A well-known result of P. Hall shows that finite solvable groups may be characterized by a permutability requirement on Sylow subgroups. The notion of a generalized Sylow tower group (GSTG) arises when this permutability condition on Sylow subgroups is replaced by a suitable normalizer condition. In an earlier paper, one of the authors showed that the nilpotent length of a GSTG cannot exceed the number of distinct primes which divide the order of the group. The present investigation utilizes the ‘type’ of a GSTG to obtain improved bounds for the nilpotent length of a GSTG. It is also shown that a GSTG with nilpotent length \( n \) possesses a Hall subgroup of nilpotent length \( n \) which is a Sylow tower group.

Let \( G \) be a finite group with order \( p_1^{a_1} \cdots p_r^{a_r} \), where \( p_1, \ldots, p_r \) are distinct primes and \( a_1, \ldots, a_r \) are positive integers. For each integer \( i, 1 \leq i \leq r \), let \( G_i \) denote a Sylow \( p_i \)-subgroup of \( G \). The collection of subgroups \( \mathcal{S} = \{G_1, \ldots, G_r\} \) is then called a complete set of Sylow subgroups of \( G \). If the elements of \( \mathcal{S} \) are pairwise permutable as subgroups (that is, if \( G_iG_j = G_jG_i \) holds for all \( i \) and \( j \)) then \( \mathcal{S} \) will be called a Sylow basis for \( G \). The notion of a generalized Sylow tower group arises when the permutability condition for a Sylow basis is replaced by a normalizer condition. Thus, we say that a finite group \( G \) is a generalized Sylow tower group (GSTG) if and only if some complete set of Sylow subgroups \( \mathcal{S} \) of \( G \) satisfies the normalizer condition \((N)\): if \( G_i \) and \( G_j \) are distinct elements of \( \mathcal{S} \), at least one of these subgroups normalizes the other. It should be noted that not every complete set of Sylow subgroups of a GSTG need satisfy condition \((N)\).

A well-known result of P. Hall states that a finite group is solvable if and only if the group possesses a Sylow basis. If a complete set of Sylow subgroups \( \mathcal{S} \) of a group \( G \) satisfies condition \((N)\) then any two elements of \( \mathcal{S} \) are permutable as subgroups and \( \mathcal{S} \) is a Sylow basis for \( G \). Consequently every generalized Sylow tower group must be solvable.

A finite group \( G \) is called a Sylow tower group (STG) if every nontrivial epimorphic image of \( G \) possesses a nontrivial normal Sylow subgroup. Equivalently, the group \( G \) is an STG if the prime divisors \( p_1, \ldots, p_r \) of the order of \( G \) can be labelled in such a way that a Sylow \( p_i \)-subgroup of \( G \) normalizes a Sylow \( p_j \)-subgroup of \( G \) when-
ever $i > j$. It is clear from this definition that a Sylow tower group is necessarily a generalized Sylow tower group. An example of a GSTG which is not a Sylow tower group was given in ([1]; p. 638).

In order to handle generalized Sylow tower groups it will be necessary to introduce the ‘type’ of a GSTG. Suppose $G$ is some given GSTG and let $\mathcal{T}$ be a Sylow basis for $G$. Since any two Sylow bases for $G$ are conjugate ([3]; p. 665), $\mathcal{T}$ satisfies the normalizer condition (N). Let $R$ be a relation on the set of all primes with the property that either $pRq$ or $qRp$ (or both) holds for any primes $p$ and $q$. If the Sylow $p_i$-subgroup of $G$ in $\mathcal{T}$ normalizes the Sylow $p_j$-subgroup of $G$ in $\mathcal{T}$ whenever $p_iRq$ holds, then $G$ will be called a GSTG of type $R$. It follows directly from the conjugacy of Sylow bases that the type of a GSTG is independent of the choice of a Sylow basis. It should be noted that a group can be a GSTG of more than one type.

It was shown in [1] that the class of all generalized Sylow tower groups of a given type $R$ is a formation. In addition, any subgroup of a GSTG of type $R$ was shown to be a GSTG of type $R$. We list the main results about GSTG's in [1] for easy reference.

**Theorem 1.7** [1]. If $G$ is a GSTG then the nilpotent length of $G$ does not exceed the number of distinct prime divisors of the order of $G$.

**Theorem 1.8** [1]. If $G$ is a GSTG and the nilpotent length of $G$ is equal to the number of distinct prime divisors of the order of $G$ then $G$ is a Sylow tower group.

All groups mentioned are assumed to be finite. The following notations will be used. For a group $G$

- $c(G)$ denotes the set of distinct prime divisors of the order of $G$
- $\pi(G)$ denotes the number of distinct prime divisors of the order of $G$
- $\omega(G)$ denotes the nilpotent (Fitting) length of $G$.

If $H$ is a subgroup of $G$ then $N_c(H)$ means the normalizer of $H$ in $G$ and $C_c(H)$ means the centralizer of $H$ in $G$.

If $p_i$ is a prime, $G_i$ will denote a Sylow $p_i$-subgroup of $G$.

The following lemma will be used in several of our arguments.

**Lemma 1.** If $G$ is a nontrivial GSTG then at least one of the following holds:

1. $G$ contains a nontrivial normal Sylow subgroup $P$ with $C_c(P) \subseteq P$
(2) \( G \) contains nontrivial normal subgroups having relatively prime orders.

Proof. Let \( G \) be a GSTG and suppose that \( \mathcal{S} = \{ G_1, \cdots, G_n \} \) is a Sylow basis for \( G \). Since a GSTG is necessarily solvable, \( G \) possesses a nontrivial minimal normal subgroup \( M \) with order a power of some prime \( p \). Let \( Q \) denote the maximum normal \( p \)-subgroup of \( G \) and suppose that \( G_i \) is the Sylow \( p \)-subgroup of \( G \) belonging to \( \mathcal{S} \). Since \( \mathcal{S} \) satisfies the normalizer condition (N), either

\[
G_i \subseteq N_o(G_k) \quad \text{or} \quad G_k \subseteq N_o(G_i)
\]

must hold for each integer \( k, 1 \leq k \leq n \). We distinguish two cases.

First suppose that \( G_i \subseteq N_o(G_k) \) holds for some \( k, 1 < k \leq n \). Since \( Q \) is normal in \( G \) and has order prime to the order of \( G_k \), \( G_k \) centralizes \( Q \). Hence \( C = C_o(Q) \) is not a \( p \)-group. Set \( Q_0 = Q \cap C_o(Q) \) and consider the factor group \( C/Q_0 \). Since \( C/Q_0 \) is a nontrivial solvable group, \( C/Q_0 \) contains a nontrivial minimal normal subgroup \( L/Q_0 \) with order a power of some prime \( p \). Let \( T/Q_0 \) be the maximum normal \( p \)-subgroup of \( C/Q_0 \). Since \( Q \) is the maximum normal \( q \)-subgroup of \( G \) it follows that \( p \neq q \). If \( N \) is a Sylow \( p \)-subgroup of \( T \) then \( T \) is the direct product of \( N \) and \( Q_0 \). The normality of \( T \) in \( G \) then implies the normality of \( N \) in \( G \). Therefore \( G \) has nontrivial normal subgroups with relatively prime orders.

Now suppose that \( G_i \subseteq N_o(G_k) \) holds for all integers \( k, 1 < k \leq n \). Since \( \mathcal{S} \) satisfies the normalizer condition (N), \( G_k \subseteq G_o(G_i) \) must then hold for all integers \( k \). Thus \( G_i \) is a normal Sylow \( q \)-subgroup of \( G \). Then \( G_i \cap C_o(G_i) \) is a normal Sylow \( q \)-subgroup of \( C_o(G_i) \) and \( C_o(G_i) \) has a normal \( q \)-complement \( W \). If \( W \) is nontrivial then \( W \) is a normal subgroup of \( G \) having \( q' \)-order and (2) holds. If \( W \) is trivial then \( G_i \cap C_o(G_i) = C_o(G_i) \). In this case \( C_o(G_i) \subseteq G_i \) and (1) holds.

Theorem S. Let \( G \) be a GSTG. If \( H_1, \cdots, H_n \) are pairwise permutable Hall subgroups of \( G \) with \( G = H_1 \cdots H_n \), then the nilpotent length of \( G \) does not exceed the sum of the nilpotent lengths of \( H_1, \cdots, H_n \).

Proof. (By induction on the order of \( G \).) Since the product \( H_1 \cdots H_n \) is a Hall subgroup of \( G \) permutable with \( H_i \) we may assume that \( n = 2 \). First suppose that \( G \) possesses nontrivial normal subgroups \( A \) and \( B \) with \( A \cap B = 1 \). Then \( G \) is isomorphic to a subgroup of the direct of \( G/A \) and \( G/B \). Hence

\[
\mathcal{L}(G) = \max \{ \mathcal{L}(G/A), \mathcal{L}(G/B) \}
\]
and it suffices to show that
\[ \phi(G/A) \leq \phi(H_1) + \phi(H_2) \quad \text{and} \quad \phi(G/B) \leq \phi(H_1) + \phi(H_2). \]

Since \( G/A \) is the product of the permutable Hall subgroups \( H_1A/A \) and \( H_2A/A \) of \( G/A \), the induction hypothesis gives
\[ \phi(G/A) \leq \phi(H_1A/A) + \phi(H_2A/A). \]

Since \( H_1A/A \) and \( H_2A/A \) are epimorphic images of \( H_1 \) and \( H_2 \) (respectively), we know that \( \phi(H_iA/A) \leq \phi(H_i) \) for \( i = 1, 2 \). Therefore \( \phi(G/A) \leq \phi(H_1) + \phi(H_2) \). The same argument applied to \( G/B \) will show \( \phi(G/B) \leq \phi(H_1) + \phi(H_2) \). This verifies the theorem in this case.

Now suppose that \( G \) possesses a unique minimal normal subgroup. Then Lemma 1 shows that \( G \) contains a nontrivial normal Sylow subgroup \( P \) with \( C_G(P) \subseteq P \). Since the theorem is trivially true if \( G = P \) we may assume this is not the case. Set
\[ \bar{G} = G/P, \bar{H}_i = H_iP/P, \bar{H}_2 = H_2P/P \]

and consider the nontrivial GSTG \( \bar{G} \). The induction hypothesis applied to \( \bar{G} = \bar{H}_1\bar{H}_2 \) gives \( \phi(\bar{G}) \leq \phi(\bar{H}_1) + \phi(\bar{H}_2) \). We first observe that \( \phi(G) = \phi(\bar{G}) + 1 \). Since \( P \) is a nilpotent normal subgroup of \( G \), \( P \) must lie in the Fitting subgroup \( F \) of \( G \). If \( P \neq F \) then \( F \) contains a nonidentity element of order prime to the order of \( P \) which belongs to the centralizer in \( G \) of \( P \). This contradicts \( C_G(P) \subseteq P \). Therefore \( P = F \) and \( \phi(G) = \phi(\bar{G}) + 1 \).

Since \( H_1 \) and \( H_2 \) are Hall subgroups of \( G \) satisfying \( G = H_1H_2 \), the Sylow subgroup \( P \) must lie in \( H_1 \) or \( H_2 \). We may suppose that \( H_1 \) contains \( P \). If \( P = H_1 \) then \( \bar{G} = \bar{H}_2 \) and so \( \phi(\bar{G}) = \phi(\bar{H}_2) \leq \phi(H_2) \). Then \( \phi(G) = \phi(\bar{G}) + 1 \leq \phi(H_1) + \phi(H_2) \), which is what we wanted to show. If \( P \neq H_1 \) then the argument used in the preceding paragraph can be repeated to show \( \phi(H_1) = \phi(\bar{H}_1) + 1 \). It follows from this that
\[ \phi(G) = \phi(\bar{G}) + 1 \leq \phi(\bar{H}_1) + 1 + \phi(\bar{H}_2) \leq \phi(H_1) + \phi(H_2). \]

This completes the argument.

It seems interesting to ask if this theorem has a converse, in the following sense. Does a GSTG \( G \) necessarily possess pairwise permutable proper Hall subgroups \( H_1, \ldots, H_n \) satisfying \( G = H_1 \cdots H_n \) so that \( \phi(G) = \phi(H_1) + \cdots + \phi(H_n) \) holds? The answer is obviously no, since any nilpotent group is a GSTG. If we insist that the group \( G \) not be nilpotent, the answer to the question is still no. This can easily be verified using the example of an \( N \)-group which is not a Sylow tower group (see [1]; p. 638). We now mention some consequences of the theorem.
Let $G$ be a GSTG and suppose that $\mathcal{S} = \{G_1, \cdots, G_n\}$ is a Sylow basis for $G$. Since $G_1, \cdots, G_n$ are pairwise permutable Sylow subgroups of $G$ satisfying $G = G_1 \cdots G_n$, Theorem $S$ gives

$$\epsilon(G) \leq \epsilon(G_1) + \cdots + \epsilon(G_n) = \pi(G).$$

Consequently Theorem 1.7 [1] follows from Theorem $S$.

Now we show how the type of a GSTG can be used to improve the bound on the nilpotent length of a GSTG given by Theorem 1.7 [1]. It will be helpful to first introduce some terminology. Let $R$ be a relation on the set of all primes and let $\sigma$ denote some given set of primes. Then $\sigma$ will be called a complete $R$-symmetric set provided both $pRq$ and $qRp$ hold for all primes $p, q$ belonging to $\sigma$. If $\sigma$ contains a single prime then $\sigma$ is (trivially) a complete $R$-symmetric set. It is clear from this that any set of primes can be written as a union of complete $R$-symmetric subsets. The set $\sigma$ will be called an $R$-cyclic set if $\sigma$ contains distinct primes $p, q$, and $w$ such that $pRq$, $qRp$, and $wRp$ hold.

**Corollary 1.** Let $G$ be a GSTG of type $R$. If $\sigma_1, \cdots, \sigma_d$ are complete $R$-symmetric subsets of $c(G)$ such that the union of the $\sigma_i$ is $c(G)$ then $\epsilon(G) \leq d$.

**Proof.** Let $\mathcal{S} = \{G_1, \cdots, G_n\}$ be a Sylow basis for $G$. For each $i, 1 \leq i \leq d$, define the subgroup $H_i$ of $G$ to be the product of all Sylow $p_i$-subgroups $G_k$ for which $p_k \in \sigma_i$. Since $pRq$ and $qRp$ hold for all distinct primes $p$ and $q$ from $\sigma_i$, each $H_i$ is seen to be a nilpotent Hall $\sigma_i$-subgroup of $G$. Since the union of the $\sigma_i$'s is $c(G)$, clearly $G = H_1 \cdots H_d$. The theorem then shows that

$$\epsilon(G) \leq \epsilon(H_1) + \cdots + \epsilon(H_d) = d.$$

**Corollary 2.** Let $G$ be a GSTG of type $R$. If $\sigma_1, \cdots, \sigma_d$ are disjoint $R$-cyclic subsets of $c(G)$ then $\epsilon(G) \leq \pi(G) - d$.

**Proof.** Let $\mathcal{S} = \{G_1, \cdots, G_n\}$ be a Sylow basis for $G$. For each integer $i, 1 \leq i \leq d$, define the subgroup $H_i$ of $G$ to be the product of all Sylow $p_i$-subgroups $G_k$ for which $p_k \in \sigma_i$. It is clear from the definition that the $H_i$ are pairwise permutable Hall $\sigma_i$-subgroups of $G$. Since the product $H = H_1 \cdots H_d$ has a Hall complement in $G$, it suffices to show that $\epsilon(H) \leq \pi(H) - d$. Since the $\sigma_i$'s are disjoint sets, this will follow if $\epsilon(H_i) \leq \pi(H_i) - 1$ holds for each $i, 1 \leq i \leq d$. Let $p, q$, and $w$ be distinct primes in $\sigma_i$ satisfying $pRq$, $qRp$ and $wRp$. Consider a Hall $\{p, q, w\}$-subgroup $T$ of $H_i$. If $T$ has a normal Sylow $p$-subgroup then $pRq$ shows that $T$ has a nilpotent Hall
\{p, q\}\text{-subgroup}. Then Corollary 1 shows $\ell(T) \leq \pi(T) - 1$. It now follows from Theorem S that $\ell(H_i) \leq \pi(H_i) - 1$. Consequently we may assume $T$ has no nontrivial normal Sylow subgroup. By Lemma 1, $T$ then has nontrivial normal subgroups $A$ and $B$ with $A \cap B = 1$. Since $T/A$ and $T/B$ are GSTG’s of type $R$, induction shows that $\ell(T/A) \leq 2$ and $\ell(T/B) \leq 2$. Using the fact that $T$ is isomorphic to a subgroup of the direct product of $T/A$ and $T/B$ we obtain

$$\ell(T) \leq 2 = \pi(T) - 1.$$ 

Theorem S applied to $H_i$ now gives $\ell(H_i) \leq \pi(H_i) - 1$. Therefore we have shown that $\ell(H_i) \leq \pi(H_i) - 1$ holds for arbitrary $i$ and the assertion follows.

The next consequence of the theorem is Theorem 1.8 [1].

**COROLLARY 3.** Let $G$ be GSTG with $\ell(G) = \pi(G)$. Then $G$ is a Sylow tower group of exactly one type $R$, in the sense that the relation $R$ is uniquely determined for pairs of primes $p$, $q$ in $c(G)$.

**Proof.** Let $\mathcal{S} = \{G_1, \ldots, G_n\}$ be Sylow basis for $G$. Define the relation $R$ on the set of all primes as follows: $R$ is reflexive and for distinct primes $p$ and $q$, $pRq$ holds if and only if either $p$ or $q$ does not divide the order of $G$ or both $p$ and $q$ do divide the order of $G$ and the Sylow $p$-subgroup of $G$ belonging to $\mathcal{S}$ normalizes the Sylow $q$-subgroup of $G$ belonging to $\mathcal{S}$. Clearly $G$ is of type $R$. Since $\ell(G) = \pi(G)$, Corollary 1 shows that both $pRq$ and $qRp$ hold for no distinct primes $p$, $q \in c(G)$. In addition, Corollary 2 shows that $pRq$, $qRw$, and $wRp$ hold for no distinct primes $p$, $q$, and $w$ from $c(G)$. Since either $pRq$ or $qRp$ holds for any primes $p$, $q \in c(G)$, the restriction of $R$ to $c(G)$ must be a linear order. Therefore $G$ is a Sylow tower group of type $R$. Suppose that $G$ is also a STG of type $S$ and the restriction of $S$ to $c(G)$ differs from the restriction of $R$ to $c(G)$. Then $G$ would necessarily have a nilpotent Hall $\{p, q\}$-subgroup for some distinct $p$, $q \in c(G)$. The conjugacy of Hall $\{p, q\}$-subgroups in $G$ then implies that $\ell(G) \leq \pi(G) - 1$, a contradiction. Therefore $G$ is a STG of exactly one type, in the sense mentioned.

We next give an example to show that the nilpotent length of a GSTG cannot be found from the type alone. Let $A$ be the holomorph of a cyclic group of order $7$ and let $B$ denote the Hall $\{7, 3\}$-subgroup of $A$. Define the group $G_1$ as the direct product of $A$ and a symmetric group of degree $3$ and define $G_2$ as the wreath product of $B$ by a cyclic group of order $2$. Both $G_1$ and $G_2$ are Sylow tower groups of type $7 < 3 < 2$ and no distinct Sylow subgroups of $G_1$ or $G_2$ centralize one another. Hence, for a given relation $R$ on the set.
of all primes, $G_i$ is a GSTG of type $R$ if and only if $G_2$ is a GSTG of type $R$. Yet the nilpotent length of $G_1$ is 2 and the nilpotent length of $G_2$ is 3.

**Theorem T.** Let $G$ be a GSTG with nilpotent length $k$. Then $G$ contains a Hall subgroup which is a Sylow tower group and has nilpotent length $k$.

*Proof.* By Theorem 1.8 [1] it is sufficient to show that $G$ contains a Hall subgroup $L$ with $\sigma(L) = \pi(L) = k$. We proceed by induction on the order of $G$.

Suppose $G$ contains a proper subgroup $W$ with $\sigma(W) = k$. The induction hypothesis then shows that $W$ contains a Hall subgroup $T$ with $\sigma(T) = \pi(T) = k$. Choose a Hall subgroup $L$ of $G$ with $T \subseteq L$ and $c(T) = c(L)$. Then $\sigma(G) = k = \sigma(T) \leq \sigma(L) \leq \sigma(G)$ shows that $\sigma(L) = \pi(L) = k$. This proves the theorem in the case where $G$ contains a proper subgroup with nilpotent length $k$. Now suppose that every proper subgroup of $G$ has nilpotent length strictly less than $k$.

Since $G$ is a GSTG, either $G$ possesses nontrivial normal subgroups $A$ and $B$ with $A \cap B = 1$ or $G$ contains a nontrivial normal Sylow subgroup $P$ with $C_G(P) \subseteq P$. We consider these possibilities separately. First suppose that $A$ and $B$ are distinct minimal normal subgroups of $G$. If the Frattini subgroup $\phi$ of $G$ is trivial then $G$ contains a maximal subgroup $M_1$ not containing $A$ and a maximal subgroup $M_2$ not containing $B$. Then $M_i$ complements $A$ in $G$ and $M_i$ complements $B$ in $G$. Since we have assumed that all proper subgroups of $G$ have nilpotent length less than $k$, the isomorphism of $G/A$ and $M_1$ gives $\sigma(G/A) < k$. Similarly one sees that $\sigma(G/B) < k$. Since $G$ is isomorphic to a subgroup of the direct product of $G/A$ and $G/B$, it follows that $\sigma(G) = \max\{\sigma(G/A), \sigma(G/B)\}$ is less than $k$, a contradiction. Therefore $G$ has nontrivial Frattini subgroup. Since $\sigma(G/\phi) = \sigma(G) = k$, the induction hypothesis shows that $G$ contains a Hall subgroup $L$ satisfying $\sigma(L/\phi) = \pi(L/\phi) = k$. Now

$$k = \sigma(L/\phi) \leq \sigma(L) \leq \sigma(G) = k$$

shows $\sigma(L) = k$. Hence $L = G$. Since the Frattini subgroup of $G$ contains no Sylow subgroup of $G$, $\pi(G) = \pi(L) = \pi(L/\phi) = k$. Therefore $\sigma(G) = \pi(G) = k$, which completes the argument in this case.

Now suppose $G$ contains a nontrivial normal Sylow subgroup $P$ with $C_G(P) \subseteq P$. It follows that $P$ must be the Fitting subgroup of $G$. Therefore $\sigma(G) = \sigma(G/P) + 1$ or $G = P$. In the latter case the theorem is trivially true. If $\sigma(G) = \sigma(G/P) + 1$, the induction
hypothesis shows that $G/P$ contains a nontrivial Hall subgroup $L/P$ satisfying $\langle L/P \rangle = \pi(L/P) = k - 1 = \langle G/P \rangle$. Clearly $L$ is then a Hall subgroup of $G$ with $\pi(L) = k$. Since $C_G(P) \subseteq P$, $P$ must be the Fitting subgroup of $L$. Hence $\langle L \rangle = \langle L/P \rangle + 1$. Therefore

$$\langle L \rangle = k = \pi(L).$$

This completes the proof of the theorem.

Let $G$ be a given GSTG and suppose $\mathcal{S}$ is a Sylow basis for $G$. Define the relation $R$ on the set of all primes as follows: for any primes $p$ and $q$ (possibly equal), $pRq$ holds if and only if $p \in c(G)$ or $q \in c(G)$ or both $p$ and $q$ belong to $c(G)$ and the Sylow $p$-subgroup of $G$ in $\mathcal{S}$ normalizes the Sylow $q$-subgroup of $G$ in $\mathcal{S}$. Clearly $G$ is a GSTG of type $i$. If $H$ is a Hall subgroup of $G$ which is a Sylow tower group and $H$ satisfies $\langle H \rangle = \pi(H) = \langle G \rangle$, then the restriction of $R$ to $c(H)$ is a transitive relation (see the proof of Corollary 3). This leads to the following bound for the nilpotent length of $G$ in terms of the relation $R$ defined above. The nilpotent length of the GSTG $G$ cannot exceed $n$, where $n$ is the largest integer such that the restriction of $R$ to some subset of $c(G)$ having $n$ elements is a transitive relation.

**References**


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