PACIFIC JOURNAL OF MATHEMATICS
Vol. 46, No. 2, 1973

STRONG QUASI-CONVEXITY

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Quasi-convexity and strong quasi-convexity are conditions on integrands of multiple integrals which are associated with lower semi-continuity. Equivalent definitions are given which, at least for parametric integrands, are more geometric in character than the original definitions.

About twenty years ago Morrey derived a necessary condition, strong quasi-convexity, for the lower semi-continuity of certain multiple integrals [2, 3, p. 113]. Morrey also obtained conditions under which strong quasi-convexity is equivalent to quasi-convexity, a differentialless analogue of the Legendre-Hadamard condition. We will use an idea of Cesari to rewrite Morrey's condition to make it more closely resemble the usual definition of convexity, and will then extend Morrey's results relating 'convexity in the Jacobians' with quasi-convexity [3, p. 124].

Let $N \geqq \nu$ be natural numbers, let $E = R^N$ and let

$$P = L(R^\nu, E) \approx E^\nu .$$

Let f be real-valued and continuous on $R^\nu \times E \times P$ and let G and H always denote bounded open subsets of R^ν.

If $z \in C(G, E)$ and if z is Lipschitzian let

$$I_G(z) = \int_G f(x, z(x), z'(x))dx .$$

If $I_G(z) \leqq I_G(w)$ whenever w is Lipschitzian and $\|w - z\|$ is sufficiently small, then z furnishes I_G with a relative minimum. If I_G is lower semicontinuous for each G, then I is said to be lower semicontinuous.

If A is an open set, or the closure of an open set in R^ν, let $|A|$ be its ν-dim Lebesgue measure.

Part (ii) of the following theorem is weaker than [3, p. 113] and the proof of (i) is, except for a trivial modification, the same as that for part (ii).

THEOREM 1. *Let ζ be Lipschitzian on G into E and vanish on ∂G.*

(i) *If $z \in C'(H, E)$ furnishes I_H with a relative minimum and if $y \in G \cap H$ then*

$$\int_G f(y, z(y), z'(y) + \zeta'(u))du \geqq f(y, z(y), z'(y)) \mid G \mid .$$

(ii) *If I is lower semicontinuous, if $y \in R^\nu$, $w \in E$ and $p \in P$, then*

$$\int_G f(y, w, p + \zeta'(u))du \geqq f(y, w, p) \mid G \mid .$$

We notice that the proof makes use of the fact that almost all points of ∂G are points of density for Comp G so that $\zeta'(x) = 0$ for almost all x in ∂G. This fact is also used in the proof of Theorem 2.

By definition, [3, p. 114], f is strongly quasi-convex if for each bounded open set G in R^ν, for each $y \in G$, $w \in E$ and $p \in P$,

$$\int_G f(y, w, p + \zeta'(x))dx \geqq f(y, w, p) \mid G \mid$$

whenever ζ is Lipschitzian on G into E and vanishes on ∂G.

Let $\mathfrak{Z}$ be the collection of all quasilinear functions on R^ν into E which have compact support. If $\zeta \in \mathfrak{Z}$ let $\Delta(\zeta)$ be the collection of simplexes on each of which ζ is linear and whose union is K_ζ = support ζ. If $\delta \in \Delta(\zeta)$ let $\zeta'(\delta) = \zeta'(u)$ for any $u \in \operatorname{Int} \delta$.

THEOREM 2. *A necessary and sufficient condition that f be strongly quasi-convex is that*

$$f(y, w, p) \leqq \sum_{\delta \in \Delta(\zeta)} \frac{\mid \delta \mid}{\mid K_\zeta \mid} f(y, w, p + \zeta'(\delta))$$

for each $\zeta \in \mathfrak{Z}$, $y \in R^\nu$, $w \in E$ and $p \in P$.

The necessity is an immediate consequence of the definitions, while the sufficiency results from the fact that $\mathfrak{Z}$ is big enough to approximate all Lipschitzian functions with compact support.

LEMMA 1. *Let ϕ be quasilinear on a polyhedral region $\pi \subset R^\nu$ into R^ν. If ϕ^i vanishes on $\partial\pi$ for some $i = 1, \cdots, \nu$ then*

$$\int_\pi \frac{d\phi}{du} du = 0 .$$

The integral is equal to the integral of the topological index, which is identically zero since $\phi(\partial\pi)$ is contained in a hyperplane.

Let Λ: $E^\nu \to \wedge^\nu E$ be defined by $\Lambda(p) = p_1 \wedge \cdots \wedge p_\nu$ if

$$p_1, \cdots, p_\nu \in E .$$

LEMMA 2. *Let $p \in P$ and suppose that $\zeta \in \mathfrak{Z}$. Then*

$$\frac{1}{|K_\zeta|} \sum_{\delta \in \Delta(\zeta)} \Lambda(p + \zeta'(\delta)) \, |\delta| = \Lambda(p) \, .$$

Proof. Let $1 \leqq \lambda_1 < \cdots < \lambda_\nu \leqq N$. If $\alpha \in \wedge^\nu E$ let $\alpha^\lambda = \alpha^{\lambda_1 \cdots \lambda_\nu}$. If ϕ maps a set S into E let $\phi^\lambda = (\phi^{\lambda_1}, \cdots, \phi^{\lambda_\nu})$. If

$$x = p_1 u^1 + \cdots + p_\nu u^\nu$$

then $dx/du = p$. Thus

$$\begin{aligned} \sum_{\delta \in \Delta(\zeta)} \Lambda(p + \zeta'(\delta))^\lambda \, |\delta| &= \int_{K_\zeta} \frac{d(x^\lambda + \zeta^\lambda)}{du} \, du \\ &= \int_{K_\zeta} \frac{dx^\lambda}{du} \, du = \Lambda(p)^\lambda \, |K_\zeta| \end{aligned}$$

by Lemma 1 and a standard expansion of the determinant

$$\frac{d(x^\lambda + \zeta^\lambda)}{du} \, .$$

The lemma follows.

Let F be real-valued and continuous on $\wedge^\nu E$. Then F is said to be *almost convex*[1] if

$$F(\Lambda(p)) \leqq \sum_{\delta \in \Delta(\zeta)} \frac{|\delta|}{|K_\zeta|} F(\Lambda(p + \zeta'(\delta))$$

for each $p \in E^\nu$ and $\zeta \in \mathfrak{Z}$. If F is convex then, by Lemma 2,

$$\begin{aligned} F(\Lambda(p)) &= F\Big(\frac{1}{|K_\zeta|} \sum_{\delta \in \Delta(\zeta)} \Lambda(p + \zeta'(\delta)) \, |\delta|\Big) \\ &\leqq \sum_{\delta \in \Delta(\zeta)} \frac{|\delta|}{|K_\zeta|} F(\Lambda(p + \zeta'(\delta))) \end{aligned}$$

so that F is almost convex.

THEOREM 3. *Let F be real-valued and continuous on $\wedge^\nu E$. Then $F \circ \Lambda$ is strongly quasi-convex if and only if F is almost convex.*

Proof. Let F be almost convex. If $f = F \circ \Lambda$ and $\zeta \in \mathfrak{Z}$, then by Lemma 2

[1] This definition differs slightly from that in [**1**, p. 30].

$$\begin{aligned} f(p) &= F\Big(\frac{1}{|K_\zeta|}\sum_{\delta \in \Delta(\zeta)} \Lambda(p + \zeta'(\delta))\,|\delta|\Big) \\ &\leqq \sum_{\delta \in \Delta(\zeta)} \frac{|\delta|}{|K_\zeta|} F(\Lambda(p + \zeta'(\delta))) \\ &= \sum_{\delta \in \Delta(\zeta)} \frac{|\delta|}{|K_\zeta|} f(p + \zeta'(\delta)) \end{aligned}$$

so that f is strongly quasi-convex.

Now suppose f is strongly quasi-convex. A repetition of the above argument gives

$$\begin{aligned} F(\Lambda(p)) &= f(p) \\ &\leqq \sum_{\delta \in \Delta(\zeta)} \frac{|\delta|}{|K_\zeta|} f(p + \zeta'(\delta)) \\ &= \sum_{\delta \in \Delta(\zeta)} \frac{|\delta|}{|K_\zeta|} F(\Lambda(p + \zeta'(\delta))) \end{aligned}$$

so that F is almost convex.

To study the notion of quasi-convexity it is convenient to use the following notation:

$$\text{if} \quad p \in P,\ \lambda \in R_\nu = (R^\nu)' \quad \text{and} \quad \xi \in E\,,$$

let $p + \lambda\xi \in P$ be defined by

$$(p + \lambda\xi)^i_\alpha = p^i_\alpha + \lambda_\alpha \xi^i$$

$i = 1, \cdots, N$ and $\alpha = 1, \cdots, \nu$.

We say that f is *quasi-convex* [2, p. 112] if

(i) for fixed p and ξ, ϕ is convex over R_ν where

$$\phi(\lambda) = f(p + \lambda\xi)\,,$$

and

(ii) for fixed p and λ, ψ is convex over E where

$$\psi(\xi) = f(p + \lambda\xi)\,.$$

If f is strongly quasi-convex, then f is quasi-convex [2, p. 114].

THEOREM 4. *The following statements are equivalent*:

(a) *f is quasi-convex*,

(b) *if p and ξ are fixed and $\phi(\lambda) = f(p + \lambda\xi)$ then ϕ is convex, and*

(c) *if p and λ are fixed and $\psi(\xi) = f(p + \lambda\xi)$ then ψ is convex.*

Proof. Suppose that (b) holds. Let p and λ be fixed. Now let ξ and η be in E and $t \in (0, 1)$. Let $q = p + \lambda(t\xi + (1 - t)\eta)$, let

$\zeta = \eta - \xi$ and let $\phi(\mu) = f(q + \mu\zeta)$. Since ϕ is convex and

$$(1 - t)(t\lambda) + t(-(1 - t)\lambda) = 0 \,,$$

$$\begin{aligned} f(q) = \phi(0) &\leqq t\phi(-(1 - t)\lambda) + (1 - t)\phi(t\lambda) \\ &= tf(p + \lambda\xi) + (1 - t)f(p + \lambda\eta) \,. \end{aligned}$$

Hence (a) and (b) are equivalent. A dual argument shows that (a) and (c) are also equivalent.

Since quasi-convexity of f depends only upon the behavior of f relative to its third argument, we shall suppress the first two arguments in what follows.

THEOREM 5. *Let F be real-valued and continuous on $\wedge^\nu E$. Then a necessary and sufficient condition that $F \circ \Lambda$ be quasi-convex is that $F \mid \wedge^\nu T$ be convex for each* $(\nu + 1)$-dim *subspace T of E.*

Proof. Suppose that $f = F \circ \Lambda$ is quasi-convex and that T is a $(\nu + 1)$-dim subspace of E. Let ρ and σ be linearly independent elements of $\Lambda^\nu T$ and let $t \in (0, 1)$. Since $\dim T = \nu + 1$ ρ and σ are necessarily simple and there exist ν-planes R and S in T associated with ρ and σ. Furthermore, $R \cap S$ is a $(\nu - 1)$-plane. Let $p_1, \cdots, p_{\nu-1}$ be a basis for $R \cap S$. There exist r and s in T such that $\rho = p_1 \wedge \cdots \wedge p_{\nu-1} \wedge r$ and $\sigma = p_1 \wedge \cdots \wedge p_{\nu-11} \wedge s$. Let $\lambda \in R_\nu$, $\lambda = (0, \cdots, 0, 1)$ and let $\mu = -\lambda$. Let $\xi = (r - s)/2$ and let $p_\nu = r - \xi = s + \xi$. Then

$$\begin{aligned} t\rho + (1 - t)\sigma &= p_1 \wedge \cdots \wedge p_{\nu-1} \wedge (t(p_\nu + \xi) + (1 - t)(p_\nu - \xi)) \\ &= \Lambda(p + (t\lambda + (1 - t)\mu)\xi) \end{aligned}$$

so that

$$\begin{aligned} F(t\rho + (1 - t)\sigma) &= f(p + (t\lambda + (1 - t)\mu)\xi) \\ &\leqq tf(p + \lambda\xi) + (1 - t)f(p + \mu\xi) \\ &= tF(\rho) + (1 - t)F(\sigma) \,. \end{aligned}$$

To prove sufficiency let $p_1, \cdots, p_\nu$ and ξ be in E so that $p = (p_1, \cdots, p_\nu)$ can be interpreted as an element of P and let T be a $(\nu + 1)$-dim subspace of E containing all these points. Let λ and $\mu \in R_\nu$ and $t \in (0, 1)$. Since $\xi \wedge \xi = 0$,

$$\Lambda(p + (t\lambda + (1 - t)\mu)\xi) = t\Lambda(p + \lambda\xi) + (1 - t)\Lambda(p + \mu\xi) \,.$$

Hence

$$\begin{aligned} f(p + (t\lambda + (1 - t)\mu)\xi) &\leqq tF(\Lambda(p + \lambda\xi)) + (1 - t)F(\Lambda(p + \mu\xi)) \\ &= tf(p + \lambda\xi) + (1 - t)f(p + \mu\xi) \,. \end{aligned}$$

References

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Received April 28, 1972. This research was partially supported by US-AFOSR 69-1662.

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