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ISOMORPHIC CLASSES OF THE SPACES $C_\sigma(S)$

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Jerison introduced the Banach spaces $C_\sigma(S)$ of continuous real or complex-valued odd functions with respect to an involutory homeomorphism $\sigma: S \to S$ of the compact Hausdorff space S. It has been conjectured that any Banach space of the type $C_\sigma(S)$ is isomorphic to a Banach space of all continuous functions on some compact Hausdorff space. This conjecture is shown to be true if either (1) S is a Cartesian product of compact metric spaces or (2) S is a linearly ordered compact Hausdorff space and σ has at most one fixed point.

Introduction. Let S always denote a compact Hausdorff space. $C(S)$ well denote the Banach space of real or complex-valued continuous functions on S equipped with the supremum norm. A homeomorphism $\sigma: S \to S$ is involutory if $\sigma(\sigma(s)) = s$ for each $s \in S$. Jerison [2] introduced the Banach space $C_\sigma(S) = \{f \in C(S): f(\sigma(s)) = -f(s)\}$ of odd functions with respect to an involutory homeomorphism $\sigma: S \to S$. If X and Y are Banach spaces then X is *isomorphic* (*isometric*) to Y, and we will write $X \sim Y$ ($X \approx Y$), if there is a bounded (norm preserving) one-to-one linear operator from X onto Y.

A special case of a conjecture due to A. Pełczyński [8] is as follows: for any Banach space $C_\sigma(S)$ there is a compact Hausdorff space T with $C_\sigma(S) \sim C(T)$. In this paper we prove this conjecture when S is either a Cartesian product of compact metric spaces or a linearly ordered compact Hausdorff space (in the second case we assume σ has at most one fixed point). The results and techniques of this paper generalize, and provide shorter proofs of, some results of Samuel [11].

1. Linearly ordered spaces. A topological space A is a *linearly ordered topological space* if the topology on A is the order topology ([4], page 57) arising from some linear ordering on the set A. Examples of linearly ordered spaces are the closed interval [0,1], every space of ordinal numbers, every totally disconnected compact metric space ([5], Corollary 2a), and every compact subset of a linearly ordered space.

THEOREM 1. *Let S be an infinite linearly ordered compact Hausdorff space. If σ is an involutory homeomorphism on S with at most one fixed point, then $C_\sigma(S) \sim C(T)$ for some compact Hausdorff space T.*

Proof. The function $\Psi: S \to S$ defined by $\Psi(s) = \min\{s, \sigma(s)\}$ is continuous on S. Set $T = \Psi(S)$; the compact set T contains exactly one point from each of the pairs $\{s, \sigma(s)\}$ and thus $T \cup \sigma(T) = S$ and $T \cap \sigma(T)$ contains at most the fixed point of σ. If $T \cap \sigma(T) = \emptyset$, then $C_\sigma(S)$ is isometric to $C(T)$ via the restriction map. If $T \cap \sigma(T) = \{t_0\}$, where t_0 is the fixed point of σ, then restriction of the functions in $C_\sigma(S)$ to T is an isometry of $C_\sigma(S)$ onto the closed hyperplane $C(T, t_0) = \{f \in C(T): f(t_0) = 0\}$ of $C(T)$. By [1], $C(T, t_0) \sim C(T)$ if T contains a convergent sequence with distinct terms. Since T is infinite, it contains a strictly monotone sequence (t_n). This sequence converges either to its supremum or to its infimum and thus $C(T, t_0) \sim C(T)$.

REMARK. The first part of the proof shows that if $\sigma: S \to S$ is an arbitrary involutory homeomorphism on a linearly ordered compact Hausdorff space S, T is as in the proof, and $T_0 = \{s \in S: \sigma(s) = s\}$, then $C_\sigma(S) \approx C(T, T_0) = \{f \in C(S): f(T_0) \subset \{0\}\}$.

If S is a countable compact metric space, then S is linearly ordered since it is homeomorphic to a closed subset of the Cantor set ([5], page 286). Thus the following result due to Samuel [11] is an easy consequence.

COROLLARY 2. *Suppose S is a countably infinite compact metric space and $\sigma: S \to S$ is an involutory homeomorphism on S with at most one fixed point. Then $C_\sigma(S) \sim C(S)$.*

Proof. If T is an infinite compact metric space, then $C(T) \sim C(T) \oplus C(T)$ ([10], page 514) where $\oplus$ denotes the Cartesian product normed by taking the maximum of the norms of the two coordinates. Now, if T is as in Theorem 1 so that $S = T \cup \sigma T$ and $T \cap \sigma T$ has at most one point, it follows that $C(S) \sim C(T) \oplus C(\sigma(T))$: that is immediate if $T \cap \sigma(T) = \emptyset$; if $T \cap \sigma(T) = \{t_0\}$, then we have the string of isomorphisms $C(S) \sim C(S, t_0) \approx C(T, t_0) \oplus C(\sigma(T), t_0) \sim C(T) \oplus C(\sigma(T))$. Thus $C_\sigma(S) \sim C(T) \sim C(T) \oplus C(T) \sim C(T) \oplus C(\sigma(T)) \sim C(S)$ if S is countably infinite compact metric and σ has at most one fixed point.

REMARK. In general, even for an involutory homeomorphism $\sigma: S \to S$ having no fixed points on an ordinal space S, it is not true that $C_\sigma(S) \sim C(S)$. We are indebted to J. J. Schäffer for the following example. Let ω_1 be the first uncountable ordinal number and let $S = \{\alpha: \alpha$ an ordinal and $1 \leqq \alpha \leqq \omega_1 \cdot 2\}$. Let $F_1 = \{\alpha \in S: \alpha \leqq \omega_1\}$ and $F_2 = \{\alpha \in S: \alpha > \omega_1\}$. Then $\tau: \alpha \to \omega_1 + \alpha: F_1 \to F_2$ is a homeomorphism, and we define the involutory homeomorphism $\sigma: S \to S$ by

$\sigma(\alpha) = \tau(\alpha)$ if $\alpha \in F_1$, $\sigma(\alpha) = \tau^{-1}(\alpha)$ if $\alpha \in F_2$. Then $C_\sigma(S)$ is isomorphic to $C(F_1)$. However, $C(F_1)$ is not isomorphic to $C(S)$ ([12], Theorem 2).

2. **Products of compact metric spaces.** We begin this section with some terminology and preliminary facts from [9]. A subspace Z of a Banach space X is *complemented* if there is a bounded linear projection $P: X \to X$ with range Z, i.e., $P^2 = P$ and $P(X) = Z$. For Banach spaces Y and X, Y is *a factor of* X if there is a complemented subspace Z of X with $Y \sim Z$. If $\sigma: S \to S$ is an involutory homeomorphism, then the operator $P: C(S) \to C(S)$ defined by $(Pf)(s) = (1/2)[f(s) - f(\sigma(s))]$ projects $C(S)$ onto the subspace of odd functions $C_\sigma(S)$. Thus $C_\sigma(S)$ is a factor of $C(S)$.

D will denote the two point discrete space $\{0, 1\}$ and, for each cardinal number m, D^m will denote the generalized Cantor set which is the Cartesian product of m copies of D. We will need the following isomorphism criterion due to A. Pełczyński ([9], Proposition 8.3): if X is a Banach space and X is a factor of $C(D^m)$ and $C(D^m)$ is a factor of X, then $X \sim C(D^m)$.

Following [9], we say that a space S is an *almost Milutin space* if, for some cardinal number m, there is a continuous onto map $\theta: D^m \to S$ such that the subspace $X = \{f \circ \theta: f \in C(S)\}$ of $C(D^m)$ is complemented. If T is a closed subset of the space S, an *extension operator* is a bounded linear operator $E: C(T) \to C(S)$ such that, for each $f \in C(T)$, $Ef \mid T = f$ where "$\mid$" denotes the restriction. A compact Hausdorff space T is an *almost Dugundji space* if, for every embedding $i: T \to S$ of T into a compact Hausdorff space S, there is an extension operator $E: C(i(T)) \to C(S)$. Every Cartesian product of compact metric spaces (in particular, every space D^m) is both an almost Milutin and an almost Dugundji space ([9], Theorems 5.6 and 6.6). The *weight* of a space S is the smallest cardinal number m such that there is a base for the topology of S consisting of m open sets. If S is either an almost Milutin or an almost Dugundji space, then $C(S)$ is a factor of $C(D^m)$, where m is the weight of S (see the proof of [9], Proposition 8.4).

PROPOSITION 3. *Let S be either an almost Milutin space or an almost Dugundji space and let $\sigma: S \to S$ be an involutory homeomorphism on S. Suppose there is a closed subset F of S with $\sigma(F) \cap F = \varnothing$ such that F is homeomorphic to D^m, where m is the weight of S. Then $C_\sigma(S) \sim C(S)$.*

Proof. Since $C_\sigma(S)$ is a factor of $C(S)$ and $C(S)$ is a factor of $C(D^m)$, $C_\sigma(S)$ is a factor of $C(D^m)$. Thus, by Pełczyński's criterion, it suffices to show that $C(D^m)$ is a factor of $C_\sigma(S)$. Since F and $\sigma(F)$

are disjoint and each is homeomorphic to D^m, $F \cup \sigma(F)$ is homeomorphic to the almost Dugundji space D^{m+1}. Hence there exists an extension operator $E\colon C(F \cup \sigma(F)) \to C(S)$. Let σ' be the restriction of σ to the invariant set $F \cup \sigma(F)$ and let $P\colon C(S) \to C_\sigma(S)$ be the above-defined projection onto the odd functions. Then $C_{\sigma'}(F \cup \sigma(F))$ is isomorphic to the range of the projection Q defined on $C_\sigma(S)$ by $Qf = PE(f \mid (F \cup \sigma(F)))$. Since $C_{\sigma'}(F \cup \sigma(F))$ is trivially isometric to $C(F)$, which is isometric to $C(D^m)$, it follows that $C(D^m)$ is a factor of $C_\sigma(S)$.

LEMMA 4. *If S is an infinite product of nontrivial compact metric spaces and $\sigma\colon S \to S$ is an involutory homeomorphism on S that is not the identity, then $C_\sigma(S) \sim C(S)$.*

Proof. Let $S = \prod_{i \in I} S_i$, where each S_i has at least two points. A basis for the topology of S is given by the open sets U of the form $U = (\prod_{i \in I\setminus A} S_i) \times (\prod_{i \in A} U_i)$ where A is a finite subset of I and U_i is an open set in S_i for $i \in A$. If I is infinite, then the weight m of S is the cardinality of I. So it suffices, by Proposition 3, to construct a closed set F in S which is homeomorphic to D^m with $\sigma(F) \cap F = \emptyset$. There exists $s \in S$ with $\sigma(s) \neq s$; choose a basic neighborhood U of s with $\sigma(U) \cap U = \emptyset$. Then $U = (\prod_{i \in I\setminus A} S_i) \times (\prod_{i \in A} U_i)$ for some finite set A in I. For each i, let $\{t_i^1, t_i^2\}$ be any pair of distinct points in S_i if $i \in I\setminus A$, and just any pair of points in U_i if $i \in A$. Let $F = \prod_{i \in I} \{t_i^1, t_i^2\}$. Then F is homeomorphic to D^m and $\sigma(F) \cap F = \emptyset$.

LEMMA 5. *If S is an uncountable compact metric space and σ is an involutory homeomorphism on S such that $\{s\colon \sigma(s) = s\}$ is countable, then $C_\sigma(S) \sim C(S)$.*

Proof. Let P be the set of condensation points of S, i.e., $s \in P$ iff every neighborhood of s is uncountable. By the Cantor-Bendixson Theorem ([5], page 253), the complement of P is countable. Thus P is uncountable and there is a point $s \in P$ with $\sigma(s) \neq s$. Let F_0 be a closed neighborhood of s with $\sigma(F_0) \cap F_0 = \emptyset$. Since F_0 is an uncountable compact metric space, it must contain a closed subset F homeomorphic to $D^{\aleph_0}$ ([5], page 445). Clearly $\sigma(F) \cap F = \emptyset$. Since the weight of S is $\aleph_0$, the conclusion follows from Proposition 3.

THEOREM 6. *If S is a product of compact metric spaces and σ is an involutory homeomorphism on S that is not the identity, then $C_\sigma(S) \sim C(T)$ for some compact Hausdorff space T.*

Proof. If S is an infinite product of nontrivial compact metric spaces, then $C_\sigma(S) \sim C(S)$ by Lemma 4. If S is a finite product of compact metric spaces, then S is compact metric. Let T be the quotient space obtained from S by identifying the fixed points of σ. Let σ' denote the involutory homeomorphism on T which is induced by σ; it has at most one fixed point. Then $C_\sigma(S) \approx C_{\sigma'}(T)$, and $C_{\sigma'}(T) \sim C(T)$ by Lemma 5 if T is uncountable; by Corollary 2 if T is countably infinite. The conclusion is obvious if T is finite.

We conclude with an application to the problem of the isomorphic classification of complemented subspaces of the Banach spaces of type $C(S)$. This result is due to Samuel [11].

COROLLARY 7. *Let X be a subspace of $C(S)$, where S is a compact metric space. If X is the range of a norm-1 projection on $C(S)$, then $X \sim C(T)$ for some compact metric space T.*

Proof. By [7] or [3] (see also [6]), we have $X \approx C_\sigma(K)$ where σ is an involutory homeomorphism on a certain subspace K of a Hausdorff quotient space of S. Since a Hausdorff quotient of a compact metric space is metric, $C_\sigma(K) \sim C(T)$ for some compact metric space T by the preceding theorem.

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