^* = -x$ and $y^* = y$. Clearly \bar{S} contains xBx , since it contains some ideal. If $ax + t \in \bar{S}$ for $t \in xBx$, then $ax \in \bar{S}$. But $f(x, y) \in \bar{S}$ implies $\deg_x f \neq 1$, so $a = 0$ and $\bar{S} = xBx$.

If \bar{S} is commutative, then $\bar{S} = S$ is a field. In this case, as the next lemma will show, it is always possible to reduce to the situation where R has an identity, a well-defined prime characteristic, and the nonzero elements of S are units in R .

LEMMA 3.9. *If S is a field then $R = eR \oplus (1 - e)R$ where e is the identity of S and a central idempotent in R . Further, $(1 - e)R$ is a 2-torsion free ideal, nil of index 2.*

Proof. Let e be the identity of S . Since S is a Lie ideal of R , for any $x \in R$, $ex - xe \in S$, and so, $e(ex - xe) = ex - xe = (ex - xe)e$. These equalities imply that $ex = exe = xe$, or $e \in Z$. Hence $eR \oplus (1 - e)R = R$. Suppose $a = (1 - e)a$. Then $a^* = (1 - e)a^*$, so $a, a^* \in (1 - e)R$. But $eS = S \subset eR$, so $aa^* = a + a^* = 0$, and $(a + a^*)a = a^2 = 0$. Further, if $2a = 0$ then $a + a^* = 0$ implies $a = a^* \in S \subset eR$. Thus $(1 - e)R$ is a 2-torsion free nil ideal of index 2.

As a result of Lemma 3.9 there is no great loss in assuming that S consists of units of R and that the identity of S is the identity of R . Henceforth we shall make this assumption.

We claim that we also may as well assume that R is semi-prime. If $2R = 0$, then by Lemma 3.6 R is semi-prime. Assume that $\text{char } S \neq 2$ but $2r = 0$ for some $r \in R$. Then $2(r + r^*) = 0$, so $r + r^* = 0$. Since $2r = 0$ we have $r = r^* \in S$, so $r = 0$. Therefore R is 2-torsion free, and so, R/N is semi-prime and 2-torsion free, where N is the ideal of Lemma 3.6. Consider $x + N \in S(R/N)$. Thus $x + N = x^* + N$, and so, $x - x^* \in N$. Hence $2x - (x + x^*) = x - x^* \in N$. We may conclude that $2x + N$ is the image of $x + x^* \in S$. Since $2 \in S$, it is a unit in R , and we have $S(R/N) \cong S$. Thus in all cases we can reduce to R semi-prime. With this assumption, we come to our last result.

THEOREM 3.10. *If S is a field, $*$ is not the identity map, and R is semi-prime, then one of the following must hold:*

- (1) R is a field and $[R: S] = 2$
- (2) $R = S \oplus S$ and $*$ interchanges co-ordinates
- (3) R is a division ring with $[R: Z] = 4$
- (4) $R = S_2$ and $\text{char } S \neq 2$.

Proof. If R is not simple let m be a maximal ideal in R . Because nonzero elements of S are units in R , $m \cap S = 0$. As we have seen before, $x \in m \cap m^*$ implies $xx^*, x + x^* \in m \cap S = 0$, so $x^2 = 0$. Further, $m \cap m^*$ must be 2-torsion free, so $(m \cap m^*)^3 = 0$. But R is semi-prime, so $m \cap m^* = 0$. Since m is maximal $m + m^* = R$. Thus $R = R/m \oplus R/m^*$. If $x \in m^*$ then $x + m = (x + x^*) + m$, so $R/m =$

$(S + m)/m \cong S$. Hence R/m is a field isomorphic to S . Since R/m^* is anti-isomorphic to R/m , as in Theorem 3.7, we have possibility (2) of the theorem.

Assume now that R is simple. Suppose first that $S \not\supset Z$ and let $c \in Z$ with $c - c^* \neq 0$. For any $r \in R$, $(c - c^*)r = c(r + r^*) - (c^*r + r^*c) \in cS + S$. Since $c - c^* \in Z$ we have that R is commutative, so a field. As $*$ is not the identity map, S is its fixed field and $[R:S] = 2$, giving possibility (1) of the theorem.

Since S is a Lie ideal of R , by Theorem 1.5 of [3], either $S \subset Z$ or $\text{char } R = 2$ and $\dim_Z R \leq 4$. Even if $S \subset Z$, since $x^2 - (x + x^*)x + x^*x = 0$ for any $x \in R$, R is quadratic over Z , so $\dim_Z R \leq 4$ by Kaplansky's Theorem [2, p. 157]. Hence we may assume that R is simple of dimension 4 over its center, for otherwise R is commutative and we have case (1) as before. If $\text{char } R \neq 2$, then $S = Z$, so (3) or (4) holds. Lastly, if $\text{char } R = 2$, then if $x \in R$ and $x^2 = 0$ we have $xx^* = x^*x = 0$, since they are not units, and so $(x + x^*)^2 = 0$. Thus $x + x^* = 0$, which implies $x = x^* \in S$. Therefore, we may conclude that R has no nilpotent elements, and so, (4) can hold only if $\text{char } R \neq 2$.

Examples similar to those after Theorem 3.7 show that in case $\text{char } S \neq 2$, R may have nilpotent ideals which are not direct summands even though S is a field. Specifically, take $F[x]/(x^2)$ with $x^* = -x$. For a noncommutative example start with $T = xF[x, y] + yF[x, y]$. Let I be the ideal generated by $x^2 - x$, $y^2 - y$, and xy . Set $W = T/I$ with $*$ interchanging x and y . Let $R = W\{z_i\}$ the free algebra with constants over W in more than two indeterminates $\{z_i\}$. Let J be the ideal of R generated by all xz_i , yz_i and all squares in the ring generated by $\{z_i\}$. Extend $*$ to R/J by setting $z_i^* = -z_i$. Then the ideal N of Lemma 3.6 is the ideal generated by $\{z_i\}$ and $S = F \cdot (x + y) \cong F$.

REFERENCES

1. S. A. Amitsur, *Identities in rings with involution*, Israel J. of Math., (1969) **7**, 63-68.
2. I. N. Herstein, *Non Commutative Rings*, The Carus Mathematical Monographs, Math. Assn. of Amer., 1968.
3. ———, *Topics in Ring Theory*, University of Chicago Press, Chicago, 1969.
4. N. Jacobson, *Structure of Rings*, Amer. Math. Soc. Colloquium Publication v. 37. Amer. Math. Society, Providence, 1964.
5. M. S. Montgomery, *Lie structure of simple rings of characteristic 2*, J. Algebra **15**, (1970), 387-407.

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