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DENDRITES, DIMENSION, AND THE INVERSE ARC FUNCTION

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In this paper, the concept of an inverse arc function is introduced. An inverse arc function f is a function such that for each arc L in the range of f, there exists an arc L_1 in the domain of f such that $f(L_1) = L$. It is proved that a dendrite D is the continuous image of an inverse arc function f with domain an arc L if and only if D has only a finite number of endpoints. Other results are obtained telling what dendrites can be ranges of continuous inverse arc functions having dendrites as domains.

The dimension raising ability of a continuous inverse arc function whose domain is a dendrite is questioned. It is proved that if D is a dendrite with only a countable number of endpoints, then there does not exist a continuous inverse arc function f with domain D such that $\dim f(D) \geqq 2$. If a dendrite D has uncountably many endpoints, then the question is left unanswered.

Basic theorems and definitions used are as stated in [3], [4], [5], and [6]. In particular, a continuum M is a dendrite provided it is locally connected and contains no simple closed curve. A continuum is a compact closed connected set. Other characterizations of a dendrite are also used. Topological spaces considered are all separable metric spaces. If x and y are distinct points, then xy will denote an arc with end points x and y.

DEFINITION. Let $f\colon X \rightarrow Y$ be a function from X onto Y. Then, f is an inverse arc function if and only if for each arc $L \subset Y$ there exists an arc $L_1 \subset X$ such that $f(L_1) = L$.

In this paper the class $\mathscr{D}$ of all dendrites is partitioned into two subclasses, $\mathscr{H}$ and $\mathscr{K}$, such that

$$\mathscr{H} = \{X\colon X \in \mathscr{D} \text{ and there exists a continuous inverse arc function, } f, \text{ with domain an arc } A \text{ and } f(A) = X\}$$

and

$$\mathscr{K} = \mathscr{D} - \mathscr{H} .$$

Then, it is shown that

$$\mathscr{H} = \{X\colon X \in \mathscr{D} \text{ and } X \text{ has only a finite number of endpoints}\} .$$

and

$$\mathscr{K} = \{X\colon X \in \mathscr{D} \text{ and } X \text{ has infinitely many endpoints}\}\,.$$

Further related results are found by studying the question: "Can each member $X \in \mathscr{D}$ be the domain of a continuous inverse arc function, f, with $\dim f(X) \geqq 2$"? It is shown that each member, X, of the class of all dendrites with only countably many endpoints cannot be the domain of a continuous inverse arc function, f, with $\dim f(X) \geqq 2$. Thus, there remains open an interesting question for further study. That is, if D is a dendrite with uncountably many endpoints, then does there exist a continuous inverse arc function, f, with domain D such that $\dim f(D) \geqq 2$? This function, if it exists, would necessarily be a dimension raising function since the $\dim D = 1$.

The following examples point out that there are continuous inverse arc functions that do raise dimension.

EXAMPLE 1. Let n be any natural number. Professor Bing [1] has shown that there are n-dimensional hereditarily indecomposable continua in E_{n+1}. Let M_n be such a hereditarily indecomposable continuum in E_{n+1}. Let P_n be a locally connected continuum in E_{n+1} such that $M_n \subset P_n$. By the Hahn-Mazurkiewicz theorem there exists a continuous function, h, with domain the unit interval I such that $h(I) = P_n$. Let f be the restriction of h to $h^{-1}(M_n)$. The domain of f has dimension less than or equal to 1 since $\dim I = 1$ and the dimension of M_n (the range of f) is n. Thus, f is a dimension raising continuous function when $n \geqq 2$. Since M_n contains on arcs, then vacuously f is a continuous inverse arc function that does raise dimension when $n \geqq 2$.

EXAMPLE 2. Let S be any n-dimensional space such that each point of S is contained in some arc in S. Let

$$\mathscr{A} = \{L\colon L \subset S \text{ and } L \text{ is an arc}\}\,.$$

Let

$$T = \{(x, L)\colon L \in \mathscr{A},\ x \in L\}$$

and $P\colon T \to S$ be the projection function such that $P(x, L) = x$. A basis, σ, for a topology for T can be defined using P. Let

$$\sigma = \{A_L\colon L \in \mathscr{A},\ A_L \subset T \cap (S \times \{L\}),\ P(A_L) \cap L \text{ is an interval on } L \text{ that is open relative to } L\}$$

and τ bo the topology for T generated by σ. The function $P\colon T \to S$ is a continuous inverse arc function from T onto S where the topology for T is τ. By observing that the boundary of each member of σ has at most two points, it then follows that dim $T = 1$. Since dim $T = 1$, then for any natural number $n \geqq 2$, P is a dimension raising continuous inverse arc function.

Example 2 points out that given any n-dimensional space S such that each point of S is contained in some arc in S, then there exists

(1) a 1-dimensional space T and

(2) a continuous inverse arc function, $P\colon T \to S$, onto S.

2. Inverse arc functions onto dendrites. A simple property about inverse arc functions is stated first as Theorem 1.

THEOREM 1. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are two inverse arc functions, then $gf\colon X \to Z$ is an inverse arc function.*

LEMMA 1. *Let D be a dendrite, $p \in D$, and K the set of endpoints of D. Then*

$$D = \bigcup_{x \in K} px \, .$$

Theorem 2 says that a dendrite with only a finite number of endpoints is a member of $\mathscr{H}$.

THEOREM 2. *If D is a dendrite with only a finite number of endpoints, $z \in D$, and $A = ab$ an arc, then there exists a continuous inverse arc function, f, with domain A, range D, and*

$$f(a) = f(b) = z \, .$$

Proof. Without loss of generality consider the arc A as the unit interval $I = [0, 1]$. Suppose that D has n endpoints and denote these endpoints as $K = \{x_1, \cdots, x_n\}$. Lemma 1 implies that

$$D = \bigcup_{i=2}^{n} x_1 x_i \, .$$

This proof can be done by induction. If $n = 2$ and we assume that $z \notin K$, then partition I with the partition $P = \{a_0, a_1, a_2, a_3\}$ where $a_0 = 0$, $a_1 = 1/3$, $a_2 = 2/3$, and $a_3 = 1$.

Define

(1) f_1: $[0, 1/3] \to D$ such that f_1 is a homeomorphism and $f_1(0) = z$, $f_1(1/3) = x_1$, and $f_1([0, 1/3]) = zx_1$,

(2) f_2: $[1/3, 2/3] \to D$ such that f_2 is a homeomorphism and $f_2(1/3) = x_1$, $f_2(2/3) = x_2$, and $f_2([1/3, 2/3]) = x_1x_2$, and

(3) f_3: $[2/3, 1] \to D$ such that f_3 is a homeomorphism and $f_3(2/3) = x_2$, $f_3(1) = z$, and $f_3([2/3, 1]) = x_2z$.

Now, let $f\colon [0, 1] \to D$ be the function defined such that $f(x) = f_i(x)$ for the appropriate natural number i. Clearly, f is a continuous inverse arc function with domain I such that $f(I) = D$ and $f(0) = f(1) = z$.

If $z \in K$ then partition $[0, 1]$ with $P = \{a_0, a_1, a_2\}$ where $a_0 = 0$, $a_1 = 1/2$, and $a_2 = 1$. Without loss of generality suppose $z = x_1$.

Define

(1) $f_1\colon [0, 1/2] \to D$ such that f_1 is a homeomorphism and $f_1(0) = x_1 = z$, $f_1(1/2) = x_2$, and $f([0, 1/2]) = x_1x_2$, and

(2) $f_2\colon [1/2, 1] \to D$ such that f_2 is a homeomorphism and $f_2(1/2) = x_2$, $f_2(1) = x_1 = z$, and $f_2([1/2, 1]) = x_2x_1$.

Now, let $f\colon [0, 1] \to D$ be the function defined such that $f(x) = f_i(x)$ for the appropriate natural number i. Clearly, f is a continuous inverse arc function with domain I such that $f(I) = D$ and $f(0) = f(1) = z$.

Therefore, if $n = 2$, then the theorem is true.

Suppose that the theorem is true for $n = k - 1$. Now, let

$$D = \bigcup_{i=2}^{k} x_1x_i \,.$$

Let p be the first point on

$$x_kx_1 \cap \left(\bigcup_{i=2}^{k-1} x_1x_i\right)$$

from x_k to x_1. The induction hypothesis says that there exists

$$g\colon I \longrightarrow \bigcup_{i=2}^{k-1} x_1x_i$$

a continuous inverse arc function onto

$$\bigcup_{i=2}^{k-1} x_1x_i$$

such that $g(0) = g(1) = p$. Let $I_1 = [0, 2]$ and partition $[1, 2]$ with the partition $P = \{a_0, a_1, \cdots, a_{2k}\}$ where $a_0 = 1$, $a_1 = 2k + 1/2k, \cdots$, $a_j = 2k + j/2k, \cdots, a_{2k} = 2$ and denote $\Delta_j = [a_{j-1}, a_j]$, $j = 1, 2, \cdots, 2k$.

Define

(1) $f_1\colon \Delta_1 \to D$ such that f_1 is a homeomorphism, $f_1(a_0) = p$, $f_1(a_1) = x_k$, and $f_1(\Delta_1) = px_k$,

(2) $f_j\colon \Delta_j \to D$ such that f_j is a homeomorphism, $f_j(a_{j-1}) = x_k$, $f_j(a_j) = x_{j/2}$, and $f_j(\Delta_j) = x_kx_{j/2}$ if $1 < j < 2k$ and j is even,

(3) $f_j\colon \Delta_j \to D$ such that f_j is a homeomorphism, $f_j(a_{j-1}) = x_{j-1/2}$, $f_j(a_j) = x_k$, and $f_j(\Delta_j) = x_{j-1/2}x_k$ if $1 < j < 2k$ and j is odd, and

(4) $f_{2k}\colon \Delta_{2k} \to D$ such that f_{2k} is a homeomorphism, $f_{2k}(a_{2k-1}) = x_k$, $f_{2k}(a_{2k}) = p$, and $f_{2k}(\Delta_{2k}) = x_kp$.

Let $h: I_1 \to D$ be the function defined such that

$$h(x) = g(x) \qquad \text{if } x \in [0, 1]$$

or

$$h(x) = f_i(x) \text{ for the appropriate natural number } i \text{ if } x \in [1, 2] .$$

Again, h is a continuous function defined on the arc I_1 such that $h(I_1) = D$. Let L be any arc in D. If

$$L \subset \bigcup_{i=2}^{k-1} x_1x_i$$

then the induction hypothesis implies that there exists an arc

$$L_1 \subset [0, 1]$$

such that $g(L_1) = L$. Since h extends g then $h(L_1) = L$. Otherwise, if L is an arc in D such that

$$L \not\subset \bigcup_{i=2}^{k=1} x_1x_i ,$$

then there exists a natural number j such that $L \subset x_jx_k$. Now, note that $f_{2j}(\varDelta_{2j}) = x_kx_j$ and since f_{2j} is a homeomorphism on $\varDelta_{2j}$ then there exists an arc $L_1 \subset \varDelta_{2j}$ such that $f_{2j}(L_1) = L$. Since h extends f_{2j}, then $h(L_1) = L$. Therefore, h is an inverse arc function.

Let $z \in D$. If $z = p$ then h is the desired function since h is a continuous inverse arc function with domain an arc, range D, and $h(0) = h(2) = z$. Otherwise, suppose $z \neq p$ and $\varepsilon > 0$. Consider the closed interval $[-\varepsilon, 2 + \varepsilon]$.

Define

(1) h_1: $[-\varepsilon, 0] \to D$ such that h_1 is a homeomorphism onto zp, $h_1(-\varepsilon) = z$, and $h_1(0) = p$, and

(2) h_2: $[2, 2 + \varepsilon] \to D$ such that h_2 is a homeomorphism onto pz, $h_2(2) = p$, and $h_2(2 + \varepsilon) = z$.

As before, let s: $[-\varepsilon, 2 + \varepsilon] \to D$ be the function defined such that

$$s(x) = h(x) \qquad \text{if } x \in [0, 2]$$

or

$$s(x) = h_i(x) \text{ for the appropriate natural number } i \text{ if } x \in [-\varepsilon, 0] \cup [2, 2 + \varepsilon] .$$

Again, s is a continuous function defined on $[-\varepsilon, 2 + \varepsilon]$ such that $s([-\varepsilon, 2 + \varepsilon]) = D$. Since s extends h, then s is also an inverse arc function. The function s is defined such that

$$s(-\varepsilon) = s(2+\varepsilon) = z .$$

Let A be any arc denoted $A = ab$. Let $r: A \to [-\varepsilon, 2+\varepsilon]$ be a homeomorphism onto $[-\varepsilon, 2+\varepsilon]$. Now define f to be the function $rs: A \to D$ and then f is the function desired to prove this theorem by induction.

COROLLARY 1. *If D is a dendrite with only a number of endpoints, $A = ab$ an arc, and $z_1, z_2 \in D$, then there exists a continuous inverse arc function, f, with domain A, range D, $f(a) = z_1$, and $f(b) = z_2$.*

Proof. This corollary can be proved in a similar way to that used to prove Theorem 2.

Lemma 2 can be easily proved and is an aid in the proof of Theorem 3.

LEMMA 2. *If $f: X \to Y$ is a continuous inverse arc function such that $f(X) = Y$, then for each arc $ab \subset Y$, there exists an arc $a_0b_0 \subset X$ such that*

(1) $f(a_0b_0) = ab$,

(2) $f(a_0) = a$, $f(b_0) = b$, *and*

(3) *if* $x \in a_0b_0$, $x \neq a_0$, *and* $x \neq b_0$, *then* $f(x) \neq a$ *and* $f(x) \neq b$.

THEOREM 3. *Let D be a dendrite with infinitely many endpoints and I the unit interval. Then there does not exist $f: I \to D$ a continuous inverse arc function such that $f(I) = D$.*

Proof. Let y be a cut point of D and $D - y = A \cup B$ sep. Without loss of generality suppose that B contains infinitely many endpoints of D. Pick a point $z \in A$ and a countable infinite subset of endpoints contained in B and name this subset, $\{y_n\}$. Now suppose that there exists $f: I \to D$ a continuous inverse arc function such that $f(I) = D$. Define

$$K = \bigcup_{n=1}^{\infty} zy_n .$$

Using Lemma 2 we can obtain for each n an arc $p_nx_n \subset I$ such that $f(p_nx_n) = zy_n$ having the properties stated in Lemma 2. Let

$$\mathscr{A}_1 = \{p_nx_n: n = 1, 2, \cdots\}$$

and define $\mathscr{A}_2 \subset \mathscr{A}_1$ such that $p_1x_1 \in \mathscr{A}_2$. Suppose that for some n it has been decided for each $k < n$ whether or not $p_kx_k \in \mathscr{A}_2$. Then $p_nx_n \in \mathscr{A}_2$ if and only if $p_nx_n \cap p_kx_k = \varnothing$ for each $k < n$ such that

$p_k x_k \in \mathscr{A}_2$. By induction $\mathscr{A}_2$ is defined.

We show that no more than one member of $\mathscr{A}_1$ intersects another member of $\mathscr{A}_1$. Suppose that $p_i x_i$ and $p_j x_j$ are distinct members of $\mathscr{A}_1$ such that $p_i x_i \cap p_j x_j \neq \emptyset$. Since for each k, no point of

$$p_k x_k - \{p_k, x_k\}$$

maps onto either z or y_y, then the only way that $p_i x_i \cap p_j x_j \neq \emptyset$ is for $p_i = p_j$ and without loss of generality $x_i < p_i < x_j$. (Note that if $p_i = p_j < x_i < x_j$ or if $x_j < x_i < p_i = p_j$ we we will reach a contradiction by again arguing as is in this paragraph below.) That is, the arcs $p_i x_i$ and $p_j x_j$ are the the closed intervals $[x_i, p_i]$ and $[p_j, x_j]$ respectively with $p_i = p_j$. The case is, that no other $p_k x_k \in \mathscr{A}_1$ is such that $p_k x_k \cap p_i x_i \neq \emptyset$, for if we assume so, then without loss of generality $p_k x_k = [x_k, p_k]$ and $p_i x_i = [x_i, p_i]$ is true which implies either $[x_i, p_i] \subset [x_k, p_k]$ or $[x_k, p_k] \subset [x_i, p_i]$. If $[x_i, p_i] \subset [x_k, p_k]$, then $f(x_i p_i) = z y_i \subset f(x_k p_k) = z y_k$. This says that y_i is contained in an arc of D and is not an and point of that arc. This is a contradiction to the definition of y_i being an endpoint of D. Similarly, a contradiction can be reached if we assume that $[x_k, p_k] \subset [x_i, p_i]$. Thus, no more than one member of $\mathscr{A}_1$ intersects another member of $\mathscr{A}_1$.

Suppose that $\mathscr{A}_2$ is finite. The method of definition of $\mathscr{A}_2$ implies that there exists N such that for each $n > N$,

$$p_n x_n \in \mathscr{A}_1 - \mathscr{A}_2 .$$

For convenience, denote $\mathscr{A}_2 = \{p_1 x_1, \cdots, p_k x_k\}$. Since

$$P_{N+1} x_{N+1} \in \mathscr{A}_1 - \mathscr{A}_2 ,$$

then there exists only one member of $\mathscr{A}_2$ intersecting $p_{N+1} x_{N+1}$ If there were more than one, then this would contradict the argument in the preceding paragraph. Likewise, $p_{N+2} x_{N+2}$ intersects only one member of $\mathscr{A}_2$. The member of $\mathscr{A}_2$ that $p_{N+2} x_{N+2}$ intersects is distinct from the one that $p_{N+1} x_{N+1}$ intersects for if not, then again the argument in the preceding paragraph would be contradicted. After k considerations, the set of arcs $L = \{p_{N+1} x_{N+1}, \cdots, p_{N+k} x_{N+k}\}$, has the property that each member of L intersects one and only one distinct member of $\mathscr{A}_2$. This exhausts $\mathscr{A}_2$ and implies that $p_{N+k+1} x_{N+k+1}$ intersects no member of $\mathscr{A}_2$. This contradicts the definition of $\mathscr{A}_2$ and therefore $\mathscr{A}_2$ is infinite.

Since we know that $\mathscr{A}_2$ is an infinite set of mutually exclusive arcs in I, then we can pick a null sequenc $\mathscr{A} \subset \mathscr{A}_2$. Denote $\mathscr{A}$ by $\mathscr{A} = \{p_{n_i} x_{n_i}\}$ and let z_0 be a member of the limit set of $\mathscr{A}$. Assume that $f(z_0) \neq z$. Let U be any open set such that $f(z_0) \in U$ and $z \notin U$. Since f is continuous, then there exists an open set V such that

$z_0 \in V$ and $f(V) \subset U$. Because $\mathscr{A}$ is a null sequence and z_0 is in the limit set of $\mathscr{A}$, then there exists an i such that $p_{n_i}x_{n_i} \subset V$. Thus, $z = f(p_{n_i}) \subset f(V) \subset U$ which contradicts the definition of U. Thus, $f(z_0) = z$.

The set A is an open set containing z and since $f(z_0) = z$ there exists an open set V such that $z_0 \in V$ and $f(V) \subset A$. Again, because $\mathscr{A}$ is a null sequence and z_0 is in the limit set of $\mathscr{A}$ there exists an i such that $p_{n_i}x_{n_i} \subset V$. Therefore,

$$y_{n_i} = f(x_{n_i}) \subset f(V) \subset A$$

which contradicts $y_{n_i} \in B$. This final contradiction completes the proof of the theorem.

THEOREM 4. *The subclasses* $\mathscr{H}$ *and* $\mathscr{K}$ *of* $\mathscr{D}$ *are characterized as*

$$\mathscr{H} = \{X\colon X \in \mathscr{D} \text{ and } X \text{ has only a finite number of endpoints}\}$$

and

$$\mathscr{K} = \{X\colon X \in \mathscr{D} \text{ and } X \text{ has infinitely many endpoints}\} .$$

Proof. The proof of theorem is a consequence of Theorems 2 and 3.

Theorem 4 classifies dendrites into the class of all dendrites that are the range of an inverse arc function whose domain is an arc and the class of all those dendrites that cannot be the range of an inverse arc function with domain an arc. The remaining theorems in this section tell us what dendrites can be ranges of continuous inverse arc functions having dendrites as domains. In addition, Theorem 5 will be used as a tool to prove Theorem 8 in § 3.

LEMMA 3. *If D is a dendrite and H is an uncountable collection of arcs, each contained in D, then there exists $z \in D$ such that z is contained in uncountably many members of H.*

THEOREM 5. *Let S be a topological space that contains a subspace M where*

$$M = \bigcup_{y \in L} py ,$$

L is an uncountable subset of S, $p \in S$, and if y_1 and y_2 are distinct members of L, then $y_1 \notin py_2$ and $y_2 \notin py_1$. If D is a dendrite with only countably many endpoints, then there does not exist $f\colon D \to S$, a continuous inverse arc function such that $f(D) = S$.

Proof. If we assume the contrary, then for each arc py, $y \in L$, there exists an arc $L_y = o_p x_y \subset D$ with the properties given by Lemma 2. That is, $f(L_y) = py$, $f(o_p) = p$, and $f(x_y) = y$. Let $H = \{L_y\colon y \in L\}$. Lemma 3 implies that there exists a point $z \in D$ such that z is contained in uncountably many members of H. Let

$$K = \{L_y\colon L_y \in H,\ z \in L_y\} .$$

Note that K is uncountable and that for each

$$L_y = o_p x_y \in K,\ z x_y \subset L_y .$$

If we denote the endpoints of D as $\{d_n\}$, then Lemma 1 allows us to denote

$$D = \bigcup_{n=1}^{\infty} z d_n .$$

Since K is uncountable there will exist $L_{y_1}, L_{y_2} \in K$, a natural number N, and $y_1 \neq y_2$ such that x_{y_1} and x_{y_2} are points on the arc zd_N. Without loss of generality suppose $x_{y_1} < x_{y_2}$ on zd_N from z to d_N. Thus, $zx_{y_1} \subset zx_{y_2}$ and

$$f(zx_{y_1}) \subset f(zx_{y_2}) \subset f(L_{y_2}) = py_2 .$$

This says that $f(x_{y_1}) = y_1 \in py_2$ which contradicts hypothesis of the theorem.

COROLLARY 2. *If D is a dendrite with uncountably many endpoints and D_1 a dendrite with only countably many endpoints, then there does not exist $f\colon D_1 \to D$, a continuous inverse arc function such that $f(D_1) = D$.*

Proof. Pick a point $p \in D$. Denote D by

$$D = \bigcup_{y \in L} py$$

where L is the set of endpoints of D. The hypothesis says that L is uncountable. Theorem 5 then implies the desired result.

THEOREM 6. *If D is a dendrite with an infinite number of endpoints and D_1 is a dendrite with only a finite number of endpoints, then there does not exist $f\colon D_1 \to D$, a continuous inverse arc function such that $f(D_1) = D$.*

Proof. Assume that such a function f does exist. Theorem 2 implies that there exists $g\colon A \to D_1$, a continuous inverse arc function such that A is an arc and $g(A) = D_1$. Then $fg\colon A \to D$ is a continuous inverse arc function such that $fg(A) = D$, which contradicts Theorem 3.

LEMMA 4. *If D is a dendrite and A is any arc contained in D, then D is retractible to A.*

THEOREM 7. *If S is a topological space that is retractible to an arc L_1 and D is any dendrite with only a finite number of endpoints, then there exists $f: S \to D$, a continuous inverse arc function such that $f(S) = D$.*

Proof. Since S is retractible to L_1 there exists a continuous function $g: S \to S$ such that $g(S) = L_1$ and g is the identity function on L_1. Theorem 2 says that there exists $h: L_1 \to D$, a continuous inverse arc function such that $h(L_1) = D$. Let $f: S \to D$ be the continuous function from S onto D such that $f = hg$. Let L be any arc in D. Since h is an inverse arc map, there exists an arc $L_2 \subset L_1$ such that $h(L_2) = L$. The definition of f implies that

$$f(L_2) = h(g(L_2)) = h(L_2) = L .$$

Thus, f is an inverse arc map and the theorem is proved.

COROLLARY 3. *If D_1 is any dendrite and D is any dendrite with only a finite number of endpoints, then there exists $f: D_1 \to D$,* a *continuous inverse arc function such that $f(D_1) = D$.*

Proof. The proof is obtained by using Lemma 4 and Theorem 7.

Let D and D_1 be dendrites with infinitely many endpoints. The question that is interesting in this case is: "Does there exist $f: D_1 \to D$, a continuous inverse arc function such that $f(D_1) = D$?" The only portion of this question that is answered in this paper is when D_1 has countably many endpoints and D has uncountably many endpoints. For this, Corollary 2 reveals the answer to be no. Thus, there are three cases for further study.

3. A dimension related problem. This section gives a partial answer to the dimension raising ability of a continuous inverse arc function whose domain is a dendrite.

THEOREM 8. *If D is a dendrite with only a countable number of endpoints, then there does not exist a continuous inverse arc function, f, with domain D such that* $\dim f(D) \geqq 2$.

Proof. Let D be a dendrite and suppose that there does exist a continuous inverse arc function, f, with domain D and $\dim f(D) \geqq 2$. Pick a point $p \in f(D)$ at which $f(D)$ has dimension larger than one. The space $f(D)$ has a convex metric ρ, [2]. Using this convex metric to generate the topology of $f(D)$ and using the definition of dimension, there exists $\varepsilon > 0$ such that the boundary, B of the ε-sphere with center at p is uncountable. Again using the metric ρ, we can construct an arc py for each $y \in B$ such that if $y_1, y_2 \in B$, $y_1 \neq y_2$, then $y_1 \notin py_2$ and $y_2 \notin py_1$. Let

$$M = \bigcup_{y \in B} py$$

and then note that Theorem 5 says that f cannot exist and a contradiction is reached.

COROLLARY 4. *If S is a Peano continuum that is the continuous inverse arc image of the unit interval I, then* $\dim S \leqq 1$.

Proof. Since I is a dendrite with only two endpoints, then Theorem 8 applies.

At this time I have not been able to discover whether or not dimension can be raised on a dendrite with uncountably many endpoints by a continuous inverse arc function. However, Theorem 9 does answer this question in the special case when $\dim f(D) = n$ and $f(D)$ can be imbedded in E_n.

THEOREM 9. *If D is a dendrite, f is an inverse arc function with domain D,* $\dim f(D) = n$, *and $f(D)$ can be imbedded in E_n, then $n \leqq 1$.*

Proof. If we suppose the contrary, then $n \geqq 2$. Thus, [3], $f(D)$ contains a nonempty open subset of E_n and in particular contains an n-cube, $n \geqq 2$. Because of this, $f(D)$ contains an uncountable collection, G, of mutually exclusive arcs. Since f is an inverse arc function, we can pick an arc $L_X \subset f^{-1}(X)$ for each $X \in G$. Thus, $H = \{L_X \colon X \in G\}$ is an uncountable collection of mutually exclusive arcs contained in D which contradicts Lemma 3.

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