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DECOMPOSITION OF PLANE CONVEX SETS, PART I.

RUTH SILVERMAN

The class K of plane convex bodies has the property that the sum of any two members of the class is again a member of the class. This paper characterizes $I(K)$, the subclass consisting of all indecomposable members of K, as the class of all triangles and line segments.

This was stated by Gale several years ago, but a proof was never published.

A compact convex set in n-dimensional real linear space R^n will be called a *convex body*. Let K_1 and K_2 be two convex bodies in R^n. Their *vector sum*, $K_1 + K_2$, is the convex body given by:

$$K_1 + K_2 = \{x + y \mid x \in K_1 \quad \text{and} \quad y \in K_2\} .$$

If $C = K_1 + K_2$, where C, K_1, and K_2 are convex bodies, then K_1 and K_2 are called *summands* of C. If $\lambda > 0$ then any translate of λC is said to be *homothetic* to C.

A convex body C is said to be written as a sum in a *nontrivial* way if neither summand is homothetic to C nor a one-pointed set. We remark that every convex body can be expressed trivially as a sum, for, if C is a convex subset of R^n, $x \in R^n$, and $\lambda \in (0, 1)$, then

$$C = (x + \lambda C) + (-x + (1 - \lambda)C) .$$

A convex body is said to be *decomposable* if it admits a summand which is neither homothetic to it nor a one-pointed set; otherwise, the set is called *indecomposable*. Thus a decomposable set is one that can be expressed as a sum of two convex sets in a nontrivial way. The results of this paper will be concerned with the decomposition of convex bodies.

This paper contains a proof that the only indecomposable plane convex bodies are triangles and line segments. This result was conjectured by Gale in 1954 [4], but a proof was never published, although the partial result that the only indecomposable plane convex polygons are triangles and line segments appears as an exercise in Yaglom and Boltyanskii [9]. The author proved this result in 1964. Independently of the author Meyer [7] proved this result in 1969.

1. Preliminary definitions and results. Consider the class F^n of all functions f on R^n such that

(1) f is nonnegative; for every x in $R^n, f(x) \geqq 0$

(2) f is subadditive; for every x, y in $R^n, f(x + y) \leqq f(x) + f(y)$

(3) f is positively homogeneous; for every x in R^n, $t \geqq 0, f(tx) = tf(x)$.

The set $\boldsymbol{F}^n$ is a convex cone whose apex is the 0 function. If f, f_1, and f_2 are all members of $\boldsymbol{F}^n$, and $f = f_1 + f_2$, we will call f_1 a *summand* of f. We will use the word *homothetic* to describe functions in a manner analogous to its previous use for sets. If $f \in \boldsymbol{F}^n, \lambda > 0$, and h is a linear function on R^n, then $F_1 = \lambda f + h$ will be called *homothetic* to f. A function f in $\boldsymbol{F}^n$ will be called *irreducible* if it admits only homothetic and linear summands. Linear functions thus play a role with respect to functions analogous to the role of one-pointed sets with respect to sets.

Any $f \in \boldsymbol{F}^n$ has the property that for some compact convex set B in R^n, and all $z \in R^n, f(z) = \sup_{z' \in B} \langle z, z' \rangle$. f is called the *support function* of the set B. Let K be the *unit ball* of f, i.e., the set in R^n defined by $K = \{x \mid f(x) \leqq 1\}$. We define the polar of K to be $\{z \mid \sup_{z' \in K} \langle z, z' \rangle \leqq 1\}$. Clearly, if f is the support function of B, and K is the unit ball of f, then B is the polar of K. If B is a compact convex set in R^n, B has a translate B' with support function $f_{B'} \in \boldsymbol{F}^n$.

The set B' is homothetic to B exactly when the corresponding support functions have the property that $f_{B'}$ is homothetic to f_B. B is indecomposable as a set exactly when f is irreducible as a function. (See well-known material on polar bodies in, for example, Fenchel [2].)

In this paper we will obtain results about decomposition of sets by studying their support functions and making use of the preceding remark, as well as, in some cases, by studying the sets directly.

The elementary result that a set K is polygonal exactly when P, its polar, is polygonal, will be repeatedly used in the sequel.

2. Decomposition of general convex sets. In the special case of functions on R^2, the properties of support functions enable us to reduce the problem in dimension by one; i.e., to study certain functions on the real line.

$$\text{Let} \quad L_+ = \{(t, 1) \mid t \mid \text{real})\} \quad \text{and} \quad L_- = \{(t, -1) \mid t \text{ real}\}\ .$$

Suppose $\{\varphi_1, \varphi_2\}$ is an (unordered) pair of real-valued functions on the real line. We will call this pair *admissible* if there is a function f in $\boldsymbol{F}^2$ with the property that $f \mid L_+ = \varphi_2$ and $F \mid L_- = \varphi_1$. If $f \in \boldsymbol{F}^2$ is the support function of the set B, and $\{\varphi_1, \varphi_2\}$ is the admissible pair consisting of the restrictions to L_-, L_+ of f, then $\{\varphi_1, \varphi_2\}$ is called the *supporting admissible pair* of B.

We remark that the one-sided derivatives of a convex function φ_i exist everywhere, and the two-sided derivatives exist everywhere except on a countable set. Defining, where necessary, $\varphi_i'(x) = \varphi_{i+}'(x) = D_+\varphi_i(x)$ (right derivate), φ_i' is defined everywhere and is nondecreasing. This definition of the "derivative" of a convex function will be used throughout this paper without making explicit reference to the convention as stated above.

The following characterization of admissible functions is the basis for our results on decomposability.

THEOREM 1. *The function pair $\{\varphi_1, \varphi_2\}$ is admissible if and only if it satisfies all three of the following conditions*:

(1) *$\varphi_i(t)$ is a nonnegative convex function of the real variable t, $i = 1, 2$.*

(4) *There are nonnegative numbers m_{-1} and m_1 such that $m_1 = \sup \varphi_i'$ and $-m_{-1} = \inf \varphi_i'$.*

(5) *There are nonnegative numbers α and β such that*

$$\lim_{x\to\infty} [\varphi_1(x) + \varphi_2(x) - 2m_1x] = \alpha ,$$

and

$$\lim_{x\to\infty} [\varphi_1(x) + \varphi_2(x) + 2m_{-1}x] = \beta .$$

The proof of Theorem 1 depends on the following lemma, whose proof will be referred to the appendix for clarity of the exposition.

LEMMA 1. *The function pair φ_1, φ_2 is admissible if and only if it satisfies the following three conditions*:

(1) *$\varphi_i(t)$ is a nonnegative convex function of the real variable t, $i = 1, 2$.*

(2) *There are nonnegative numbers m_1 and m_{-1} such that*

$$\lim_{t\to\infty} \frac{\varphi_i(t)}{t} = m_1 \quad \textit{and} \quad \lim_{t\to-\infty} \frac{\varphi_i(t)}{-t} = m_{-1}, i = 1, 2 .$$

(3) *For all nonzero t_1 and t_2, and $i = 1, 2$.*

$$\varphi_i(t_1 + t_2) - \varphi_i(t_1) \leqq |t_2| m_{\operatorname{sgn} t_2} \leqq \varphi_1(t + t_2) + \varphi_2(-t_1) ,$$

where $\operatorname{sgn} t_2$ *is defined to be* 1 *if* $t_2 > 0$, -1 *if* $t_2 < 0$.

Proof of Theorem. We note that the numbers m_1 and m_{-1} will be shown equal to the similarly designated numbers in condition (2) of Lemma 1. We prove first that if $\{\varphi_1, \varphi_2\}$ satisfies (1), (4), and (5), it is an admissible pair. It suffices to show that conditions (2) and (3) of Lemma 1 are satisfied. Since $\varphi_i'(x)$ is nondecreasing, if $x > 0$,

$\varphi_i(x)/x \leqq \varphi_i(0)/x + \varphi_i'(x)$. Letting $x \to \infty$, $\overline{\lim}_{x\to\infty} \varphi_i(x)/x \leqq m_1$.

On the other hand, since $\lim_{x\to\infty} \varphi_i'(x) = m_1$, for any $h > 0$, there exists C such that when $x > C$, then $\varphi_i'(x) > m_1 - h$. Pick $y > C$. Then for $x > y > C$, $\varphi_i(x) \geqq \varphi_i(y) + (x - y)(m_1 - h)$. $\underline{\lim}_{x\to\infty} \varphi_i(x)/x \geqq m_1 - h$, for every $h > 0$. Therefore, $\lim_{x\to\infty} \varphi_i(x)/x = m_1$, $i = 1, 2$. Thus $\{\varphi_1\}$ satisfies (2).

For all $t_1, t_2, t_2 \neq 0$, $-m_{-1} \leqq \varphi_i(t_1 + t_2) - \varphi_i(t_1)/t_2 \leqq m_1$. Therefore, $\{\varphi_1, \varphi_2\}$ satisfies the left-hand inequality of (3) for $t_2 \neq 0$, and trivially for $t_2 = 0$.

To show the pair $\{\varphi_1, \varphi_2\}$ satisfies the right-hand inequality of (3) is equivalent to showing that

$$F(x, y) = \varphi_1(x) + \varphi_2(y) - |x + y|\, m_{\operatorname{sgn}(x+y)} \geqq 0$$

for all real x and y. Suppose, first, that $x + y \geqq 0$. Then

$$F(x, y) = [\varphi_1(x) - m_1 x] + [\varphi_2(y) - m_1 y] .$$

Each of the two functions in brackets has a nonpositive derivative, and therefore is a nonincreasing function.

If $x \geqq y$, then

$$\begin{aligned} F(x, y) &\geqq [\varphi_1(x) - m_1 x] + [\varphi_2(x) - m_1 x] = \varphi_1(x) + \varphi_2(x) - 2m_1 x \\ &\geqq \lim_{x\to\infty} [\varphi_1(x) + \varphi_2(x) - 2m_1 x] = \alpha \geqq 0 . \end{aligned}$$

Similarly, if $x + y \leqq 0$, then $F(x, y) \geqq \beta \geqq 0$. It follows that $\{\varphi_1, \varphi_2\}$ satisfies (3) and hence is an admissible pair.

To prove the converse, it suffices to show that admissibility of $\{\varphi_1, \varphi_2\}$ implies (4) and (5). Since $\{\varphi_1, \varphi_2\}$ is admissible, by the left-hand side of condition (3) of Lemma 1 for $\Delta x > 0$, every x, $i = 1, 2$,

$$\varphi_i(x + \Delta x) \leqq (\Delta x) m_1 + \varphi_i(x) ,$$

and

$$\varphi_i(x) \leqq (\Delta x) m_{-1} + \varphi_i(x + \Delta x) ,$$

so

$$-m_{-1} \leqq \frac{\varphi_i(x + \Delta x) - \varphi_i(x)}{\Delta x} \leqq m_1 .$$

φ_i is convex, and has a nondecreasing derivative almost everywhere, therefore, whenever it exists, $-m_{-1} \leqq \varphi_i'(t) \leqq m_1$. Since $\varphi_i'(t)$ is bounded from below and above, it has a glb and a lub. That these are actually equal to $-m_{-1}$ and m_1 is seen easily; by convexity of φ,

$$\frac{\varphi_i(t) - \varphi_i(0)}{t} \leqq \varphi_i'(t) ,$$

so

$$\lim_{x\to\infty}\frac{\varphi_i(t)-\varphi_i(0)}{t}=\lim_{t\to\infty}\frac{\varphi_i(t)}{t}=m_1$$
$$\leqq \varliminf_{t\to\infty}\varphi_i'(t)\leqq \lim_{t\to\infty}\varphi_i'(t)\ .$$

Therefore, $\lim_{t\to\infty}\varphi_i'(t)$ actually equals m_1. The proof is similar for the greatest lower bound; so (5) is satisfied.

By the right-hand inequality of condition (3), letting $y=x$, $\varphi_2(x)+\varphi_1(x)-2|x|m_{\mathrm{sgn}\,2x}\geqq 0$ for every x. If $x>0$, $G(x)=\varphi_2(x)+\varphi_1(x)-2xm_1$ is a nonincreasing function, so $\lim_{x\to\infty}G(x)=\alpha$ exists and is nonnegative.

Similarly, for $x<0$, $G(x)=\varphi_2(x)+\varphi_1(x)+2xm_{-1}$ is a nondecreasing function, so $\lim_{x\to-\infty}G(x)=\beta$ exists and is nonnegative.

We next prove a useful lemma.

LEMMA 2. *A pair of nonnegative convex functions, differing from a pair of admissible functions on at most a bounded interval, is itself an admissible pair.*

Proof. Suppose $\{\varphi_1,\varphi_2\}$ an admissible pair, $\{\sigma_1,\sigma_2\}$ a pair of nonnegative convex functions, such that $\sigma_i(t)=\varphi_i(t)$ if $t\notin[a,b]$. Condition (1) of Theorem 1 is satisfied by hypothesis. For any $t>b$, $\sigma_i'(t)=\varphi_i'(t)\leqq m_1$, so by convexity of σ_i, for any $t'<t$, $\sigma_i'(t')\leqq m_1$. So for all t, $\sigma_i'(t)\leqq m_1$. Similarly $\sigma_i'(t)\geqq -m_{-1}$. Therefore, condition (4) is satisfied. Since condition (5) depends on limiting values only, it is clearly satisfied. Therefore, by Theorem 1, $\{\sigma_1,\sigma_2\}$ is an admissible pair.

We are now ready to prove the key theorem on admissible pairs.

THEOREM 2. *An admissible function pair which is the restriction to lines L_- and L_+ of the support function of a nonpolygonal plane convex set is the sum in a nontrivial manner of two other admissible pairs.*

For clarity of exposition, this proof is postponed to the appendix.

We can now characterize the indecomposable plane convex bodies.

We first state the well-know result (see, for example, Yaglom and Boltyanskii, [9]; Problem 4-12):

THEOREM 3. *Every convex polygon can be written as the sum of triangles and lines segments. Triangles and line segments are indecomposable.*

We therefore have our characterization:

THEOREM 4. *The only indecomposable plane convex bodies are triangles and line segments.*

Proof. Immediate from Theorems 2 and 3.

APPENDIX 1

Proof of Lemma 1. We prove first that if $\{\varphi_1, \varphi_2\}$ is admissible, conditions (1), (2), and (3) are satisfied. Let f be a member of F^2 such that $f|_{L+} = \varphi_2$ and $f|_{L-} = \varphi_1$.

(1) This is immediate from the nonnegativity and convexity of f.

(2) f is continuous.
Therefore,

$$\lim_{t\to\infty}\frac{\varphi_i(t)}{t} = \lim_{t\to\infty} f\left[1, \frac{(-1)^i}{t}\right] = f(1, 0) \geqq 0 .$$

Similarly,

$$\lim_{t\to-\infty}\frac{\varphi_i(t)}{-t} = f(-1, 0) \geqq 0 .$$

Thus the numbers $f(1, 0)$ and $f(-1, 0)$ play the roles of m_1 and m_{-1} respectively.

(3) For all nonzero t_1 and t_2,

$$\varphi_i(t_1 + t_2) \leqq f[t_1, (-1)^i] + f(t_2, 0) = \varphi_i(t_1) + |t_2| m_{\operatorname{sgn} t_2}$$

and

$$\begin{aligned} |t_2| m_{\operatorname{sgn} t_2} &= f(t_2, 0) \\ &\leqq f[t_1 + t_2, 1] + f(-t_1, -1) = \varphi_2(t_1 + t_2) + \varphi_1(-t_1) . \end{aligned}$$

This proves that all three conditions are satisfied when $\{\varphi_1, \varphi_2\}$ is admissible.

To prove the converse, assume φ_1 on L_- and φ_2 on L_+ satisfy all three of the above conditions. We extend the functions $\{\varphi_1, \varphi_2\}$ to a function f on R^2, in the obvious fashion. If $a_1 \neq 0$, define $T_{a_1}(t) = \varphi_2(t)$ if $a_1 > 0$, and $T_{a_1}(t) = \varphi_1(-t)$ if $a_1 < 0$. Then, letting v be a unit vector in the horizontal direction, and u a unit vector in the vertical direction,

$$f(a_1 u + a_2 v) = |a_1| \cdot T_{a_1}\left(\frac{a_2}{a_1}\right) .$$

If $a_1 = 0$, but $a_2 \neq 0, f(a_2 v) = |a_2| m_{\operatorname{sgn} a_2}$. (Of course, $f(0)$ is defined to be 0.)

The function f is clearly nonnegative and positively homogeneous. The proof that f is subadditive is quite long, and is achieved by considering subcases, according to whether the vectors $x = \alpha_1 u + \alpha_2 v$, $y = \beta_1 u + \beta_2 v$, and their sum, $x + y$, (u and v as above), fall on, above, or below the v axis.

Case 1. All three vectors are multiples of v, i.e., $\alpha_1 = \beta_1 = 0$. Subadditivity is immediate if α_2 and β_2 are of the same sign. If not, suppose $m_1 \geqq m_{-1}$. We need check only the case where $\alpha_2 \geqq 0$, $\beta_2 \leqq 0$, and $\alpha_2 + \beta_2 \geqq 0$. In this case $|\alpha_2 + \beta_2| \leqq |\alpha_2|$, so

$$f(x + y) = m_1|\alpha_2 + \beta_2| \leqq m_1|\alpha_2| + m_{-1}|\beta_2| = f(x) + f(y) .$$

Case 2. Neither x nor y is a multiple of v, but their sum is, i.e., $\alpha_1 = -\beta_1 \neq 0$.

Without loss of generality, assume $\alpha_2 + \beta_2 > 0$ and $\alpha_1 > 0$. Then $f(x) = |\alpha_1|T_{\alpha_1}(\alpha_2/\alpha_1), f(y) = |\beta_1|T_{\beta_1}(\beta_2/\beta_1)$, and $f(x + y) = m_1|\alpha_2 + \beta_2|$. By the right side of inequality (3), letting $t_2 = (\alpha_2 + \beta_2)/\alpha_1$ and $t_1 = (-\beta_2/\alpha_1)$,

$$f(x + y) \leqq |\alpha_1|\varphi_2\left(\frac{\alpha_2}{\alpha_1}\right) + |\alpha_1|\varphi_1\left(\frac{-\beta_2}{\beta_1}\right) = f(x) + f(y) .$$

Case 3. One of the two vectors is a multiple of v; say $\alpha_1 = 0$ and $\beta_1 = 0$. Without loss of generality, assume $\alpha_2 > 0$, $\beta_1 > 0$. By the left side of inequality (3),

$$f(x + y) \leqq |\beta_1| \cdot \varphi_1\left(\frac{\beta_2}{\beta_1}\right) + |\beta_1| \cdot \left|\frac{\alpha_2}{\beta_1}\right| m_1 = f(x) + f(y) .$$

Case 4. All three vectors are on the same side of the line through v; say $\alpha_1 > 0$ and $\beta_1 > 0$. Since $0 < \alpha_1/(\alpha_1 + \beta_1) < 1$, by convexity of φ_2,

$$f(x + y) \leqq |\alpha_1 + \beta_1|\frac{\alpha_1}{\alpha_1 + \beta_1}\varphi_2\left(\frac{\alpha_2}{\alpha_1}\right) + \frac{\beta_2}{\alpha_1 + \beta_1}\varphi_2\left(\frac{\beta_2}{\beta_1}\right) = f(x) + f(y) .$$

Case 5. Finally, we consider the case where two vectors are on one side of the line through v, the third on the other. Without loss of generality, assume $\alpha_1 < 0, \beta_1 > 0, |\alpha_1| < \beta_1$. By the left side of inequality (3), letting $t_1 = (\beta_2/\beta_1)$ and $t_2 = (\alpha_2\beta_1 - \alpha_1\beta_2)/(\alpha_1 + \beta_1)\beta_1$

$$f(x + y) \leqq |\alpha_1 + \beta_1|\varphi_2\left(\frac{\beta_2}{\beta_1}\right) + \frac{|\alpha_2\beta_1 - \alpha_1\beta_2|}{|\beta_1|}m_{\operatorname{sgn} t_2} .$$

Then applying the right hand side of inequality (3), the right hand side of the preceding is not greater than

$$|\alpha_1 + \beta_1|\varphi_2\Big(\frac{\beta_2}{\beta_1}\Big) + |\alpha_1|\Big[\varphi_2\Big(\frac{\beta_2}{\beta_1}\Big) + \varphi_1\Big(\frac{-\alpha_2}{\alpha_1}\Big)\Big] = f(x) + f(y) .$$

Proof of Theorem 2. We do not need the full strength of the nonpolygonality; we need merely that $\varphi_1'(x)$ or $\varphi_2'(x)$ assumes at least four different positive values, or four different negative values. This clearly is implied by the hypothesis. Without loss of generality, assume $\varphi_1'(x)$ assumes at least four different positive values. Pick x_1, x_2, x_3, and x_4 such that

$$0 < \varphi_1'(x_1) < \varphi_1'(x_2) < \varphi_1'(x_3) < \varphi_1'(x_4) \leqq m_1 .$$

Let $\sigma_i(x) = 1/2[\varphi_i(x) + y_i(x)]$, and $\psi_i(x) = 1/2[\varphi_i(x) - y_i(x)]$, where $y_i(x)$ will be defined so that $\sigma_i(x)$ and $\psi_i(x)$ are both admissible. Let $y_2(x) = 0$ for every x. Let $y_1'(x) = 0$ if $x < x_1$ or if $x \geqq x_4$. For $x \in [x_1, x_4]$, $y_1'(x)$ is defined as follows:

$$y_1'(x) = \begin{cases} a[\varphi_1'(x) - \varphi_1'(x_1)], & \text{if} \quad x_1 \leqq x < x_2 \\ a[\varphi_1'(x_2) - \varphi_1'(x_1)] - b[\varphi_1'(x) - \varphi_1'(x_2)], & \text{if} \quad x_2 \leqq x < x_3 , \\ a[\varphi_1'(x_2) - \varphi_1'(x_1)] - b[\varphi_1'(x_3) - \varphi_1'(x_2)] & \\ \qquad\qquad + c[\varphi_1'(x) - \varphi_1'(x_3)], & \text{if} \quad x_3 \leqq x < x_4 . \end{cases}$$

We then let $y_1(x) = \int_{x_1}^{x} y_1'(t)dt$. The numbers a, b, and c, are selected to satisfy conditions that $D_-y_1(x_4) = 0$, $\int_{x_1}^{x_4} y_1'(t)dt = 0$, and $y_1'(t)$ neither increases nor decreases faster than $\varphi_1'(t)$ increases.

As a result of these conditions, $0 < a \leqq \varphi_1'(x_1)/m_1 \leqq 1$, $0 < b \leqq \varphi_1'(x_1)/m_1 \leqq 1$ and $0 < c \leqq \varphi_1(x_1)/m_1 \leqq 1$.

We now check that $\{\sigma_1, \sigma_2\}$ and $\{\psi_1, \psi_2\}$ are admissible pairs.

Functions σ_2 and ψ_2 certainly satisfy the conditions of Lemma 2. For σ_1 and ψ_1 we must check that the two functions are nonnegative convex functions on $[x_1, x_4]$, and that $\sigma_1' - (x_1) \leqq \sigma_1' + (x_1)$, $\psi_1' - (x_1) \leqq \psi_1' + (x_1)$, $\varphi_1' - (x_4) \leqq \sigma_1' + (x_4)$, and $\psi_1' - (x_4) \leqq \psi_1' + (x_4)$. If $x_1 \leqq x < x_4$,

$$\begin{aligned} |y_1(x)| &\leqq \frac{\varphi_1'(x_1)}{m_1}(x - x_1)[\varphi_1'(x) - \varphi_1'(x_1)] \\ &\leqq \frac{\varphi_1'(x_1)}{m_1}(x - x_1)\varphi_1'(x) \leqq \frac{\varphi_1'(x_1)}{m_1}(x - x_1)m_1 \\ &\leqq \varphi_1'(x_1)[x - x_1] + \varphi_1'(x_1) \leqq \varphi_1(x) , \end{aligned}$$

so, $\sigma_1(x)$ and $\psi_1(x)$ are nonnegative.

Since a, b, and c are positive, σ_1' is clearly nondecreasing on $[x_1, x_2] \cup [x_3, x_4]$ and ψ_1' is nondecreasing on $[x_2, x_3]$. The inequalities $b \leqq 1$, $a \leqq 1$, and $c \leqq 1$ imply that σ_1', ψ_1' and ψ_1' are nondecreasing on $[x_2, x_3]$, $[x_1, x_2]$, and $[x_3, x_4]$ respectively.

Since σ_1' and ψ_1' are nondecreasing on each interval $[x_j, x_{j+1}]$, $j = 1, 2, 3$, the functions σ_1 and ψ_1 are convex on each of these intervals. To prove the two functions are convex on the entire line, it suffices to show that the left-hand derivative does not exceed the right-hand derivative for each function at each of the four points x_j, $j = 1, 2, 3, 4$. By definition of $y_1(x)$,

$$\sigma_1' + (x_2) - \sigma_1' - (x_2) = 1/2(1 + a)[\varphi_1' + (x_2) - \varphi_1' - (x_2)] \geqq 0 .$$

The rest follow similarly.

Therefore, by Lemma 2, $\{\sigma_1, \sigma_2\}$ and $\{\psi_1, \psi_2\}$ are admissible pairs. It is clear that σ_1 and ψ_1 are not multiples of φ_1, so the decomposition is nontrivial.

Appendix 2

The results and methods preceding were also used to characterize $I(\boldsymbol{K})$ when $\boldsymbol{K}$ consists of all planar compact sets with a given symmetry property. As the results are all easily obtainable, they are presented in summary only, without proofs. The interested reader can communicate with the author for the proofs.

A support function will be called *centrally symmetric* if it is the support function for a centrally symmetric compact convex set. A centrally symmetric support function with nonpolygonal unit ball is the sum in a nontrivial manner of two other centrally symmetric support functions. Since every centrally symmetric plane convex polygon can be written as the sum of line segments, we have:

THEOREM 1A. *Let $\boldsymbol{K}$ be the family of all centrally symmetric compact convex sets in the plane. Then $I(\boldsymbol{K})$ is exactly the family of all line segments.*

COROLLARY 1A. *A seminorm on R^2 is extreme if and only if it is the absolute value of a linear function on R^2. Corollary* 1A *was proved in a different manner by E. K. McLachlan.*

Generalizing Theorem 1A, we have:

THEOREM 2A. *Let $\boldsymbol{K}$ be the family of all planar compact convex sets with n-fold rotational symmetry. Then $I(\boldsymbol{K})$ is exactly the family of all regular n-gons.*

We also obtain:

THEOREM 3A. *Let $\boldsymbol{K}$ be the family of planar compact convex sets with an axis of symmetry parallel to the x axis. Then $I(\boldsymbol{K})$ is exactly* *the family of all quadrilaterals with diagonals parallel to the x and y axis (and degenerate forms of these quadrilaterals, i.e., horizontal line segments, vertical line segments, and isosceles triangles with vertical base).*

We also obtain:

THEOREM 4A. *Let $\boldsymbol{K}$ be the family of all planar compact convex sets with two axes of symmetry, parallel to the x and y axes. Then $I(\boldsymbol{K})$ is exactly the set of all rhombi with diagonals parallel to the x and y axes (and degenerate rhombi, i.e., horizontal and vertical line segments).*

The following corollary to Theorem 3A holds in R^3:

COROLLARY 2A. *Let $\boldsymbol{K}$ be the family of compact convex sets in R^3 with an axis of rotation. The $\boldsymbol{K}$-indecomposable sets are exactly double cones and degenerate double cones, which include single cones, disks, and line segments.*

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LEHIGH UNIVERSITY