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REAL PARTS OF UNIFORM ALGEBRAS

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This paper is concerned with identifying those uniform algebras B on $\Gamma = \{z: |z| = 1\}$ for which Re B—the space of real parts of the functions in B—equals Re A, where A denotes the disk algebra. It is shown that for any such algebra, there is an absolutely continuous homeomorphism Φ of Γ onto Γ so that $B = A(\Phi) = \{f(\Phi): f \in A\}$. A partial converse to this theorem also holds: If Φ is a homeomorphism of Γ onto itself which is of class C^2 with nowhere vanishing derivative, then Re $A(\Phi) =$ Re A.

For completeness, we recall that a uniform algebra on Γ is defined as a subalgebra of $C(\Gamma)$ which is closed in the norm $\|f\| = \max_\Gamma |f|$, contains the constants and separates the points of Γ. The disk algebra is the particular uniform algebra consisting of all functions in $C(\Gamma)$ which extend continuously to $\{z: |z| \leqq 1\}$ to be analytic on $D = \{z: |z| < 1\}$. Note that if Φ is a homeomorphism of Γ onto itself, then $A(\Phi)$ is a uniform algebra. In this paper we will frequently use the convention of writing $f(\theta)$ for $f(e^{i\theta})$ when f belongs to $C(\Gamma)$.

The first result along the lines discussed in this paper is due to Hoffman and Wermer [7]. They prove that if B is a uniform algebra on a compact Hausdorff space X such that Re B is closed in the norm of uniform convergence, then $B = C(X)$; in particular, the theorem holds if Re $B = C_r(X)$. A generalization of this fact by Sidney and Stout [10] says that if K is a closed subset of X and Re B_K is uniformly closed, then $B_K = C(K)$. Bernard [1, 2, 3] has provided extensions in a different direction. He shows that a Banach algebra $B \subseteqq C(X)$, with any norm, for which Re B is closed under uniform convergence must equal $C(X)$. Further, he gives some sufficient conditions on two Banach algebras $B_1 \subseteqq B_2 \subseteqq C(X)$, with Re $B_1 =$ Re B_2, to conclude that $B_1 = B_2$.

The theorems in this paper have previously been announced in Abstracts 71T-B10 and 71T-B94 of the *Notices of the American Mathematical Society*, 18 (1971). The author wishes to express his appreciation to Professor John Wermer for his many helpful suggestions and conversations during the research which led to these results.

II. Algebras on Γ with the same real parts.

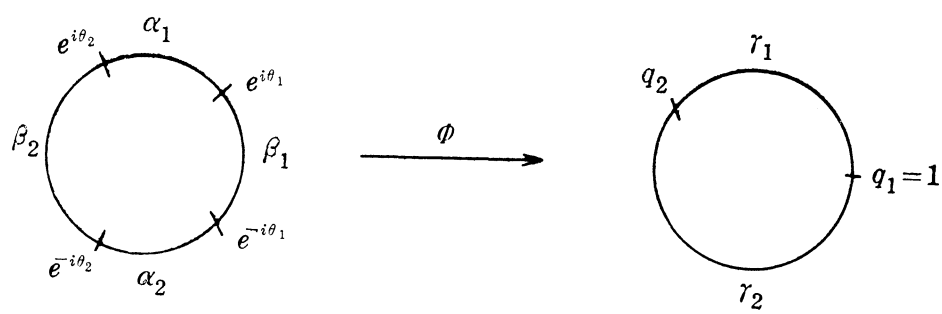
THEOREM 1. *Let B be a uniform algebra on Γ with Re $B =$ Re A. Then there exists a homeomorphism Φ of Γ onto Γ so that $B = A(\Phi)$.*

Proof. (1). Since $\cos(\theta)$ belongs to $\operatorname{Re} B$, there exists $v(\theta)$ so that $\psi(\theta) = \cos(\theta) + iv(\theta)$ belongs to B. $\operatorname{Re}\psi(\theta)$ is one-to-one and decreasing on $[0, \pi]$ and one-to-one increasing on $[-\pi, 0]$; hence $\boldsymbol{C} - \psi(\Gamma)$ has a bounded component, or else $\psi(\Gamma)$ defines a single curve traced twice, once in each direction. In the latter case, $\psi(\Gamma)$ is a Jordan arc; it then follows by Mergelyan's theorem that the function $\operatorname{Re} z$ is a uniform limit of polynomials on $\psi(\Gamma)$ and we can conclude that $\operatorname{Re}\psi(\theta) = \cos(\theta) \in B$. Similarly there exists $u(\theta)$ in $\operatorname{Re} B$ so that $\psi_1(\theta) = u(\theta) + i\sin(\theta)$ belongs to B; and either $\boldsymbol{C} - \psi_1(\Gamma)$ has bounded component or else $\sin(\theta) \in B$.

If both $\cos(\theta)$ and $\sin(\theta)$ are in B, then $B = C(\Gamma)$ and $\operatorname{Re} B \neq \operatorname{Re} A$. Thus at least one of $\boldsymbol{C} - \psi(\Gamma)$ and $\boldsymbol{C} - \psi_1(\Gamma)$ has a bounded component. We will assume that $\boldsymbol{C} - \psi(\Gamma)$ has a bounded component, which we call W. The remainder of the proof using ψ_1 involves only minor changes.

(2). The region W is bounded by two arcs on $\psi(\Gamma)$. Precisely, there are values θ_1 and θ_2 with $0 \leqq \theta_1 < \theta_2 \leqq \pi$ for which $\psi(\theta_i) = \psi(-\theta_i)$, $i = 1, 2$ and $\partial W = \psi([\theta_1, \theta_2]) \cup \psi([-\theta_2, -\theta_1])$. For θ not contained in $[\theta_1, \theta_2] \cup [-\theta_2, -\theta_1]$, $\operatorname{Re}\psi(\theta) < \operatorname{Re}\psi(\theta_2)$ or $\operatorname{Re}\psi(\theta) > \operatorname{Re}\psi(\theta_1)$.

Let τ be the Riemann map of W onto D; τ extends continuously to $\bar{W}$, mapping ∂W onto Γ in a one-to-one fashion. For convenience we may suppose $(\tau\circ\psi)(\theta_1) = 1$. We further extend τ by setting $\tau(z) = (\tau\circ\psi)(\theta_1)$ for z with $\operatorname{Re} z > \operatorname{Re}\psi(\theta_1)$ and $\tau(z) = (\tau\circ\psi)(\theta_2)$ for z with $\operatorname{Re} z < \operatorname{Re}\psi(\theta_2)$. Then, letting K denote $\psi(\Gamma)$ together with all its bounded complementary components, τ is continuous on K and is analytic on its interior. Appealing again to Mergelyn's theorem, we conclude that τ is the uniform limit on K of polynomials. Hence $\tau\circ\psi$ belongs to B. Put $\Phi = \tau\circ\psi$. Then Φ maps Γ onto Γ in the following manner: (see figure)



FIGURE

Φ takes the open arcs α_1 and α_2 homeomorphically onto γ_1 and γ_2, and the closed arcs β_1 and β_2 onto the points $q_1 = \Phi(\theta_1) = 1$, $q_2 = \Phi(\theta_2)$ respectively.

(3) For each measure ν on Γ we define a measure ν^* on Γ by $\nu^*(E) = \nu(\Phi^{-1}(E))$, and for g in B and λ on $\gamma_1 \cup \gamma_2$ we put $g^*(\lambda) = g(\Phi^{-1}(\lambda))$. Let μ be a measure on Γ with $\mu \perp B$ and $\mu^* \neq 0$. Such measures must exist as otherwise every $\mu \perp B$ would be the zero measure on $\alpha_1 \cup \alpha_2$. Then for every closed subarc E of $\alpha_1 \cup \alpha_2$ of positive linear measure, it would follow that $B|_E = C(E)$. Hence $\operatorname{Re} A|_E = C_R(E)$ which is impossible.

If g belongs to B and $|z| \neq 1$, define

$$T(g, z) = \int_{|\lambda|=1} \frac{(gd\mu)^*}{\lambda - z}(\lambda) .$$

The function T is analytic on D and also on $\{z: |z| > 1\}$. We claim that $T(g, z) = 0$ if $|z| > 1$. For, on that set,

$$T(g, z) = \sum_{n=0}^{\infty} \frac{-1}{z^{n+1}} \int_{|\lambda|=1} \lambda^n (gd\mu)^*(\lambda) .$$

The term for $n = 0$ is $\int_{|\lambda|=1} (gd\mu)^* = \int_\Gamma gd\mu = 0.$

For each $n > 1$,

$$\begin{aligned}
\int_{|\lambda|=1} \lambda^n (gd\mu)^* &= \int_{\gamma_1 \cup \gamma_2} \lambda^n g^*(\lambda) d\mu^*(\lambda) \\
&\quad + q_1^n \int_{\beta_1} gd\mu + q_2^n \int_{\beta_2} gd\mu \\
&= \int_{\alpha_1 \cup \alpha_2} \Phi^n(t) g(t) d\mu(t) \\
&\quad + \int_{\beta_1} q_1^n g(t) d\mu(t) + \int_{\beta_2} q_2^n g(t) d\mu(t) \\
&= \int_{|t|=1} \Phi^n(t) g(t) d\mu(t) \\
&= 0, \text{ since } \Phi^n g \text{ belongs to } B.
\end{aligned}$$

(4) Let ν be any measure on Γ. Put $h(\theta) =$ the ν-measure of the arc $\widehat{1e^{i\theta}}$. At any point $\lambda_0 = e^{i\theta_0}$ where $dh/d\theta(\theta_0)$ exists, we set $d\nu/d\lambda(\lambda_0) = dh/d\theta(\theta_0)$. Then for almost all values of λ_0 on Γ, we have

$$\lim_{\substack{re^{i\theta} \to \lambda_0 \\ r<1}} \left[\int_{|\lambda|=1} \frac{d\nu(\lambda)}{\lambda - re^{i\theta}} - \int_{|\lambda|=1} \frac{d\nu(\lambda)}{\lambda - \frac{1}{r} e^{i\theta}} \right] = 2\pi i \frac{d\nu}{d\lambda}(\lambda_0) .$$

The limit is taken along non-tangential curves lying within the unit disk. The statement follows from Fatou's theorem by rewriting the terms inside the brackets as a Poisson integral.

By applying this equality to the measures $(gd\mu)^*$ and using that $T(g, z) = 0$ for $|z| > 1$ we conclude that, for almost all λ_0 on $\gamma_1 \cup \gamma_2$,

$$\begin{aligned}\lim_{\substack{z\to\lambda_0 \\ |z|<1}} T(g, z) &= \lim_{\substack{z\to\lambda_0 \\ |z|<1}} \int_{|\lambda|=1} \frac{(gd\mu)^*}{\lambda - z}(\lambda) \\ &= 2\pi i g^*(\lambda_0)\frac{d\mu^*}{d\lambda}(\lambda_0) .\end{aligned}$$

The function $T(1, z)$ is the Cauchy transform of the nonzero measure μ^*. Since the Cauchy transform of a measure is zero almost everywhere-$dxdy$ if and only if the measure itself is zero[1], and since $T(1, z)$ vanishes identically for $|z| > 1$ it follows that $T(1, z)$ cannot vanish identically on D. Also, $T(1, z)$ has the boundary value $2\pi i(d\mu^*/d\lambda)(\lambda_0)$ for almost all λ_0 on $\gamma_1 \cup \gamma_2$. (Here boundary value means the limiting value along non-tangential curves approaching λ_0 from inside the unit disk). It is a theorem of Privalov and Lusin[2] that a function which is analytic on D and which has the non-tangential limiting value zero on a set of positive measure on Γ must vanish identically on D. Applying this result to $T(1, z)$ we can conclude that $d\mu^*/d\lambda(\lambda_0) \neq 0$ for almost all λ_0 on $\gamma_1 \cup \gamma_2$.

For each g contained in B, $T(g, z)$ is analytic on D and has the boundary values $2\pi i g^*(\lambda_0)d\mu^*/d\lambda(\lambda_0)$ almost everywhere on $\gamma_1 \cup \gamma_2$. Consequently, the function $G(z) = T(g, z)/T(1, z)$ is meromorphic on D and has the boundary values $g^*(\lambda_0)$ almost everywhere on $\gamma_1 \cup \gamma_2$.

(5) Now we show that each function $G(z)$ obtained in the manner just described is analytic on D. To this end, take z_0 in D so that $T(1, z_0) \neq 0$. The functional S defined on B by $S(g) = G(z_0)$ is linear. For any two functions g_1 and g_2 belonging to B, the function $t(z) = T(1, z)T(g_1g_2, z) - T(g_1, z)T(g_2, z)$ is analytic on D and has zero boundary values almost everywhere. Hence, by the theorem Privalov and Lusin cited in the last paragraph, $t \equiv 0$ and the meromorphic functions

$$\frac{T(g_1g_2, z)}{T(1, z)} \text{ and } \frac{T(g_1, z)}{T(1, z)}\frac{T(g_2, z)}{T(1, z)}$$

are identically equal whenever defined. Thus S is a multiplicative linear functional on the commutative Banach algebra B and as such has norm one. That is, $|G(z_0)| \leqq ||g||$ for every z_0 with $T(1, z_0) \neq 0$. As the zeros of $T(1, z)$ form a discrete set, all the singularities of G are removable and G extends analytically to D. Furthermore, G is bounded on D with $|G(z)| \leqq ||g||$ for all z contained in D.

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[1] See Browder [**4**] or Gamelin [**5**] for details.

[2] See Goluzin [**6**], p. 428, or Privalov [**9**].

(6). The next step is to show G belongs to A. G can be defined from g^* using the Poisson integral formula. A well-known property of that formula says that G has the nontangential boundary value $g^*(\lambda_0)$ at every point of continuity of g^*. Thus G has the boundary values g^* everywhere on $\gamma_1 \cup \gamma_2$. The following result may be used to prove that G has continuous boundary values everywhere on Γ.

THEOREM.[3] *If $f(z) \to a$ as $z \to \infty$ along a straight line and $f(z) \to b$ as $z \to \infty$ along another straight line, and $f(z)$ is regular and bounded in the angle between, then $a = b$ and $f(z) \to a$ uniformly in the angle.*

In the present case, the four limits $\lim_{\lambda \to q_i} g^*(\lambda)$, for $i, j = 1, 2$, all exist, because of the continuity of the function g. By considering neighborhoods of q_1 and q_2 in $\bar{D}$, mapping them to the upper half plane sending q_1 and q_2 in turn to the point at ∞, we can conclude that $\lim_{z \to q_1} G(z)$ and $\lim_{z \to q_2} G(z)$ both exist and that $G \in A$. In fact, we must have that $\lim_{z \to q_1} G(z) = g(\theta_1) = g(-\theta_1)$ and $\lim_{z \to q_2} G(z) = g(_2) = g(-\theta_2)$, for every function g belonging to B. But as B separates points on Γ, this is impossible unless $\theta_1 = 0$, $\theta_2 = \pi$. There, Φ is a homeomorphism of Γ onto Γ.

(7). Since Φ belongs to B, $A(\Phi) \subseteqq B$ or equivalently $A \subseteqq B(\Phi^{-1})$. As $\operatorname{Re} B = \operatorname{Re} A \neq C_R(\Gamma)$, then $B(\Phi^{-1}) \neq C(\Gamma)$. Hence by Wermer's Maximality Theorem, $B(\Phi^{-1}) = A$ or $B = A(\Phi)$.

THEOREM 2. *Let Φ be a homeomorphism of Γ onto Γ satisfying $\operatorname{Re} A(\Phi) = \operatorname{Re} A$. Then Φ is absolutely continuous.*

Proof. We also have that $\operatorname{Re} A(\Phi^{-1}) = \operatorname{Re} A$. Let m denote Lebesgue measure on Γ. We claim that if K is a closed set with $m(K) = 0$, then $m(\Phi(K)) = 0$. For, let $g \in C(\Phi(K))$. Then $g \circ \Phi|_K \in C(K)$. Since we assume that K is closed with $m(K) = 0$, it follows that $A|_K = C(K)$[4], so that there exists G belonging to A with $G|_K = g \circ \Phi|_K$. Thus $g|_{\Phi(K)} = G \circ \Phi^{-1}|_{\Phi(K)}$. In particular, if $g \in C_R(\Phi(K))$, then $g \in \operatorname{Re} A(\Phi^{-1})|_{\Phi(K)} = \operatorname{Re} A|_{\Phi(K)}$. Hence $\operatorname{Re} A|_{\Phi(K)}$ is closed in the uniform norm. This implies that $A|_{\Phi(K)} = C(\Phi)K))$[5]. If $m(\Phi(K)) > 0$, A would necessarily contain a nonzero function which vanished on a set of positive measure on Γ. This is impossible since the functions in A satisfy the Jensen inequality $\log|f(0)| \leqq 1/2\pi \int_\Gamma \log|f| d\theta$. Hence we must have $m(\Phi(K)) = 0$.

[3] Titchmarsh [**11**], 5,64.
[4] This is a theorem of Rudin and Carleson. See [**5**] for details.
[5] This result is due to Sidney and Stout [**10**].

Now if we define a measure μ on Γ by $\mu(E) = m(\Phi(E))$ we have that μ is absolutely continuous with respect to m. To see this, let $m(E) = 0$. There exists an F_σ set $F \subseteqq E$ such that $\mu(E \backslash F) = 0$. We can write $F = \bigcup_{n=1}^{\infty} K_n$ where K_n is closed and $K_n \subseteqq K_{n+1}$ for each n. From the last paragraph, it follows that $\mu(K_n) = 0$. Hence $\mu(F) = 0$ and $m(\phi(E)) = \mu(E) = 0$.

For an arc $\widehat{\alpha\beta}$, $\mu(\widehat{\alpha\beta}) = |\Phi(\beta) - \Phi(\alpha)|$. The absolute continuity of μ implies that for $\varepsilon > 0$, there is a $\delta > 0$ so that $m(E) < \delta \Rightarrow \mu(E) < \varepsilon$. Hence if $\widehat{\alpha_i\beta_i}$, $1 \leqq i \leqq N$, are disjoint arcs with $\sum_{i=1}^{N} m(\widehat{\alpha_i\beta_i}) < \delta$ then $\sum_{i=1}^{N} |\Phi(\beta_i) - \Phi(\alpha_i)| < \varepsilon$. Therefore Φ is absolutely continuous.

REMARKS. (1) Theorem 1 cannot be extended freely: there exist examples of sets X, and two uniform algebras A and B defined on X satisfying $\operatorname{Re} A = \operatorname{Re} B$, but for which no homeomorphism Φ yields $B = A(\Phi)$.

(2) If, on Γ, $B = \bar{A}$, the set of complex conjugates of the functions in A, the map $z \to \bar{z}$ is a suitable choice for Φ.

III. Sufficient conditions on Φ to conclude $\operatorname{Re} A(\Phi) = \operatorname{Re} A$.

DEFINITION. Let $\operatorname{Re} A^\dagger = \{f : f$ is continuous on $\boldsymbol{R}$, with period 2π, and so that the function F, defined on Γ by $F(e^{i\theta}) = f(\theta)$, belongs to $\operatorname{Re} A\}$.

DEFINITION. For u in $\operatorname{Re} A$, define $|||u||| = ||u|| + ||\tilde{u}||$ where $||\cdot||$ is the uniform norm on Γ, $u + i\tilde{u}$ belongs to A and $\tilde{u}(0) = 0$. For f in $(\operatorname{Re} A)^\dagger$, $\tilde{f}$ denotes the function in $(\operatorname{Re} A)^\dagger$ such that $G(e^{i\theta}) = f(\theta) + i\tilde{f}(\theta)$ belongs to A and $\operatorname{Im} G(0) = 0$. We put $||f|| = \max_{\boldsymbol{R}} |f|$ and

$$|||f||| = ||f|| + ||\tilde{f}|| .$$

With $|||\cdot|||$ as a norm, $\operatorname{Re} A$ and $(\operatorname{Re} A)^\dagger$ have the structure of real Banach spaces in which smooth functions are dense.

LEMMA 1. *Suppose ϕ is a homeomorphism of $[-\pi, \pi]$ onto $[\alpha, 2\pi + \alpha]$, with inverse ψ, and satisfying*

(i) $\phi(0) = 0$,

(ii) *ϕ is of class C^2, $\phi'(-\pi) = \phi'(\pi)$, $\psi'(\alpha) = \psi'(\alpha + 2\pi)$,*

(iii) *$|\phi'(t)| \geqq \gamma > 0$, $|\psi'(t)| \geqq \delta > 0$ and $|\psi''(t)| \leqq M$ for all t in $[-\pi, \pi]$.*

Then, for a smooth function f in $(\operatorname{Re} A)^\dagger$, $f \circ \phi$ which is defined on $[-\pi, \pi]$, if extended periodically to $\boldsymbol{R}$, lies in $(\operatorname{Re} A)^\dagger$. There exists a constant $C > 0$, depending only on γ, δ and M, so that

$$|\widetilde{f \circ \phi}(0) - \tilde{f}(0)| \leqq C ||f|| .$$

Consequently,

$$|\widetilde{f\circ\phi}(0)| \leqq (C+1)|||f|||\ .$$

Proof. We can assume, with no loss of generality, that $\phi'(t) \geqq \gamma$ on $[-\pi, \pi]$. The proof if ϕ is a decreasing function is almost identical. Our assumption means that $\phi(-\pi) = \alpha < 0$.

Let f be a smooth function in $(\operatorname{Re} A)^{\dagger}$. As $\phi(-\pi) + 2\pi = \phi(\pi)$, then $f\circ\phi$ can be extended to $\boldsymbol{R}$, continuously with period 2π. Since f and ϕ are smooth and $\phi'(-\pi) = \phi'(\pi)$, $f\circ\phi$ is smooth and thus belongs to $(\operatorname{Re} A)^{\dagger}$.

Assume first that $f(0) = 0$. The integral formula for evaluating harmonic conjugates gives[6]

$$\begin{aligned} \tilde{f}(0) &= -\frac{1}{2\pi}\int_0^{\pi} \frac{f(t) - f(-t)}{\tan\frac{t}{2}}dt \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(t)}{\tan\frac{t}{2}}dt \qquad (1) \\ &= -\frac{1}{2\pi}\int_{\alpha}^{2\pi+\alpha} \frac{f(t)}{\tan\frac{t}{2}}dt\ . \end{aligned}$$

$$\begin{aligned} \text{Also } \widetilde{f\circ\phi}(0) &= -\frac{1}{2\pi}\int_0^{\pi} \frac{(f\circ\phi)(t) - (f\circ\phi)(-t)}{\tan\frac{t}{2}}dt \\ &= -\frac{1}{2\pi}\int_0^{\pi} \frac{(f\circ\phi)(t)}{\tan\frac{t}{2}}dt - \frac{1}{2\pi}\int_{-\pi}^{0} \frac{(f\circ\phi)(t)}{\tan\frac{t}{2}}dt \\ &= -\frac{1}{2\pi}\int_0^{2\pi+\alpha} \frac{f(u)}{\tan\frac{\psi(u)}{2}}\psi'(u)du \qquad (2) \\ &\quad -\frac{1}{2\pi}\int_{\alpha}^{0} \frac{f(u)}{\tan\frac{\psi(u)}{2}}\psi'(u)du \\ &= -\frac{1}{2\pi}\int_{\alpha}^{2\pi+\alpha} \frac{f(u)\psi'(u)}{\tan\frac{\psi(u)}{2}}du\ . \end{aligned}$$

All these integrals are absolutely convergent because of the smoothness of f and ϕ, and because $f(0) = \phi(0) = 0$. We use the fact that $x \cot x$ has a series expansion

[6] Zygmund [**13**] vol. I, p. 131.

$$x \cot x = \sum_{n=0}^{\infty} A_n x^n$$

which converges for $|x| < \pi$. Hence

$$\frac{1}{\tan x} = \frac{1}{x} + a(2x) \tag{3}$$

where $a(t)$ is real analytic for $|t| < 2\pi$. Substitution of (3) into (1) and (2) yields

$$\tilde{f}(0) = -\frac{1}{2\pi}\int_{\alpha}^{2\pi+\alpha} f(t)\left[\frac{1}{\frac{t}{2}} + a(t)\right]dt$$

and

$$\widetilde{f\circ\phi}(0) = -\frac{1}{2\pi}\int_{\alpha}^{2\pi+\alpha} f(t)\left[\frac{1}{\frac{\psi(t)}{2}} + a(\psi(t))\right]\psi'(t)dt\ .$$

Since $-2\pi < \alpha < 0$, $a(t)$ is continuous on $[\alpha, 2\pi + \alpha]$, and since $-\pi \leqq \psi(t) \leqq \pi$, these two integrals are absolutely convergent. Subtracting them, we get

$$\begin{aligned} |\widetilde{f\circ\phi}(0) - \tilde{f}(0)| \leqq \frac{1}{2\pi}\|f\|\int_{\alpha}^{2\pi+\alpha}\Big\{2\Big|\frac{\psi'(t)}{\psi(t)} - \frac{1}{t}\Big| \\ + |a(\psi(t))|\psi'(t) + |a(t)|\Big\}dt\ . \end{aligned} \tag{4}$$

We estimate each term in the integrand. For the first term, we write $\psi(t) = t\psi_1(t)$. ψ_1 is smooth on $[\alpha, 2\pi + \alpha]$ and $\psi_1(0) \neq 0$. For each t in $[\alpha, 2\pi + \alpha]$,

$$\begin{aligned} \psi(t) &= \int_0^1 \frac{d}{ds}\psi(st)ds \\ &= \int_0^1 t\psi'(st)ds \\ &= t\int_0^1 \psi'(st)ds\ . \end{aligned}$$

Thus

$$\psi_1(t) = \int_0^1 \psi'(st)ds \geqq \delta\ .$$

Also,

$$\psi_1'(t) = \int_0^1 \frac{d}{dt}\psi'(st)ds$$
$$= \int_0^1 s\psi''(st)ds$$

(5)
$$\Longrightarrow |\psi_1'(t)| \leqq M\int_0^1 sds = \frac{M}{2}$$
$$\Longrightarrow \left|\frac{\psi'(t)}{\psi(t)} - \frac{1}{t}\right| = \left|\frac{\psi_1'(t)}{\psi_1(t)}\right| \leqq \frac{M}{2\delta}$$
$$\Longrightarrow \int_\alpha^{2\pi+\alpha}\left|\frac{\psi'(t)}{\psi(t)} - \frac{1}{t}\right|dt \leqq \frac{M\pi}{\delta} .$$

For the second term in (4),

(6)
$$\int_\alpha^{2\pi+\alpha}\Big||a(\psi(t))|\psi'(t)dt = \int_{-\pi}^{\pi}|a(t)|dt .$$

For the last term, we note that

$$2\pi + \alpha = \phi(\pi)$$
$$= \phi(\pi) - \phi(0)$$
$$= \pi\phi'(r) \text{ for some } r \in [0, \pi] .$$
$$\Longrightarrow 2\pi + \alpha \geqq \gamma\pi$$
$$\Longrightarrow \alpha \geqq \gamma\pi - 2\pi .$$

Similarly,

$$-\alpha = -\phi(-\pi)$$
$$= \phi(0) - \phi(-\pi)$$
$$= \pi\phi'(r_1) \text{ for some } r_1 \in [-\pi, 0] .$$
$$\Longrightarrow -\alpha \geqq \gamma\pi$$
$$\Longrightarrow 2\pi + \alpha \leqq 2\pi - \gamma\pi .$$

Thus,

(7)
$$\int_\alpha^{2\pi+\alpha}|a(t)|dt \leqq \int_{-2\pi+\gamma\pi}^{2\pi-\gamma\pi}|a(t)|dt .$$

The inequalities (4), (5), (6) and (7) imply

(8)
$$|\widetilde{f\circ\phi}(0) - \tilde{f}(0)| \leqq ||f||\frac{1}{2\pi}\Big[2\pi\frac{M}{\delta} + \int_{-\pi}^{\pi}|a(t)|dt + \int_{-2\pi+\gamma\pi}^{2\pi-\gamma\pi}|a(t)|dt\Big] .$$

Put $C/2$ equal to the coefficient of $||f||$ in (8). Note that C depends only on M, δ and γ. That inequality applies to smooth functions in

$(\operatorname{Re} A)^\dagger$ vanishing at $x = 0$. Now, let f be any smooth function in $(\operatorname{Re} A)^\dagger$. We can apply (8) to $f - f(0)$. Since $\tilde{f}(0) = \widetilde{(f - f(0))}(0)$ and $\widetilde{f\circ\phi}(0) = \widetilde{[(f - f(0))\circ\phi]}(0)$, we conclude that

$$\begin{aligned} |\widetilde{f\circ\phi}(0) - \tilde{f}(0)| &\leqq \|f - f(0)\|\frac{C}{2} \\ \Longrightarrow |\widetilde{f\circ\phi}(0) - \tilde{f}(0)| &\leqq C\|f\| \\ \Longrightarrow |\widetilde{f\circ\phi}(0)| &\leqq C\|f\| + |\tilde{f}(0)| \\ &\leqq C\|f\| + \|\tilde{f}\| \\ &\leqq (C+1)|||f||| . \end{aligned}$$

DEFINITION. For $\beta > 0$ define C_β by
$C_\beta = \{\Phi\colon \Phi$ is a homeomorphism of Γ onto Γ of class C^2 satisfying:

$$\begin{aligned} \beta \leqq \left|\frac{d}{dt}\Phi(e^{it})\right| &\leqq \frac{1}{\beta} \\ \beta \leqq \left|\frac{d}{dt}\Phi^{-1}(e^{it})\right| &\leqq \frac{1}{\beta} \\ \left|\frac{d^2}{dt^2}\Phi(e^{it})\right| &\leqq \frac{1}{\beta} \\ \left|\frac{d^2}{dt^2}\Phi^{-1}(e^{it})\right| &\leqq \frac{1}{\beta}\Big\} . \end{aligned}$$

LEMMA 2. *There exists $K_\beta > 0$, depending only on β such that for all Φ belonging to C_β with $\Phi(1) = 1$ and for all smooth functions F in Re A, we have*

$$|\widetilde{F(\Phi)}(1)| \leqq K_\beta |||F||| .$$

Proof. Define ϕ on $[-\pi, \pi]$ by $\Phi(e^{it}) = e^{i\phi(t)}$. Then ϕ is of class C^2 and $\phi(0) = 0$. If $\Psi = \Phi^{-1}$ and $\Psi(e^{iu}) = e^{i\psi(u)}$ for $u \in \phi([-\pi, \pi])$, then ψ is the inverse of ϕ. Furthermore

$$|\phi'(t)| = \left|\frac{d}{dt}\Phi(e^{it})\right| ,$$

$$|\psi'(t)| = \left|\frac{d}{dt}\Psi(e^{it})\right| ,$$

and

$$|\psi''(t)| = \left|\frac{d^2}{dt^2}\Psi(e^{it})\right| + \left|\frac{d}{dt}\Psi(^{it})\right|^2 .$$

If we set $\gamma = \beta, \delta = \beta$ and $M = 1/\beta + 1/\beta^2$, then ϕ satisfies the hypotheses of Lemma 1, and if we apply the lemma to ϕ, the constant C obtained in (8) depends only on β.

If F is a smooth function in $\operatorname{Re} A$ and if f is defined on $\boldsymbol{R}$ by $f(\theta) = F(e^{i\theta})$, then f is a smooth function in $(\operatorname{Re} A)^{\dagger}$ with $|||F||| = |||f|||$ and $\widetilde{f\circ\phi}(0) = \widetilde{F\circ\Phi}(1)$. Hence (9) yields that

$$|\widetilde{F(\Phi)}(1)| \leqq (C+1)|||F||| .$$

Therefore the lemma is proved by defining $K_\beta = C + 1$.

LEMMA 3. *If Φ belongs to C_β, then*

$$|\widetilde{F(\Phi)}(1)| \leqq K_\beta |||F|||$$

for all smooth functions F contained in Re A.

Proof. Define λ by $\Phi(1) = e^{i\lambda}$ and set $\Phi_1 = e^{-i\lambda}\Phi$. Note that $\Phi_1(1) = 1$. If $\Psi = \Phi^{-1}$ and $\Psi_1 = \Phi_1^{-1}$ then $\Psi_1(e^{it}) = \Psi(e^{i(t+\lambda)})$. We check that $\Phi_1 \in C_\beta$ by noting that

$$\left|\frac{d}{dt}\Phi_1(e^{it})\right| = \left|\frac{d}{dt}\Phi(e^{it})\right|$$
$$\left|\frac{d^2}{dt^2}\Phi_1(e^{it})\right| = \left|\frac{d^2}{dt^2}\Phi(e^{it})\right|$$
$$\left|\frac{d}{du}\Psi_1(e^{iu})\right| = \left|\frac{d}{du}\Psi(e^{i(u+\lambda)})\right|$$
$$\left|\frac{d^2}{du^2}\Psi_1(e^{iu})\right| = \left|\frac{d^2}{du^2}\Psi(e^{i(u+\lambda)})\right| .$$

Thus, by Lemma 2, $|\widetilde{G(\Phi_1)}(1)| \leqq K_\beta |||G|||$ for all smooth functions G belonging to Re A. Now, given F smooth in Re A, put $G(e^{i\theta}) = F(e^{i(\theta+\lambda)})$. Then $G(\Phi_1) = F(\Phi)$, $\widetilde{G(\Phi_1)} = \widetilde{F(\Phi)}$ and $|||F||| = |||G|||$. Hence

$$|\widetilde{F(\Phi)}(1)| \leqq K_\beta |||F||| .$$

LEMMA 4. *For $\Phi \in C_\beta$ and ζ with $|\zeta| = 1$,*

$$|\widetilde{F(\Phi)}(\zeta)| \leqq K_\beta |||F|||$$

for all smooth functions F in Re A.

Proof. Fix $\zeta = e^{i\lambda}$ on Γ. Define Φ^* by $\Phi^*(e^{i\theta}) = \Phi(e^{i(\theta+\lambda)})$. By comparing the derivatives of Φ, Φ^* and their respective inverses in a manner similar to that in Lemma 3, we can show that $\Phi^* \in C_\beta$. Therefore, for a smooth function F in Re A,

$$|\widetilde{F(\Phi^*)}(1)| \leqq K_\beta |||F||| .$$

But $\widetilde{F(\Phi)}(\zeta) = \widetilde{F(\Phi^*)}(1)$. Hence

$$|\widetilde{F(\Phi)}(\zeta)| \leqq K_\beta |||F||| .$$

LEMMA 5. *If Φ belongs to C_β for some $\beta > 0$, then* $\mathrm{Re}\, A(\Phi) \subseteqq \mathrm{Re}\, A$.

Proof. From Lemma 4, it follows that if F is a smooth function in $\mathrm{Re}\, A$,

$$||\widetilde{F(\Phi)} \leqq K_\beta |||F||| .$$

Hence

$$\begin{aligned} |||F(\Phi)||| &= ||F(\Phi)|| + ||\widetilde{F(\Phi)}|| \\ &\leqq ||F|| + K_\beta |||F||| \leqq (K_\beta + 1)|||F||| . \end{aligned}$$

Defining T by $T(F) = F(\Phi)$, we see that T is a bounded linear transformation defined on smooth functions in $\mathrm{Re}\, A$ and mapping into $\mathrm{Re}\, A$. The domain of T is dense in $\mathrm{Re}\, A$, so that T has a norm preserving extension which maps $\mathrm{Re}\, A \to \mathrm{Re}\, A$; we also denote this extension by T. If u belongs to $\mathrm{Re}\, A$, there are smooth function F_n in $\mathrm{Re}\, A$ with $|||F_n - u||| \to 0$, and therefore $|||F_n(\Phi) - Tu||| \to 0$. These imply that $||F_n - u|| \to 0$ and $||F_n(\Phi) - Tu|| \to 0$. Hence $Tu = u(\Phi)$. That is $u(\Phi)$ belongs to $\mathrm{Re}\, A$ for every u in $\mathrm{Re}\, A$.

THEOREM 3. *Let Φ belong to C_β for some $\beta > 0$. Then* $\mathrm{Re}\, A(\Phi) = \mathrm{Re}\, A$.

Proof. We apply Lemma 5 to Φ and Φ^{-1}, both of which are contained in C_β, to conclude $\mathrm{Re}\, A(\Phi) \subseteqq \mathrm{Re}\, A$ and $\mathrm{Re}\, A(\Phi^{-1}) \subseteqq \mathrm{Re}\, A$. Therefore $\mathrm{Re}\, A(\Phi) = \mathrm{Re}\, A$.

COROLLARY. *Let Φ be a homeomorphism of Γ onto Γ of class C^2 with $d/dt(\Phi(e^{it})) \neq 0$ for all t. Then $Re\, A(\Phi) = Re\, A$.*

REMARK. The assumption that $d/dt(\Phi(e^{it}) \neq 0$ cannot be eliminated, as there do exist examples of homeomorphisms Φ of Γ onto Γ which are of class C^∞, but for which $\mathrm{Re}\, A(\Phi) \neq \mathrm{Re}\, A$.

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