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BOUNDED ENTIRE SOLUTIONS OF ELLIPTIC EQUATIONS

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Let

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} . \tag{1.1}$$

Consider the equation

$$Lu(x) = f(x) . \tag{1.2}$$

It is shown, under some general conditions on the coefficients of L, that if $f(x)$ is locally Hölder continuous and

$$f(x) = O(|x|^{-2-\mu}) \quad \text{as} \quad |x| \longrightarrow \infty \qquad (\mu > 0) \tag{1.3}$$

then there exists a bounded solution of (1.2) in R^n when $n \geqq 3$. If $n = 2$ then bounded entire solutions may not exist, but there exists a nonnegative solution of (1.2) in R^2 which is bounded above by $O(\log |x|)$. An application of these results to the Cauchy problem is given in the final section of the paper.

If in (1.3) $\mu = 0$ then already the equation $\Delta u = f$ $(n \geqq 3)$ may not have an entire bounded solution; an example is given by Meyers and Serrin [4].

2. Existence of a bouned solution.

We shall need the following conditions:

$$\sum_{i,j=l}^{n} a_{ij}(x)\xi_j\xi_i > 0 \quad if \ x \in R^n, \xi \in R^n, \xi \neq 0 , \tag{2.1}$$

(2.2) $a_{ij}(x)$, $b_i(x)$ are bounded, locally Hölder continuous in R^n $(1 \leqq i, j \leqq n)$,

$$\text{For some } \delta > 0, R > 0, 0 < \rho < 1 , \tag{2.3}$$

$$(2 + \delta)|x|^{-2} \sum_{i,i=i}^{n} a_{ij}(x)x_i x_j \leqq \rho \sum_{i=1}^{n} a_{ii}(x) + \sum_{i=1}^{n} x_i b_i(x) \ \text{if} \ |x| > R ,$$

$$\sum_{i=1}^{n} a_{ii}(x) \geqq \gamma > 0 \text{ for all } x \in R^n \qquad (\gamma \text{ constant}) . \tag{2.4}$$

Notice that (2.1) and (2.4) both follow from the condition of uniform ellipticity

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqq \gamma_0 |\xi|^2 \text{ for all } x \in R^n, \ \xi \in R^n \qquad (\gamma_0 \text{ positive constant}) . \tag{2.5}$$

Denote the eigenvalues of $(a_{ij}(x))$ by $\lambda_1(x) \leqq \cdots \leqq \lambda_n(x)$. Then the condition in (2.3) means that

$$(2 + \delta)\tilde{\lambda}(x) \leqq \rho\,[\lambda_1(x) + \cdots + \lambda_n(x)] + \sum_{i=j}^{n} x_i b_i(x) \tag{2.6}$$

for some $\lambda_1(x) \leqq \tilde{\lambda}(x) \leqq \lambda_n(x)$.

We finally impose on $f(x)$ the condition:

$$f(x) = O(|x|^{-2-\nu}) \quad \text{as} \quad |x| \longrightarrow \infty \qquad (\nu > 0)\,. \tag{2.7}$$

THEOREM 1. *Suppose that either the conditions* (2.1)–(2.4) *or the conditions* (2.5), (2.2) *and* (2.3) *with* $\rho = 1$ *hold. Then for any locally Hölder continuous function* $f(x)$ *satisfying* (2.7) *there exists a unique bounded solution* $u(x)$ *of* (1.2) *in* R^n *satisfying* $u(x) \to 0$ *if* $|x| \to \infty$.

Proof. We shall construct a function $v(r)$ for $r > R$ such that

$$Lv(r) \leqq -|f(x)| \quad \text{if} \quad r = |x| > R\,, \tag{2.8}$$

$$v'(r) < 0 \quad \text{if} \quad r > R\,. \tag{2.9}$$

It is easily seen that

$$\begin{aligned} Lv(r) = {} & \frac{1}{r^2}\Big[\sum_{i,j} a_{ij}(x)x_i x_j\Big]v''(r) \\ & + \frac{v'(r)}{r}\Big[\sum_i a_{ii}(x) - \frac{1}{r^2}\sum_{i,j} a_{ij}(x)x_i x_j + \sum_i x_i b_i(x)\Big]\,. \end{aligned}$$

If (2.9) holds then, by (2.3),

$$\begin{aligned} Lv(r) \leqq {} & \Big[v''(r) + (1+\delta)\frac{v'(r)}{r}\Big]\frac{1}{r^2}\sum_{i,j} a_{ij}(x)x_i x_j \\ & + \frac{(1-\rho)v'(r)}{r}\sum_i a_{ii}(x)\,. \end{aligned} \tag{2.10}$$

Take $\mu > 0$ such that $\mu < 1$, $\mu < \nu$, $\mu \leqq \delta$ and take $0 < R_0 < R$. Consider the function

$$v(r) = B\int_r^\infty \frac{ds}{s^{1+\mu}}\int_{R_0}^\infty \tau^{1+\mu}\frac{d\tau}{\tau^{2+\nu}}$$

for any constant $B > 0$. Then $v(r)$ satisfies (2.9), and

$$v''(r) + (1+\mu)\frac{v'(r)}{r} = -\frac{B}{r^{2+\nu}}\,,$$

$$v'(r) < -\frac{BC'}{r^{1+\mu}}\,,$$

$$0 < v(r) < \frac{BC}{r^{\mu}} \tag{2.11}$$

if $r > R$, where C', C are positive constants independent of B. Recalling (2.10) and assuming that (2.3), (2.4) hold, we get

$$Lv(r) \leqq -\frac{BC'(1-\rho)}{r^{2+\mu}}\sum_i a_{ii}(x) \leqq -|f(x)| \quad \text{if} \quad |x| = r > R$$

provided B is sufficiently large. If instead of (2.3), (2.4) one assumes that (2.5) and (2.3) with $\rho = 1$ hold, then again one derives from (2.10) the inequality $Lv(r) \leqq -|f(x)|$.

Consider the exterior Dirichlet problem

$$\begin{aligned} L\phi_0(x) &= f(x) && \text{in} \quad |x| > R\,, \\ \phi_0 &= 0 && \text{on} \quad |x| = R\,, \\ \phi_0(x) &\to 0 && \text{if} \quad |x| \to \infty\,. \end{aligned} \tag{2.12}$$

In Meyers-Serrin [4] it is proved that there is a unique solution ϕ_0 of (2.12) if (2.7), (2.2) and (2.3) with $\rho = 1$ hold, and if $\sum a_{ij}(x)x_ix_i \geqslant |x|^2$. The last condition is equivalent to the condition (2.5). The crucial step in the proof in [4] is the construction of $v(r)$ for which $Lv(r) \leqq -|f(x)|$ and (2.11) holds. Since we have constructed such a $v(r)$ also when the assumptions (2.5), (2.3) with $\rho = 1$ are replaced by (2.3), (2.4), the proof of [4] shows that the problem (2.12) has a unique solution ϕ_0.

Consider next the Dirichlet problem

$$\begin{cases} L\phi = 0 & \text{in} \quad |x| > R\,, \\ \phi = h & \text{on} \quad |x| = R\,, \\ \phi(x) \to 0 & \text{if} \quad |x| \to \infty \end{cases} \tag{2.13}$$

where h is a continuous function. This again has a unique solution ϕ.

Take $R' > R$ and let w be the soultion of

$$\begin{cases} Lw = 0 & \text{in} \quad |x| < R'\,, \\ w = \phi & \text{on} \quad |x| = R'\,. \end{cases} \tag{2.14}$$

Finally let w_0 be the solution of

$$\begin{cases} Lw_0(x) = f(x) & \text{in} \quad |x| < R'\,, \\ w_0 = \phi_0 & \text{on} \quad |x| = R'\,. \end{cases} \tag{2.15}$$

Then $\phi + \phi_0$ and $w + w_0$ are solutions of $Lu = f$ in $|x| > R$ and $|x| < R'$ respectively, and they coincide on $|x| = R'$. If there exists a function h such that

$$\phi + \phi_0 = w + w_0 \quad \text{on} \quad |x| = R\,, \tag{2.16}$$

then $\phi + \phi_0 = w + w_0$ in $R < |x| < R'$, so that

$$u(x) = \begin{cases} \phi + \phi_0 & \text{in} \quad |x| > R\,, \\ w + w_0 & \text{in} \quad |x| < R' \end{cases}$$

defines a bounded solution of (1.2) in R^n which tends to zero as $|x| \to \infty$.

Denote by X the Banach space of continuous functions on $|x| = R$ with the sup norm, and denote by $\| \ \|$ the norm of operators in X. Denote by Wh the restriction of w to $|x| = R$. Then (2.16) reduces to

$$h - Wh = w_0 - \phi_0\,. \tag{2.17}$$

If we show that

$$\| W \| < 1 \tag{2.18}$$

then the existence of a unique solution h of (2.17) follows, and the existence part of the theorem is proved.

The function

$$\tilde{\phi}(x) = \|h\| \frac{v(r)}{v(R)} \qquad (\|h\| = \sup_{|x|=R} |h(x)|)$$

satisfies:

$$L\tilde{\phi} \leqq 0 \text{ if } |x| > R,\ \tilde{\phi} \geqq \phi \text{ if } |x| = R,\ \tilde{\phi}(x) - \phi(x) \to 0 \text{ if } |x| \to \infty\,.$$

By the maximum principle it follows that $\tilde{\phi} \geqq \phi$ if $|x| > \mathrm{R}$. Similarly $\tilde{\phi} \geqq -\phi$. Hence

$$|\phi(x)| \leqq \|h\| \frac{v(R')}{v(R)} = \sigma \|h\| \quad \text{if} \quad |x| = R'\,,$$

where $\sigma < 1$ by (2.9). Since, by the maximum principle,

$$\sup_{|x|=R} |w(x)| \leqq \sup_{|x|=R'} |\phi(x)|\,,$$

we conclude that

$$\sup_{|x|=R} |w(x)| \leqq \sigma \|h\|\,.$$

This gives (2.18).

Suppose now that $\tilde{u}(x)$ is another solution of (1.2) in R^n which tends to zero as $|x| \to \infty$. We shall prove that $\tilde{u} \equiv u$. Let $z = u - \tilde{u}$ and denote by h the restriction of z to $|x| = R$. Then $Wh = h$. Since $\|W\| < 1$, $h = 0$. It follows that $z \equiv 0$ in R^n.

From the proof of Theorem 1 we obtain the estimate

$$u(x) = 0(|x|^{-\mu}) \tag{2.19}$$

on the solution. Hence:

COROLLARY 1. *Let the assumptions of Theorem 1 hold. Then for any number N there is a unique solution of (1.2) in R^n satisfying: $u(x) \to N$ if $|x| \to \infty$; further,*

$$u(x) = N + O(|x|^{-\mu}) \qquad as \ |x| \to \infty$$

for any $\mu \leqq \delta, \mu < \nu, \mu < 1$.

COROLLARY 2. *Suppose (2.1), (2.2) hold and suppose*

$$|x| \sum_{i=1}^{n} |b_i(x)| \to 0 \quad if \quad |x| \to \infty \ , \tag{2.20}$$

$$\bar{a}_{ij} = \lim_{|x| \to \infty} a_{ij}(x) \ exists \ for \ 1 \leqq i, j \leqq n \ . \tag{2.21}$$

If the matrix $(\bar{a}_{ij})$ has at least three positive eigenvalues then the assertion of Theorem 1 and Corollary 1 are valid.

Proof. A nonsingular affine transformation $x \to Tx$ does not change the assumptions and assertions of the corollary. Such a transformation changes (a_{ij}) into $T(a_{ij})T^*$. Thus, without loss of generality one may assume that

$$\bar{a}_{ij} = 0 \quad \text{if} \quad i \neq j, \bar{a}_{ii} = 1 \quad \text{if} \quad i = 1, 2, 3, a_{ii} = 0 \quad \text{or } 1 \text{ if} \quad i \geqq 3 \ .$$

But then the conditions (2.4), (2.3) (with $\rho = 1$) are satisfied, so that Theorem 1 and Corollary 1 can be applied.

We recall a result of Gilbarg-Serrin [2; Theorem 3] asserting that if (2.2), (2.5), (2.21) hold, and if $n \geqq 3$ and

$$\sum_i |b_i(x)| = O\left(\frac{1}{|x|}\right) \quad \text{as} \quad |x| \to \infty \ ,$$

then any bounded solution of $Lu = 0$ in R^n has a limit at infinity. By the maximum principle, this yields a Liouville theorem: Any entire bounded solution of $Lu = 0$ is a constant. Hence:

COROLLARY 3. *Suppose (2.1), (2.2), (2.20), (2.21) hold, and let the matrix $(\bar{a}_{ij})$ be nonsingular. Then, any bounded solution of (1.2) in R^n, $n \geqq 3$, has the form $N + u(x)$ where $u(x)$ is the solution asserted in Theorem 1. (Recall that $u(x)$ satisfies (2.19).)*

3. The case $n = 2$. If (2.20), (2.21) hold and $n = 2$, then the condition (2.3) with $\rho = 1$ is not satisfied. We shall now study this situation. The following conditions will be imposed:

$$\begin{gathered} n = 2 \text{ and for all } x \in R^2, \xi \in R^2 \ , \\ \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geqq \nu_0 |\xi|^2 \quad (\nu_0 \text{ positive constant}) \ , \end{gathered} \tag{3.1}$$

$$\sum_{i,j} | a_{ij}(x) - \bar{a}_{ij} | \leqq \frac{C}{(1 + |x|)^\kappa} \qquad (C > 0, \kappa > 0) , \tag{3.2}$$

$$\sum_{j} | b_i(x) | \leqq \frac{C}{(1 + |x|)^{1+\kappa}} \qquad (C > 0, \kappa > 0) , \tag{3.3}$$

THEOREM 2. *Let the conditions* (2.2), (3.1)-(3.3) *hold. Then for any locally Hölder continuous function* $f(x)$ *satisfying* (2.7) *there exists a solution* $u(x)$ *of* (1.2) *in* R^2 *satisfying*

$$0 \leqq u(x) \leqq K \log (2 + |x|) \qquad (K \text{ constant}). \tag{3.4}$$

Proof. Without loss of generality we may assume that $\bar{a}_{ij} = \delta_{ij}$, $1 \leqq i, j \leqq 2$. We shall construct functions $v_1(r)$, $v_2(r)$ for $r > R_0$ (R_0 and fixed positive number) satisfying:

$$\begin{cases} Lv_1(r) \leqq 0 & \text{if } r \geqq R_0 , \\ v_1(R_0) = 0,\ v_1'(r) > 0 & \text{if } r > R_0 , \end{cases} \tag{3.5}$$

$$\begin{cases} Lv_2(r) \geqq 0 & \text{if } r \geqq R_0 , \\ v_2(R_0) = 0,\ v_2'(r) > 0 & \text{if } r > R_0 . \end{cases} \tag{3.6}$$

The inequality $Lv_1 \leqq 0$ is satisfied if

$$v_1'' + \frac{1}{r}\left(1 + \frac{c}{r^\kappa}\right)v_1' = 0, \qquad v_1' > 0 \tag{3.7}$$

where c is a sufficiently large positive constant. A solution of (3.7) which vanishes at $r = R_0$ is given by

$$\begin{aligned} v_1(r) &= \int_{R_0}^r \exp\left\{-\int_{R_0}^t \frac{c}{s^{1+\kappa}} ds\right\}\frac{dt}{t} \\ &= \int_{R_0}^r \exp\left\{\frac{c}{\kappa}(t^{-\kappa} - R_0^{-\kappa})\right\}\frac{dt}{t} . \end{aligned} \tag{3.8}$$

This function then satisfies (3.5). Similarly,

$$v_2(r) = \int_{R_0}^r \exp\left\{-\frac{c}{\kappa}(t^{-\kappa} - R_0^{-\kappa})\right\}\frac{dt}{t} \tag{3.9}$$

is a solution of (3.6). From (3.8), (3.9) it is clear that

$$c_1 \log (1 + r) \leqq v_1(r) \leqq v_2(r) \leqq c_2 \log (1 + r) \quad (c_1 > 0, c_2 > 0) \tag{3.10}$$

for all $r \geqq R_0 + 1$.

For each $R > R_0 + 1$, let u_R be the solution of

$$\begin{aligned} Lu_R &= 0 && \text{in } R_0 < |x| < R , \\ u_R &= 0 && \text{on } |x| = R_0 , \\ u_R &= v_2(R) && \text{on } |x| = R . \end{aligned}$$

From the maximum principle it follows that $u_R \geqq v_2$ if $R_0 < |x| < R$. From (3.10) we have:

$$u_R \leqq \frac{c_2}{c_1} v_1(R) \qquad \text{on } |x| = R .$$

Hence, by the maximum principle,

$$u_R \leqq \frac{c_2}{c_1} v_1 \qquad \text{if } R_0 < |x| < R .$$

Using (3.10) once more we conclude that

$$c_1 \log(1+r) \leqq u_R(x) \leqq \frac{c_2}{c_1} c_2 \log(1+r) \quad \text{if } R_0 + 1 \leqq |x| < R .$$

We can now take a subsequence $\{u_{R_m}\}$, with $R_m \to \infty$ if $m \to \infty$, that is uniformly convergent in compact subsets of $\{x; |x| \geqq R_0\}$ to a solution $w_2(x)$ of

$$\text{(3.11)} \qquad \begin{cases} Lw_2 = 0 & \text{if } R_0 < |x| < \infty , \\ w_2 = 0 & \text{on } |x| = R_0 , \end{cases}$$

and

$$\text{(3.12)} \qquad c_1 \log(1+r) \leqq w_2(x) \leqq \frac{c_2}{c_1} c_2 \log(1+r) \quad (r = |x| > R_0 + 1) .$$

Let $R' = R_0$, $R'' > R'$ and denote by S' and S'' the circles given by $|x| = R'$ and $|x| = R''$ respectively. Let w_1 be the unique solution (see [4]) of

$$\text{(3.13)} \qquad \begin{cases} Lw_1 = f & \text{in } |x| > R' , \\ w_1 = 0 & \text{on } S' , \\ w_1 \text{ bounded in } |x| > R' . \end{cases}$$

Let z_1 and z_2 be the solutions of

$$\text{(3.14)} \qquad \begin{cases} Lz_1 = f & \text{in } |x| < R'' , \\ z_1 = w_1 & \text{on } S'' , \end{cases}$$

$$\text{(3.15)} \qquad \begin{cases} Lz_2 = 0 & \text{in } |x| < R'' , \\ z_2 = w_2 & \text{on } S'' . \end{cases}$$

Denote by z_1^*, z_1^* the restriction to S' of z_1 and z_2, respectively.

We shall introduce now an operator W similar to the operator W in the proof of Theorem 1. We denote by X the Banach space of the continuous functions h on S' provided with the uniform norm. For any $h \in X$, let w be the unique solution (see [4]) of

$$(3.16)\qquad \begin{cases} Lw = 0 & \text{in } |x| > R' , \\ w = h & \text{on } S' , \\ w \text{ bounded in } |x| \geqq R' , \end{cases}$$

and let z be the solution of

$$(3.17)\qquad \begin{cases} Lz = 0 & \text{in } |x| < R'' , \\ z = w & \text{on } S'' . \end{cases}$$

Then Wh is defined as the restriction of z to S'.

By the maximum principle, for any $\varepsilon > 0$,

$$\| h \| + \varepsilon v_1(r) \geqq \pm\, w(x) \qquad \text{in } |x| > R' .$$

This implies that

$$\sup_{|x|=R''} | w(x) | \leqq \| h \| .$$

Again by the maximum principle,

$$\sup_{|x|=R'} | z(x) | \leqq \sup_{|x|=R''} | z(x) | = \sup_{|x|=R''} | w(x) | .$$

Hence, $\| Wh \| \leqq \| h \|$. Since, for $h(x) \equiv 1$, $Wh = h$, it follows that $\| W = \|1$.

Employing the function $v_1(r)$ and using the maximum principle it can be shown (see [4, p. 523]) that Liouville's theorem is valid (under the assumptions of Theorem 2), that is, every bounded solution u of $Lu = 0$ in R^2 is a constant. Now, h satisfies $Wh = h$ if and only if the corresponding w and z coincide on S', S'' and, consequently, in the region $R' < |x| < R''$; thus, $Wh = h$ if and only if the pair w, z defines a bounded entire solution u of $Lu = 0$. By Liouville's theorem it follows that $u \equiv$ const. and, in particular, $h =$ const. Thus, 1 is an eigenvalue of W and the eigenspace is one dimensional.

From the interior Schauder estimates (see, for instance, [1]) one deduces that W maps bounded subsets of X into compact subsets. Hence the Fredholm-Riesz-Schauder theorem can be applied to solve equations of the form

$$(3.18)\qquad \zeta + Wh = h.$$

Denoting by $\hat{h}$ an eigenfunctional of the adjoint W^* of W, we can assert that the equation (3.18) has a solution if and only if

$$\hat{h}(\zeta) = 0.$$

We wish to solve the equation

$$z_1^* + \lambda z_2^* + Wh = h \tag{3.19}$$

for some real number λ. We first show that

$$\hat{h}(z_2^*) \neq 0 \; . \tag{3.20}$$

Suppose $\hat{h}(z_2^*) = 0$. Then the equation

$$z_2^* + Wh = h \tag{3.21}$$

has a solution h. Denote by w, z the corresponding solutions of (3.16), (3.17). Then the functions $w + w_2$ and $z + z_2$ coincide on S'' and (by (3.21)) on S'. Since they both are solutions of $Lu = 0$ in $R' < |x| < R''$, it follows that they coincide in this region. Consequently, the function

$$u_0(x) = \begin{cases} w(x) + w_2(x) & \text{if } |x| > R' , \\ z(x) + z_2(x) & \text{if } |x| < R'' \end{cases}$$

is an entire solution of $Lu_0 = 0$. Since, by (3.12), $u_0(x) \to \infty$ if $|x| \to \infty$, u_0 must attain a minimum at some point in R^2. But then, by the maximum principle, $u_0(x) \equiv$ const.; this is impossible since $u_0(x) \to \infty$ if $|x| \to \infty$.

Having proved (3.20), we choose in (3.19)

$$\lambda = -\hat{h}(z_1^*)/\hat{h}(z_2^*) \; .$$

Then

$$\hat{h}(z_1^* + \lambda z_1^*) = 0; \tag{3.22}$$

consequently (3.19) has a solution which we shall denote by h. Denote by w, z the solutions of (3.16), (3.17) corresponding to this h. The functions

$$w + w_1 + \lambda w_2, \qquad z + z_1 + \lambda z_2$$

are solutions of $Lu = f$ in $|x| > R'$ and $|x| < R''$ respectively. They coincide on S'' and (by 3.19)) on S'; consequently, they coincide in $R' < |x| < R''$. The function

$$\hat{u}(x) = \begin{cases} w(x) + w_1(x) + \lambda w_2(x) & \text{if } |x| > R' , \\ z(x) + z_1(x) + \lambda z_2(x) & \text{if } |x| < R'' \end{cases}$$

is then an entire solution of $L\hat{u} = f$. In view of (3.12), the function $u(x) = \hat{u}(x) + K_0$ is a solution of (1.2) in R^2, satisfying (3.4), provided K_0 is a sufficiently large positive constant.

REMARK. If $L = \Delta$ then for any locally Hölder continuous function $f(x)$ with compact support K for which

$$\Phi \equiv \int_K f(x)dx \neq 0$$

there does not exist a bounded entire solution of $Lv = f$ in R^2. Indeed, suppose $\Phi > 0$ and let

$$w(x) = \frac{1}{2r}\int_K f(y) \log |x - y| dy .$$

Then $\Delta w = f$ in R^2 and

$$w(x) = \frac{\Phi}{2\pi} \log |x| + O(1) \qquad \text{if } x \to \infty .$$

If there is a bounded entire solution $v(x)$ of $\Delta v = f$ in R^2 then the function $u = w - v$ is harmonic in R^2 and $u(x) \to \infty$ if $x \to \infty$. Consequently u must attain its minimum (in R^2) at a finite point. By the maximum principle, $u(x) \equiv$ const., which is impossible.

4. An application.

Consider the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{i=l}^{n} a_i(x)\frac{\partial u}{\partial x_i} \qquad \text{if } 0 < t < \infty, x \in R^n \tag{4.1}$$

$$u(0, x) = f(x) \text{ if } x \in R^n . \tag{4.2}$$

We shall assume: $a_i(x)$ are locally Hölder continuous and

$$| a_i(x) | \leqq \frac{A}{(1 + |x|)^{2+\nu}} \qquad (\nu > 0, A > 0) , \tag{4.3}$$

$f(x)$ is continuous and

$$|f(x) - f(y)| \leqq N|x - y| \qquad (N > 0) . \tag{4.4}$$

It is then well known [1] that the problem (4.1), (4.2) has a unique solution in the class of functions $v(t, x)$ satisfying, for each $T > 0$,

$$| v(t, x) | \leqq Ce^{c|x|^2} \qquad (0 \leqq t \leqq T, x \in R^n)$$

for some positive constants C, c depending on v, T.

THEOREM 3. *Let* (4.3), (4.4) *hold, and let* $n \geqq 3$. *Then the solution* $u(t, x)$ *of* (4.1), (4.2) *satisfies*

$$\left| u(t, x) - \frac{1}{(4\pi t)^{n/2}}\int_{R^n} \exp\left\{-\frac{|x - \xi|^2}{4t}\right\} f(\xi)d\xi \right| \leqq M \tag{4.5}$$

for all $t \geqq 0, x \in R^n$ *where* M *is a constant.*

Proof. We can write $u(t, x)$ in the form (see [3])

$$u(t, x) = Ef(\xi_x(t)) \tag{4.6}$$

where E is the expectation and $\xi_x(t)$ is a solution of the stochastic integral equation

$$\xi_x(t) = x + \int_0^t a(\xi_x(s))ds + 2\int_0^t dw(s) \; ; \tag{4.7}$$

here $w(t)$ is n-dimentional Brownian motion. Similarly (for $a_i \equiv 0$)

$$\frac{1}{(4\pi t)^{n/2}} \int_{R^n} \exp\left\{-\frac{|x-\xi|^2}{4t}\right\} f(\xi)d\xi = Ef(x + 2w(t)) \; . \tag{4.8}$$

By Theorem 1 there exists a bounded solution $v_j(x)$ of

$$\Delta v_j + \sum_{i=1}^n a_i(x)\frac{\partial v_j}{\partial x_i} = |a_j(x)| \text{ in } R^n \; .$$

By Ito's formula [3],

$$E\int_0^t |a_j(\xi_x(s))|\,ds = Ev_j(\xi_x(t)) - v_j(x) \; .$$

Hence,

$$E\left|\int_0^t a_j(\xi_x(s))ds\right| \leqq C$$

where C is a constant independent of (t, x). Recalling (4.7), we conclude that

$$E\,|\xi_x(t) - x - 2w(t)| \leqq C \; . \tag{4.9}$$

Combining (4.6), (4.8) with (4.4), (4.9), the assertion of the theorem follows.

For $n = 2$ one can employ Theorem 2 and establish the inequality

$$\left|u(t, x) - \frac{1}{(4\pi t)^{1/2}} \int_{R^2} \exp\left\{-\frac{|x-\xi|^2}{4t}\right\} f(\xi)d\xi\right| \leqq M \log(2 + t + |x|) \; .$$

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