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THE BEHAVIOR OF THE NORM OF AN AUTOMORPHISM OF THE UNIT DISK

D. M. GIRARD

For $f(z)$ analytic on the closed unit disk,

$$f(z) = \sum a_k z^k \ ,$$

let

$$||f|| = \sum |a_k| \ .$$

In this paper the following result is obtained: Theorem. Let $f(z)$ be an automorphism of the unit disk:

$$f(z) = e^{i\zeta}\frac{z-\alpha}{1-\bar{\alpha}z}\,, 0 < |\alpha| < 1, \zeta \text{ real} \ .$$

Then

$$\frac{1}{\sqrt{n}}||f^n|| \sim \frac{8\sqrt{2}}{\Gamma^2\left(\frac{1}{4}\right)}(1-\beta^2)^{1/2}F\left(\frac{1}{2}, \frac{3}{4}; \frac{3}{2}; 1-\beta^2\right)$$

as $n \to \infty$ where $F = {}_2F_1$ is the hypergeometric function and

$$\beta = \frac{1-|\alpha|}{1+|\alpha|} \ .$$

1. Introduction. In a more general context we denote by A the class of all functions with absolutely convergent Fourier series and define

$$||f||_A = \sum_{-\infty<k<\infty} |\hat{f}(k)| \ .$$

If $f(z)$ is analytic on the closed unit disk, then $f(e^{it}) \in A$ and $||f|| = ||f||_A$.

The asymptotic behavior of $||f^n||_A$ has been studied in several recent papers. Kahane [5] has shown that if f is real, analytic, periodic of period 2π, and nonconstant, then there exist two positive constants C_1 and C_2 such that $C_1\sqrt{n} < ||e^{inf}||_A < C_2\sqrt{n}$. More recently in [3] the behavior of $||f^n||_A$ has been studied in the case where $f \in A$, $|f(t)| \leqq 1$ and $|f(t)| = 1$ for at most a finite number of points in $[0, 2\pi]$. Further results and connections with summability methods, the stability of difference schemes and the structure theory of A may be found, respectively, in [3], [4] and the recent monograph by Kahane [6].

2. Preliminary lemmas. In this section we give several results which will be needed in the proof of the theorem. The first is a weak form of Laplace's estimate for integrals (see, for instance [1]).

LEMMA 1. *Let* $\phi(t)$ *be real valued and twice differentiable on* $[a, b]$ *and suppose that* c *is the unique point in* $[a, b]$ *satisfying* $\phi'(c) = 0, \phi''(c) < 0$. *Then*

$$\int_a^b e^{n\phi(t)}dt = 0[e^{n\phi(c)}(n\,|\,\phi''(c)\,|)^{-1/2}]\ , \qquad n \longrightarrow \infty\ .$$

A proof of the following lemma, due to van der Corput, may be found in [8, p. 61].

LEMMA 2. *Let* $h(t)$ *be differentiable,* $h'(t)$ *monotone and suppose that* $h'(t) \geqq m > 0$ *(or that* $h'(t) \leqq -m < 0$*) in* $[a, b]$. *Then*

$$\left|\int_a^b e^{ih(t)}dt\right| < \frac{4}{m}\ .$$

The next result is a modification of Exercise 173 in [7].

LEMMA 3. *Let* s_{nk}, $na \leqq k \leqq nb$, $0 < a < b$, *be such that*
(i) $0 \leqq s_{nk} \leqq 1$,
(ii) *for each* k, $na \leqq k \leqq nb$, *and any integer* $j \neq 0$,

$$\sum_{na<m\leqq k} \exp(2\pi ij s_{nm}) = o(n)\ , \qquad n \longrightarrow \infty\ .$$

Further, let (α_k) *be a positive increasing sequence such that* $\alpha_{nb}M^{-1} = 0(n^{-1})$, $n \rightarrow \infty$ *where* $M = \sum_{na<k\leqq nb} \alpha_k$. *Then, if* $g(x)$ *is a continuous function on* $[0, 1]$ *with* $g(0) = g(1)$,

$$\lim_{n\to\infty} M^{-1} \sum_{na<k\leqq nb} \alpha_k g(s_{nk}) = \int_0^1 g(x)dx\ .$$

Proof. For any $\varepsilon > 0$, there are trigonometric polynomials

$$p(x) = \sum_{|j|\leqq N} c_j \exp(2\pi ijx),\ P(x) = \sum_{|j|\leqq N} d_j \exp(2\pi ijx)$$

such that for $x \in [0, 1]$ $p(x) \leqq g(x) \leqq P(x)$ and

$$\int_0^1 [P(x) - p(x)]dx \leqq \varepsilon\ .$$

We now write

$$M^{-1} \sum_{na<k\leqq nb} \alpha_k p(s_{nk}) = \sum_{|j|\leqq N} c_j M^{-1} \sum_{na<k\leqq nb} \alpha_k \exp(2\pi ij s_{nk})$$

and by applying Abel's summation formula and (ii) to these inner sums we get, for $j \neq 0$,

$$\begin{aligned} M^{-1} &\sum_{na<k\leqq nb} \alpha_k \exp(2\pi i j\, s_{nk}) \\ &= o(\alpha_{nb} n M^{-1}) + o(nM^{-1}) \sum_{na<k\leqq nb-1} (\alpha_k - \alpha_{k+1}) \\ &= o(\alpha_{nb} n M^{-1}) \\ &= o(1)\,, \qquad n \longrightarrow \infty\,. \end{aligned}$$

Thus

$$\lim_{n\to\infty} M^{-1} \sum_{na<k\leqq nb} \alpha_k p(s_{nk}) = c_0 = \int_0^1 p(x)dx$$

and in a similar fashion,

$$\lim_{n\to\infty} M^{-1} \sum_{na<k\leqq nb} \alpha_k P(s_{nk}) = d_0 = \int_0^1 P(x)dx\,.$$

Taking limits in the following inequality

$$\begin{aligned} M^{-1} &\sum_{na<k\leqq nb} \alpha_k p(s_{nk}) - \int_0^1 g(x)dx \\ &\leqq M^{-1} \sum_{na<k\leqq nb} \alpha_k g(s_{nk}) - \int_0^1 g(x)dx \\ &\leqq M^{-1} \sum_{na<k\leqq nb} \alpha_k P(s_{nk}) - \int_0^1 g(x)dx \end{aligned}$$

we obtain the limit of the middle term bounded below by

$$\int_0^1 [p(x) - g(x)]dx\,,$$

which is greater than $-\varepsilon$, and bounded above by

$$\int_0^1 [P(x) - g(x)]dx\,,$$

which is less than ε. The result follows.

LEMMA 4. *Let s_{nk} and $g(x)$ satisfy the conditions of Lemma 3. Then, for any polynomial $p(x)$,*

$$\lim_{n\to\infty} \sum_{na<k\leqq nb} p\left(\frac{k}{m}\right) g(s_{nk}) \frac{1}{n} = \int_a^b p(x)dx \cdot \int_0^1 g(x)dx\,.$$

Proof. It is sufficient to establish the result in the case when $p(x) = x^m$, $m \geqq 0$ an integer.

In Lemma 3 we take $\alpha_k = k^m$ and then

$$\sum_{na<k\leqq nb}\left(\frac{k}{n}\right)^m g(s_{nk})\frac{1}{n} = \frac{M}{n^{m+1}}\cdot M^{-1}\sum_{na<k\leqq nb} k^m g(s_{nk})$$

where $M = \sum_{na<k\leqq nb} k^m$. Taking limits yields the result since

$$Mn^{-m-1} = \sum_{na<k\leqq nb}\left(\frac{k}{n}\right)^m\frac{1}{n}\sim\int_a^b x^m dx\ .$$

LEMMA 5. *Let s_{nk} and $g(x)$ satisfy the conditions of Lemma 3 and let $f(x)$ be continuous on $[a, b]$. Then*

$$\lim_{n\to\infty}\sum_{na<k\leqq nb} g(s_{nk})f\left(\frac{k}{n}\right)\frac{1}{n} = \int_0^1 g(x)dx\cdot\int_a^b f(x)dx\ .$$

The proof follows directly by approximating $f(x)$ uniformly by a polynomial $p(x)$ on $[a, b]$. Our last lemma, also due to van der Corput, is Theorem 5.9 in [8].

LEMMA 6. *If $\phi(t)$ is twice differentiable and real, and $0<\lambda<\phi''(t)<\mu\lambda$ (or $\lambda<-\phi''(t)\leqq\mu\lambda$) throughout the interval (c, d) and $d\geqq c+1$, then*

$$\sum_{c<k\leqq d}\exp(2\pi i\,\phi(k)) = 0[\mu(d-c)\lambda^{1/2}] + 0[\lambda^{-1/2}]\ .$$

3. Proof of the theorem. We first show that it suffices to prove the result when $\zeta = 0$ and α is real and positive. Indeed, if we let $\operatorname{Arg}\alpha = \theta$ and

$$\left(\frac{z-|\alpha|}{1-|\alpha|z}\right)^n = \sum a_{nk}z^k$$

then

$$\left(e^{i\zeta}\frac{z-\alpha}{1-\bar{\alpha}z}\right)^n = e^{in(\theta+\zeta)}\left(\frac{ze^{-i\theta}-|\alpha|}{1-|\alpha|ze^{-i\theta}}\right)^n$$
$$= \sum a_{nk}e^{in(\theta+\zeta)-ik\theta}z^k$$

and so

$$\left\|e^{i\zeta}\left(\frac{z-\alpha}{1-\bar{\alpha}z}\right)^n\right\| = \left\|\left(\frac{z-|\alpha|}{1-|\alpha|z}\right)^n\right\|\ .$$

Thus, we will assume that $0<\alpha<1$.

We want to determine the asymptotic behavior of

$$\|f^n\| = \sum|a_{nk}|\ .$$

The essential ideas are these: using Cauchy's theorem and Laplace's method for real integrals we will show that, depending on n and α, there is only a small range of summation which is significant; then over this range of summation we will apply a modification of the method of stationary phase to further estimate the coefficients. For convenience the proof is divided into three parts.

We shall omit the phrase "for n sufficiently large" finitely many times in the course of the proof.

PART 1

Since $0 < \alpha < 1$, there is an $R > 1$ such that $f(z) = (z - \alpha)/(1 - \alpha z)$ is analytic in the disk $|z| \leqq R$ and so for any r, $0 < r \leqq R$

$$||f^n|| = \sum_{k=0}^{\infty} |a_{nk}| = \sum_{k=0}^{\infty} \left| \frac{1}{2\pi i} \int_{|z|=r} f^n(z) z^{-k-1} dz \right| .$$

In this section we show that

$$||f^n|| = \sum_{k \in T(n)} |a_{nk}| + 0(\sqrt{n}/\log n),\ n \longrightarrow \infty ,$$

where $T(n) = \{k\colon n[\beta + \varepsilon_n] \leqq k \leqq n[1/\beta - \varepsilon_n]\}$, for

$$\beta = \frac{1-\alpha}{1+\alpha}$$

and

$$\varepsilon = \frac{\log n}{\sqrt{n}} .$$

For simplicity set $n[1/\beta - \varepsilon_n] = M$. Then, if $r > 1$,

$$\begin{aligned}\sum_{k \geqq M} |a_{nk}| &= \sum_{k \geqq M} \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} f^n(re^{it})(re^{it})^{-k} dt \right| \\ &\leqq \sum_{k \geqq M} \frac{1}{2\pi} r^{-k} \int_{-\pi}^{\pi} \exp [n\Psi(t)] dt \\ &\leqq \left[\frac{r^{-M+1}}{(r-1)} \right] \int_{-\pi}^{\pi} \exp [n\Psi(t)] dt ,\end{aligned}$$

where

$$\Psi(t) = \log \left| \frac{re^{it} - \alpha}{1 - \alpha r e^{it}} \right|$$

and by writing

$$\Psi(t) = \frac{1}{2} \log \frac{r^2 + \alpha^2 - 2\alpha r \cos t}{1 + \alpha^2 r^2 - 2\alpha r \cos t}$$

the following properties are easily verifiable:

$$\Psi'(0) = \Psi'(\pi) = \Psi'(-\pi) = 0$$

and

$$\Psi''(0) = \alpha r(1 - r^2)(1 - \alpha^2)(r - \alpha)^{-2}(1 - r\alpha)^{-2}$$
$$\Psi''(\pi) = -\alpha r(1 - r^2)(1 - \alpha^2)(r + \alpha)^{-2}(1 + r\alpha)^{-2} .$$

Thus, for $r > 1$,

$$\Psi''(0) < 0, \Psi''(\pi) > 0, \Psi''(-\pi) > 0 ;$$

for $r < 1$,

$$\Psi''(0) > 0, \Psi''(\pi) < 0, \Psi''(-\pi) < 0 .$$

Applying Lemma 1 in the case when $r > 1$ yields

$$\int_{-\pi}^{\pi} \exp[n\Psi(t)]dt = 0\left[\left(\frac{r - \alpha}{1 - \alpha r}\right)^n n^{-1/2}(r - \alpha)(1 - \alpha r) \cdot \right.$$
$$\left. \cdot [\alpha r(1 - r^2)(1 - \alpha^2)]^{-1/2}\right]$$

and so

$$\sum_{k \geqq M} |a_{nk}| = 0\left[n^{-1/2}\frac{(r - \alpha)^{n+1}}{(1 - \alpha r)^{n-1}} \frac{r^{-M+1/2}}{(r - 1)^{3/2}}(r + 1)^{-1/2}\right], \qquad n \longrightarrow \infty .$$

we choose our path of integration so that $r = 1 + (n\varepsilon_n)^{-1}$ and then this last expression is asymptotic to

$$\frac{1}{\sqrt{2}} n^{-1/2}(n\varepsilon_n)^{3/2} \exp\left[(n\varepsilon_n)^{-1}\left[\left(\frac{1 + \alpha}{1 - \alpha}\right)n - M\right]\right], \qquad n \longrightarrow \infty$$

and this finally yields

$$\sum_{k \geqq M} |a_{nk}| = 0[n^{1/4}(\log n)^{3/2}]$$
$$= 0[n^{1/2}/\log n] , \qquad n \longrightarrow \infty .$$

By choosing a path of integration $|z| = 1 - (n\varepsilon_n)^{-1}$ we can, in a similar way, show that

$$\sum_{k \leqq N} |a_{nk}| = 0[n^{1/2}/\log n] , \qquad n \longrightarrow \infty ,$$

where $N = n[\beta + \varepsilon_n]$.

Part 2

Unless otherwise noted all sums in this part will be for $k \in T(n)$.

To estimate a_{nk} for $k \in T(n)$ we integrate along the unit circle so that

$$|a_{nk}| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f^n(e^{it})e^{-ikt}dt\right|$$

and by setting

$$b_{nk} = \int_0^{\pi} \exp\,[ih(t)]dt\ ,$$

where

$$h(t) = h_{nk}(t) = -in\log\left(\frac{e^{it}-\alpha}{1-\alpha e^{it}}\right) - kt$$

we have

$$(1) \qquad \sum |a_{nk}| = \sum \frac{1}{2\pi}\left|b_{nk} + \bar{b}_{nk}\right|\ .$$

We will determine the behavior of b_{nk} and show that

$$(2) \qquad \sum |b_{nk} + \bar{b}_{nk}| = 2\sqrt{2\pi}\sum |h''(t_{nk})|^{-1/2}\left|\cos\left[h(t_{nk}) - \frac{\pi}{4}\right]\right| + 0(n^{1/2}/\log n)\ , \qquad n \longrightarrow \infty$$

where t_{nk} is the root of $h'(t) = 0$.

We begin by listing some properties of $h(t)$: $h(t)$ is real valued and from the derivatives

$$h'(t) = n(1-\alpha^2)(1+\alpha^2 - 2\alpha\cos t)^{-1} - k\ ,$$
$$h''(t) = -\,[2n\alpha(1-\alpha^2)\sin t](1+\alpha^2-2\alpha\cos t)^{-2}$$

we obtain that: $h''(t) < 0$ for $0 < t < \pi$, $h''(0) = h''(\pi) = 0$ and $h'(t)$ is a decreasing function on $[0, \pi]$. For $k \in T(n)$, $h'(t)$ has a unique zero in $(0, \pi)$, say t_{nk}, given by

$$(3) \qquad \cos t_{nk} = [1 + \alpha^2 - n(1-\alpha^2)/k]/2\alpha$$

and there are two constants $C_\alpha, C'_\alpha > 0$ and independent of n and k such that $-1 < -1 + C_\alpha\varepsilon_n < \cos t_{nk} < 1 - C'_\alpha\varepsilon_n < 1$ which implies

$$0 < C'_\alpha\varepsilon_n < t_{nk} < \pi - C_\alpha\varepsilon_n < \pi\ .$$

A direct calculation, using the expression for $\cos t_{nk}$ above, shows that

$$(4) \qquad h''(t_{nk}) = -k(k/n - \beta)^{1/2}(1/\beta - k/n)^{1/2}$$

which yields, for $k \in T(n)$,

$$|h''(t_{nk})| \geqq n\varepsilon_n\beta .$$

Finally, for $x \in [0, \pi]$, $|h'''(x)| \leqq K_\alpha n$ where K_α is a constant depending on α alone.

We set $\delta_n = c\varepsilon_n$ where $0 < c < \min[C_\alpha, C'_\alpha, \beta/K_\alpha]$ and define the intervals

$$\begin{aligned} I_{nk} &= (t_{nk} - \delta_n, t_{nk} + \delta_n) , \\ I'_{nk} &= [0, t_{nk} - \delta_n] , \\ I''_{nk} &= [t_{nk} + \delta_n, \pi] . \end{aligned}$$

This choice of c guarantees that $t_{nk} - \delta_n > 0$ and $t_{nk} + \delta_n < \pi$. The equality (2) will be established in two steps. First we replace

$$\sum |b_{nk} + \bar{b}_{nk}|$$

by

$$\sum \left| \int_{I_{nk}} \exp[ih(t)]dt + \int_{I_{nk}} \exp[-ih(t)]dt \right| .$$

To do this it suffices to show that

$$\sum \left[|b_{nk} + \bar{b}_{nk}| - \left| \int_{I_{nk}} + \overline{\int_{I_{nk}}} \right| \right] = 0(n^{1/2}/\log n) ,$$

as $n \to \infty$. This expression is bounded above by

$$\begin{aligned} &2 \sum \left| b_{nk} - \int_{I_{nk}} \right| \\ &\leqq 2 \sum \left[\left| \int_{I'_{nk}} \right| + \left| \int_{I''_{nk}} \right| \right] , \end{aligned}$$

the integrands in each of these cases being $\exp[ih(t)]$.

On the interval I'_{nk} $h'(t)$ is decreasing and $h'(t) \geqq h'(t_{nk} - \delta_n) > 0$ for fixed n and k. From Lemma 2 we then infer that

$$\left| \int_{I'_{nk}} \exp[ih(t)]dt \right| \leqq 4/|h'(t_{nk} - \delta_n)| ;$$

but for some value of ζ, $t_{nk} - \delta_n < \zeta < t_{nk}$,

$$|h'(t_{nk} - \delta_n)| = \left| \int_{t_{nk}-\delta_n}^{t_{nk}} h''(t)dt \right| = |h''(\zeta)|\delta_n .$$

Using the lower and upper bounds on $|h''(t_{nk})|$ and $|h'''(t)|$ respectively, given above, we obtain

$$\begin{aligned} |h''(t_{nk}) - h''(\zeta)|/|h''(t_{nk})| &\leqq K_\alpha n\delta_n/n\varepsilon_n\beta \\ &= K_\alpha c/\beta \\ &< 1 \end{aligned}$$

and so

$$|h''(\zeta)| \geqq |h''(t_{nk})|\,|1 - |h''(t_{nk}) - h''(\zeta)|/|h''(t_{nk})||$$
$$\geqq M|h''(t_{nk})|$$

where M is a constant depending only upon α.

From this inequality we obtain

$$\sum\left|\int_{I'_{nk}} \exp[ih(t)]dt\right| = 0(\delta_n^{-1})\sum|h''(t_{nk})|^{-1}$$

as $n \to \infty$. On the interval I''_{nk} $h'(t)$ is again decreasing and in an entirely similar fashion, now using the alternative hypothesis in van der Corput's lemma, we obtain the above estimate for this interval of integration.

We now show that this last sum is 0(1). Equation (4) allows us to rewrite it as

$$\sum(k/n)^{-1}(k/n - \beta)^{-1/2}(1/\beta - k/n)^{-1/2}n^{-1}$$

and by recalling that $k/n > \beta$ for $k \in T(n)$ we can majorize this by

$$\beta^{-1}\sum(k/n - \beta)^{-1/2}(1/\beta - k/n)^{-1/2}n^{-1}\,.$$

The function $(x - \beta)^{-1/2}(1/\beta - x)^{-1/2}$ takes its minimum value at the point $\gamma = (\beta + 1/\beta)/2$ and by splitting the sum we have

$$\begin{aligned}\sum_{k/m<\gamma}&(k/m - \beta)^{-1/2}(1/\beta - k/m)^{-1/2}n^{-1}\\ &\leqq (1/\beta - \gamma)^{-1/2}\sum_{k/n<\gamma}(k/n - \beta)^{-1/2}n^{-1}\\ &= 0\left[\int_\beta^\gamma(x - \beta)^{-1/2}dx\right]\\ &= 0(1)\end{aligned}$$

and similarly the sum for $k/n \geqq \gamma$ is 0(1) as $n \to \infty$. Thus, we may write

$$\sum|b_{nk} + \bar{b}_{nk}| = \sum\left|\int_{I_{nk}}\exp[ih(t)]dt + \int_{I_{nk}}\exp[-ih(t)]dt\right| + 0(n^{1/2}/\log n)\,. \tag{5}$$

Expanding $h(t)$ about the point $t = t_{nk}$ we can write

$$\exp[ih(t)] = A(t) + G^*(t)$$

where

$$\begin{aligned}A(t) &= \exp[ih(t_{nk}) + ih''(t_{nk})(t - t_{nk})^2/2]\,\cdot\\ &\quad\cdot[1 + ih'''(t_{nk})(t - t_{nk})^3/6]\end{aligned}$$

and $G^*(t) = 0[n(t - t_{nk})^4]$, $n \to \infty$. Then

$$\left|\sum\left|\int_{I_{nk}} \exp[ih(t)]dt + \int_{I_{nk}} \exp[-ih(t)]dt\right| - \sum\left|\int_{I_{nk}} A(t)dt + \int_{I_{nk}} \bar{A}(t)dt\right|\right|$$

is bounded above by

$$\begin{aligned} &2\sum\left|\int_{I_{nk}} \exp[ih(t)]dt - \int_{I_{nk}} A(t)dt\right| \\ &= 2\sum\left|\int_{I_{nk}} G^*(t)dt\right| \\ &= 0\Big[\sum_{k \in T(n)} n\delta_n^5\Big] \\ &= 0(n^2\delta_n^5) = o(1) \end{aligned}$$

as $n \to \infty$ and so we may write, using (5),

$$\sum |b_{nk} + \bar{b}_{nk}| = \sum\left|\int_{I_{nk}} [A(t) + \bar{A}(t)]dt\right| + 0(n^{1/2}/\log n)\,. \tag{6}$$

We first note that

$$\int_{I_{nk}} A(t)dt = 2\int_0^{\delta_n} \exp[ih(t_{nk}) + ih''(t_{nk})u^2/2]du$$

and a further change of variable, $(1/2)|h''(t_{nk})|u^2 = v$, produces

$$\int_{I_{nk}} A(t)dt = \sqrt{2}\,|h''(t_{nk})|^{-1/2} \exp[ih(t_{nk})]\int_0^{w_{nk}} e^{-iv}v^{-1/2}dv$$

where $w_{nk} = |h''(t_{nk})|\delta_n^2/2$. Thus, the sum on the right hand side of equation (6) may be written as

$$\sqrt{2}\sum |h''(t_{nk})|^{-1/2}\left|\exp[ih(t_{nk})]\int_0^{w_{nk}} e^{-iv}v^{-1/2}dv + \exp[-ih(t_{nk})]\int_0^{w_{nk}} e^{iv}v^{-1/2}dv\right|\,.$$

As a last step in establishing equation (2), we replace the integrals in the expression above by the integrals $\int_0^\infty e^{-iv}v^{-1/2}dv$ and $\int_0^\infty e^{iv}v^{-1/2}dv$ whose values are, respectively, $\sqrt{\pi}e^{i\pi/4}$ and $\sqrt{\pi}e^{-i\pi/4}$. To do this it suffices to show that

$$\sum |h''(t_{nk})|^{-1/2}\left|\int_{w_{nk}}^\infty e^{-iv}v^{-1/2}dv\right| = 0(n^{1/2}/\log n)\;;$$

but this follows rather easily. Integration by parts yields an estimate for the absolute value of the integral of $2/\sqrt{w_{nk}}$. The above sum is then

$$\begin{aligned}
0[\sum |h''(t_{nk})|^{-1/2} w_{nk}{}^{-1/2}] \\
&= \delta_n^{-1} 0[\sum |h''(t_{nk})|^{-1}] \\
&= \delta_n^{-1} 0(1) \\
&= 0(n^{1/2}/\log n),\ n \longrightarrow \infty ,
\end{aligned}$$

as we have shown above. Equation (6) now becomes

$$\begin{aligned}
\sum |b_{nk} + \bar{b}_{nk}| = \sum \sqrt{2\pi}\, |h''(t_{nk})|^{-1/2} \\
\cdot |\exp[i(h(t_{nk}) - \pi/4)] + \exp[-i(h(t_{nk}) - \pi/4)]| \\
+ 0(n^{1/2}/\log n)
\end{aligned}$$

and this reduces to equation (2).

PART 3

We now complete the proof of the theorem by applying Lemma 5. By setting

$$F(x) = x^{-1/2}(x - \beta)^{-1/4}(1/\beta - x)^{-1/4}$$

we can write

$$n^{-1/2}|h''(t_{nk})|^{-1/2} = F(k/n)n^{-1}$$

by using equation (4). Further, since $0 \leqq t \leqq \pi$,

$$\begin{aligned}
-i \log [(e^{it} - \alpha)/(1 - \alpha e^{it})] &= \operatorname{Arg} [(e^{it} - \alpha)(1 - \alpha e^{it})^{-1}] \\
&= \cos^{-1}\left[\frac{(1 + \alpha^2)\cos t - 2\alpha}{1 + \alpha^2 - 2\alpha \cos t}\right]
\end{aligned}$$

and we then get, by (3),

$$-i \log [(e^{it_{nk}} - \alpha)/(1 - \alpha e^{it_{nk}})] = \cos^{-1}[\,-\,[1 + \alpha^2 - k(1 - \alpha^2)/n]/2\alpha] \,.$$

We now define

$$G(x) = \frac{1 - \alpha^2}{2\alpha}\left(x - \frac{1 + \alpha^2}{1 - \alpha^2}\right)$$

and

$$H(x) = \cos^{-1}[G(x)] - x \cos^{-1}[-G(1/x)]$$

and write

$$h(t_{nk}) = nH(k/n) \,.$$

By combining (1) and (2) we then get

$$\frac{1}{\sqrt{n}}\,||f^n|| = \sqrt{\frac{2}{\pi}}\sum|\cos[nH(k/n) - \pi/4]|\,F(k/n)\frac{1}{n} + 0(1/\log n)$$

where this sum is for $k \in T(n)$. The application of Lemma 5 to this sum is delicate. For $0 < \eta < 1/\beta$ we set

$$T'(n) = \{k\colon n[\beta + \eta] \leqq k \leqq n[1/\beta - \eta]\}$$

and write

$$\left|\sqrt{\frac{2}{\pi}}\sum_{k\in T(n)}\left|\cos[nH(k/n) - \pi/4]\right|F(k/n)n^{-1} - \left(\frac{2}{\pi}\right)^{3/2}\int_\beta^{1/\beta}F(x)dx\right|$$
$$\leqq \sum_{j=1}^{5} K_j$$

where

$$K_1 = \int_\beta^{\beta+\eta}F(x)dx\,, \qquad K_2 = \int_{1/\beta-\eta}^{1/\beta}F(x)dx\,,$$
$$K_3 = \sum_{\substack{k<n(\beta+\eta)\\ k\in T(n)}}F(k/n)n^{-1}\,, \qquad K_4 = \sum_{\substack{k>n(1/\beta-\eta)\\ k\in T(n)}}F(k/n)n^{-1}\,,$$

and

$$K_5 = \left|\left(\frac{2}{\pi}\right)^{1/2}\sum_{k\in T'(n)}|\cos[nH(k/n) - \pi/4]|\,F(k/n)n^{-1} - \left(\frac{2}{\pi}\right)^{3/2}\int_{\beta+\eta}^{1/\beta-\eta}F(x)dx\right|\,.$$

Since the integral of $F(x)$ over the interval $[\beta, 1/\beta]$ is convergent, for $\varepsilon > 0$ we can choose η sufficiently small so that both K_1 and K_2 will be less than $\varepsilon/5$. Likewise, if $\eta < 1/\beta - \beta$,

$$K_3 < \beta^{-1/2}(1/\beta - \beta - \eta)^{-1/4}\sum_{k<n(\beta+\eta)}(k/n - \beta)^{-1/4}n^{-1}$$
$$= 0\left[\int_\beta^{\beta+\eta}(x - \beta)^{-1/4}dx\right]$$
$$= o(1)\,, \qquad \eta \longrightarrow 0\,.$$

A similar dominance argument applies to K_4 and thus, if we pick η sufficiently small, we have $\sum_{j=1}^4 K_j < 4\varepsilon/5$.

With η sufficiently small and fixed, we now show that $K_5 < \varepsilon/5$ for n sufficiently large. In Lemma 5 we take $g(x) = \cos(2\pi x - \pi/4)$ and $s_{nk} = nH(k/n)/2\pi - [nH(k/n)/2\pi]$*. $F(x)$ is continuous on the interval $[\beta + \eta, 1/\beta - \eta]$ and so we must show that for each integer $j \neq 0$ and each k, $na \leqq k \leqq nb$

$$\sum_{na<m\leqq k}\exp[2\pi ijs_{nm}] = o(n)\,, \qquad n \longrightarrow \infty$$

* Here [•] denotes the greatest integer function.

where $a = \beta + \eta$, $b = 1/\beta - \eta$. For each n we apply Lemma 6 with $c = na$, $d = k$ and $2\pi\phi(x) = jH_n(x) = nj\,H(x/n)$.

First, $d - c \leqq n(1/\beta - \eta) - n(\beta + \eta) < 4\alpha\, n/(1 - \alpha^2)$. Next, with the aid of the identity

$$1 - G^2(x) = x^2(1 - G^2(1/x))$$

we compute

$$\begin{aligned} H_n''(x) &= n^{-1}H''(x/n) \\ &= [(1 - \alpha^2)/2\alpha][x\sqrt{1 - G^2(x/n)}\,]^{-1} \end{aligned}$$

from which it follows that

$$\begin{aligned} H_n''(x) &\geqq [(1 - \alpha^2)/2\alpha]/(n/\beta) \\ &= (1 - \alpha)^2/2\alpha n \end{aligned}$$

and

$$\begin{aligned} H_n''(x) &\leqq [(1 - \alpha^2)/2\alpha][\min x]^{-1} \\ &\quad \cdot [\min \sqrt{1 - G^2(x/n)}\,]^{-1} \\ &< [(1 - \alpha^2)/2\alpha][n(\beta + \eta)]^{-1} \\ &\quad \cdot [1 - G^2(1/\beta - \eta)]^{-1/2} \\ &= K_{\alpha,\eta}(1 - \alpha)^2/2\alpha n \end{aligned}$$

where $K_{\alpha,\eta}$ depends only on α and η. Thus

$$(1 - \alpha)^2/2\alpha n \leqq H_n''(x) \leqq K_{\alpha,\eta}(1 - \alpha)^2/2\alpha n$$

and for $j \geqq 1$

$$(1 - \alpha)^2/2\alpha n \leqq jH_n''(x) \leqq jK_{\alpha,\eta}(1 - \alpha)^2/2\alpha n$$

so that if we put $2\pi\lambda = (1 - \alpha)^2/2\alpha n$ and $\mu = jK_{\alpha,\eta}$, then, since j is bounded in magnitude, by Lemma 6

$$\begin{aligned} \sum_{na < m \leqq k} \exp\,[2\pi i j s_{nm}] &= 0[jK_{\alpha,\eta}[4\alpha n/(1 - \alpha^2)](1 - \alpha)/\sqrt{2\alpha n}\,] \\ &\quad + 0[n^{1/2}] \\ &= 0(n^{1/2})\,, \qquad n \longrightarrow \infty\,. \end{aligned}$$

If $j < 0$ we apply the alternate form of van der Corput's estimate to obtain the same result.

We now need only to calculate the integral. If we let

$$t = (1/\beta - x)/(x - \beta)\,,$$

then

$$\int_{\beta}^{1/\beta} F(x)dx = (1-\beta^2)^{1/2}\int_0^{\infty} t^{-1/4}(t+1)^{-1}(1+\beta^2 t)^{-1/2}dt .$$

However, this is an integral representation of the hypergeometric function. From equation (5) in [2] we have, for $\operatorname{Re} c > \operatorname{Re} b > 0$ and $|\arg z| < \pi$,

$$F(a, b; c; 1-z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^{\infty} t^{b-1}(1+t)^{a-c}(1+zt)^{-a}dt$$

and taking $a = 1/2$, $b = 3/4$, and $c = 3/2$ gives

$$\int_{\beta}^{1/\beta} F(x)dx = [\Gamma(3/4)]^2[\Gamma(3/2)]^{-1}(1-\beta^2)^{1/2}F\Big(\frac{1}{2}, \frac{3}{4}; \frac{3}{2}; 1-\beta^2\Big) .$$

Using the relations $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$,

$$\Gamma(z+1) = z\Gamma(z) \quad \text{and} \quad \Gamma(1/2) = \sqrt{\pi} \text{ gives}$$

$$[\Gamma(3/4)]^2/\Gamma(3/2) = 4\pi^{3/2}/[\Gamma(1/4)]^2$$

and so

$$(2/\pi)^{3/2}\int_{\beta}^{1/\beta} F(x)dx$$
$$= 8\sqrt{2}\,\Gamma^{-2}(1/4)(1-\beta^2)^{1/2}F\Big(\frac{1}{2}, \frac{3}{4}; \frac{3}{2}; 1-\beta^2\Big) .$$

This completes the proof of the theorem.

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