THREE THEOREMS ON IMBEDDED PRIME DIVISORS OF PRINCIPAL IDEALS

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Let B be a finitely generated integral domain over a Noetherian domain A. The first theorem shows that there are only finitely many imbedded prime divisors of principal ideals in B if and only if this holds in A. The second theorem gives a necessary and sufficient condition in order that only finitely many height one prime ideals in A ramify in B, when A is locally factorial. The third theorem characterizes local domains which contain infinitely many imbedded prime divisors of principal ideals.

1. **Introduction.** For convenience of description, let $\mathscr{I}(C)$ denote the set of imbedded prime divisors of principal ideals generated by regular elements in a ring C. Then a desirable property of a Noetherian domain A is for $\mathscr{I}(A)$ to be a finite set, for this implies that there exist nonzero elements $a \in A$ such that no principal ideal in A_a has an imbedded prime divisor, hence $A_a = A_a^{(1)}$, and so A_a has certain other nice properties. Theorem 2.4 shows that this desirable property is inherited by finitely generated extension domains of A, and its Corollary (2.8) extends this result to the case A contains nonzero zero divisors. Corollary 2.9 shows that for a large class of Noetherian rings A, $\mathscr{I}(B)$ is a finite set (where B is a certain type of finitely generated extension ring of A).

In § 3, (3.1) gives two characterizations of a local domain R such that $\mathscr{I}(R)$ is an infinite set. That $\mathscr{I}(R)$ can be an infinite set is closely related to some open problems on unmixed and quasi-unmixed local domains, and characterizations of such R are therefore important. Corollary 3.4 extends these characterizations to certain local rings, and then an example is given of a local domain L such that: $L^{(1)}$ is not a finite L-algebra; $\mathscr{I}(L)$ is a finite set; and, $\mathscr{I}(L^*)$ is an infinite set, where L^* is the completion of L. This example also gives some information concerning the open problem of whether an integrally closed local domain must be unmixed.

Section 4 is concerned with unramification of height one prime ideals. Such unramification is of importance in a numbe of problems in local ring theory, for example in the purity of the branch locus [5, (41.1)]. Theorem 4.4 gives two necessary and sufficient conditions in order that only finitely many height one prime ideals in a locally factorial Noetherian domain A ramify in a finite separably generated extension domain B of A. The paper is closed with some corollaries

to (4.4), among these are that the conditions are satisfied if either A is a Dedekind domain (4.6) or B is a flat A-algebra (4.8).

2. **Finite extension rings.** All rings in this paper are assumed to be commutative rings with a unit, and the undefined terminology in the paper is, in general, the same as that in [5].

We begin with the following definition.

DEFINITION 2.1. For a ring A, let $\mathscr{I}(A) = \{p \in \operatorname{Spec} A$; p is an imbedded prime divisor of a principal ideal generated by a regular element in $A\}$.

That $\mathscr{I}(A)$ must be infinite in many cases when A contains nonzero zero divisors follows from the following remark. It is because of this remark that we shall usually only consider rings with no imbedded prime divisors of zero in this paper.

REMARK 2.2. (Cf. [6, Lemma 6].) Let q be a prime divisor of zero in a Noetherian ring A, and let b be a regular element in A such that $(q, b)A \neq A$. Then each minimal prime divisor of $(q, b)A$ is a prime divisor of bA.

Proof. This follows immediately from [6, Lemma 6] and its proof.

The following lemma sets the stage for an easy proof of the first theorem in this paper, and provides information needed throughout the paper. Although the lemma could be adapted to the case A contains nonzero zero divisors, it is stated only for the integral domain case, since this case is sufficient for our purposes in this paper. The following known fact is needed for the proof of (2.3): If B is a Noetherian ring which is a flat extension of a Noetherian ring A and I is an ideal in A, then a prime ideal P in B is a prime divisor (resp., minimal prime divisor) of IB if and only if P is a prime divisor (resp., minimal prime divisor) of pB, for some prime divisor (resp., minimal prime divisor) p of I, in which case $P \cap A = p$ [5, (18.11)]. Also, implicitly used in the proof (and frequently throughout this paper) is the fact that if $p \in \mathscr{I}(A)$ and a is a regular element in p, then p is an imbedded prime divisor of aA [5, (12.6)] (so the intersection of infinitely many such p consists of zero divisors).

LEMMA 2.3. *Let A be a Noetherian domain.*

(1) *For each nonzero b in A, $\mathscr{I}(A_b) = \{pA_b; b \notin p \in \mathscr{I}(A)\}$, and $\mathscr{I}(A) = \{P \cap A; P \in \mathscr{I}(A_b)\} \cup \{p; p$ is an imbedded prime divisor of $bA\}$.*

(2) *For each* $n \geqq 1$, $\mathscr{S}(A_n) = \{pA_n;\ p \in \mathscr{S}(A)\}$, *where* $A_n = A[X_1, \cdots, X_n]$ *with the* X_i *indeterminates.*

(3) *If* A' *is a finite free integral extension ring of* A, *then* $\mathscr{S}(A') = \{p' \in \operatorname{Spec} A';\ p' \cap A \in \mathscr{S}(A)\}$.

(4) *If* A^* *is a flat* A-*algebra, then* $\mathscr{S}(A^*) \supseteqq \{p^* \in \operatorname{Spec} A^*;\ p^*$ *is a prime divisor of* pA^*, *for some* $p \in \mathscr{S}(A)\}$.

(5) $\mathscr{S}(A)$ *is a finite set if and only if* $A^{(1)} = \bigcap \{A_p;\ p \in \operatorname{Spec} A$ *and height* $p = 1\} \subseteqq A_b$, *for some nonzero* b *in* A.

(6) *If* B *is a finitely generated algebraic extension domain of* A, *then* $\mathscr{S}(B)$ *is a finite set if and only if* $\mathscr{S}(A)$ *is a finite set.*

Proof. (1) and (2) are straightforward, (3) and (4) follow from [5, (18.11)], and (5) is given in [7, Lemma 5.15(8)] (see (3.3) below). It clearly suffices to prove (6) in the case $B = A[b]$, for some $b \in B$. Then there exists a nonzero $a \in A$ such that $A_a[b] = B_a$ is a free integral extension domain of A_a. Therefore, it follows from (1) and (3) that $\mathscr{S}(B_a)$ is a finite set if and only if $\mathscr{S}(A)$ is, hence, by (1), $\mathscr{S}(B)$ is a finite set if and only if $\mathscr{S}(A)$ is a finite set.

THEOREM 2.4. *Let* B *be a finitely generated integral domain over a Noetherian domain* A. *Then* $\mathscr{S}(B)$ *is a finite set if and only if* $\mathscr{S}(A)$ *is a finite set.*

Proof. There are elements $X_1, \cdots, X_n$ in B which are algebraically independent over A such that B is a (finite) algebraic extension domain of $A_n = A[X_1, \cdots, X_n]$. Therefore, the conclusion follows from (2.3)(2) and (6).

To generalize (2.4) to the case where A contains nonzero divisors of zero, the following two lemmas are needed.

LEMMA 2.5. *Let* q *be a minimal prime ideal in a Noetherian ring* A. *Then there exists* $a \in A$, $\notin q$ *such that, for all* $P \in \operatorname{Spec} A$ *such that* $q \subseteqq P$ *and* $a \notin P$, $P \in \mathscr{S}(A)$ *if and only if* $P/q \in \mathscr{S}(A/q)$.

Proof. If $b \in A$, $\notin q$, then it clearly suffices to prove the lemma for A_b instead of A. Hence it may be assumed that q is nilpotent. The lemma now readily follows from [3, IV. (6.10.6)]. Specifically, the referenced result is essentially local, and passing from the language of preschemes to the language of commutative rings we find that it asserts that there exists $a \in A$, $\notin q$ such that, for all $P \in \operatorname{Spec} A$ such that $q \subseteqq P$ and $a \notin P$, altitude $A_P =$ altitude $(A/q)_{P/q}$ and Prof $A_P =$ Prof $(A/q)_{P/q}$ (since q is nilpotent), where Prof R is the length of a maximal R-sequence with R a local ring. The lemma follows from this, since $P \in \mathscr{S}(A)$ if and only if Prof $A_P = 1$ and height $P > 1$.

LEMMA 2.6. *Let A be a Noetherian ring such that each prime divisor of zero in A is minimal. Then $\mathscr{F}(A)$ is a finite set if and only if, for each minimal prime ideal q in A, $\mathscr{F}(A/q)$ is a finite set.*

Proof. Let $q_1, \cdots, q_g$ be the minimal prime ideals in A, and, for $i = 1, \cdots, g$, let a_i be as in (2.5) for q_i. Therefore, for all $P \in \operatorname{Spec} A$ such that $q_i \subseteqq P$ and $a_i \notin P$, $P \in \mathscr{F}(A)$ if and only if $P/q_i \in \mathscr{F}(A/q_i)$. Now, if $\mathscr{F}(A)$ is an infinite set, then there is an i such that $q_i = \bigcap \{P; q_i \subset P \in \mathscr{F}(A)\}$. Fix such an i, and let $q = q_i$ and $a = a_i$. Then $qA_a = \bigcap \{PA_a; qA_a \subset PA_a \in \mathscr{F}(A_a)\}$. Hence (2.5) implies $\mathscr{F}(A/q)$ is an infinite set, since $\mathscr{F}((A/q)_{a+q})$ is. Conversely, if $\mathscr{F}(A/q_i)$ is an infinite set, for some $i = 1, \cdots, g$, then fix such an i and let $q = q_i$ and $a = a_i$. Then $(0) = \bigcap \{P/q, P/q \in \mathscr{F}(A/q)\}$, so $(0)(A/q)_{a+q} = \bigcap \{(P/q)_{a+q}; (P/q)_{a+q} \in \mathscr{F}((A/q)_{a+q})\}$, hence (2.5) implies $\mathscr{F}(A)$ is an infinite set.

COROLLARY 2.7. *With A as in (2.6), $\mathscr{F}(A)$ is a finite set if and only if $\mathscr{F}(A/(\operatorname{Rad} A))$ is a finite set.*

Proof. Clear by (2.6).

COROLLARY 2.8. *Let B be a finitely generated ring over a Noetherian ring A, and assume that all prime divisors of zero in A and in B are minimal and that, for each prime divisor q of zero in B, $q \cap A$ is a prime divisor of zero. Then $\mathscr{F}(A)$ is a finite set if and only if $\mathscr{F}(B)$ is a finite set.*

Proof. Let $q_1, \cdots, q_g$ be the prime divisors of zero in B. Then, by (2.6) and (2.4), $\mathscr{F}(B)$ is a finite set if and only if $\mathscr{F}(B/q_i)$ is, for all i, if and only if $\mathscr{F}(A/(q_i \cap A))$ is, for all i, if and only if $\mathscr{F}(A)$ is a finite set.

The corollary shows that such finite extension rings of a large class of Noetherian rings have only finitely many imbedded prime divisors of principal ideals. Specifically, the following corollary holds.

COROLLARY 2.9. *Let A and B be as in (2.8). Then $\mathscr{F}(B)$ is finite in each of the following cases:*
  (1) *A is locally factorial (4.1).*
  (2) *A is integrally closed.*
  (3) *A is pseudogeometric [5, p. 131].*
  (4) *A is Japanese [2, 0. (23.1.1)].*
  (5) *The integral closure of A is a finite A-algebra.*
  (6) *A is locally Macaulay.*

(7) *A is excellent* [3, IV, (7.8.2)].

(8) *Altitude* $A \leqq 1$.

(9) *A is semi-local and altitude* $A \leqq 2$.

(10) *A is an analytically unramified semi-local ring.*

(11) *A is an unmixed semi-local domain.*

(12) *A is a local ring whose completion has no imbedded prime divisor of zero.*

Proof. By (2.8) it may be assumed, where appropriate, that $\operatorname{Rad} A = (0)$. Then it is well known that in each case $\mathscr{F}(A)$ is a finite set (for (11), see (3.5)(2) below and (2.3)(5), and for (12), see (3.4) below), hence the conclusion follows from (2.8).

One further result which is related to (2.4) and which gives a sharper conclusion when $\mathscr{F}(A)$ is finite is given in the following proposition.

PROPOSITION 2.10. *Let A and B be as in* (2.4), *and assume* $\mathscr{F}(A)$ *is a finite set. Then there exists a nonzero a in A such that* $B_a^{(1)}$ *is a finite* B_a-*algebra.*

Proof. It is known [5, (14.4)] that there exist $X_1, \cdots, X_n$ in B which are algebraically independent over A and a nonzero c in A such that B_c is a (finite) integral extension domain of $A[1/c, X_1, \cdots, X_n]$. Also, since $\mathscr{F}(A)$ is a finite set, there is a nonzero d in A such that $A^{(1)} \subseteqq A_d$ (2.3)(5). Then $A_d^{(1)} = A_d$ [7, Corollary 5.9(2)], so, with $a = cd$, $A_a^{(1)} = A_a$ [7, Corollary 5.9(2)], hence $D^{(1)} = D$ [7, Lemma 5.11(2)], where $D = A[1/a, X_1, \cdots, X_n]$. Therefore, since B_a is a finite integral extension demain of D, $B_a^{(1)}$ is a finite B_a-algebra [3, IV. (5.10.17)].

3. The local domain case. With A and B as in (2.8), if A is semi-local of altitude at most two, then $\mathscr{F}(B)$ is a finite set (2.9)(9). This leads to the question: Can $\mathscr{F}(R)$ be an infinite set when R is a local ring? The answer is well known to be "yes", if there is an imbedded prime divisor q of zero in R such that depth $q > 1$ (2.2). Quite recently the answer was shown to be "yes" even when R is a local domain. Specifically, in [1, Proposition 3.5] an example was constructed of a local domain R such that altitude $R = 3$ and R contains infinitely many height two prime ideals P such that R_P is not Macaulay (hence PR_P, and so P also, is an imbedded prime divisor of each nonzero principal ideal that is contained in it).

The following theorem gives two characterizations of a local domain R such that $\mathscr{F}(R)$ is an infinite set.

THEOREM 3.1. *The following statements are equivalent for a*

local domain (R, M):

(1) $\mathscr{F}(R)$ *is an infinite set.*

(2) *For no nonzero* $b \in R$ *is* $R^{(1)} \subsetneqq R_b$.

(3) *There exists an imbedded prime divisor* q *of zero in the M-adic completion* R^* *of* R *such that* $q = \bigcap \{p^* \in \mathscr{F}(R^*); q \subset p^*$ *and* $p^* \cap R \in \mathscr{F}(R)\}$.

Proof. The equivalence of (1) and (2) has already been noted in (2.3)(5). For the equivalence of (1) and (3), assume first that (1) holds. For each $p \in \mathscr{F}(R)$, let p^* be a minimal prime divisor of pR^*, and let $\mathscr{F}^* = \{p^*; p \in \mathscr{F}(R)\}$ (so $\mathscr{F}^* \subsetneqq \mathscr{F}(R^*)$ (2.3)(4)). Then, for each $p^* \in \mathscr{F}^*$, $R_{p^*}^*$ is not Macaulay, so $I \subsetneqq p^*$, where I is the radical ideal in R^* which defines the non-Macaulay locus of R^* (the Macaulay locus of R^* (that is, the set $\{p^* \in \operatorname{Spec} R^*; R_{p^*}^*$ is Macaulay$\}$) is open (in the Zariski topology on $\operatorname{Spec} R^*$) [3, IV. (5.11.8)]). Now infinitely many p^* must contain some (minimal) prime divisor of I, say q, and then $q \subsetneqq \bigcap \{p^*; q \subsetneqq p^* \in \mathscr{F}^*\}$. Since R^* is Noetherian, it follows that $q = \bigcap \{p^*; q \subset p^* \in \mathscr{F}^*\}$ and q is a prime divisor of zero. Since R_q^* is not Macaulay, height $q > 0$, so (3) holds.

Conversely, if (3) holds, then let $\mathscr{F} = \{p^* \cap R; q \subset p^* \in \mathscr{F}(R^*)$ and $p^* \cap R \in \mathscr{F}(R)\}$, so $\mathscr{F} \subsetneqq \mathscr{F}(R)$ and (1) holds if $\mathscr{F}$ is an infinite set. Since q is a prime divisor of zero, there are infinitely many p^*, so (1) holds if, for each $p^* \cap R \in \mathscr{F}$, p^* is a prime divisor of $(p^* \cap R)R^*$. But this is true by [5, (18.11)], since $p^* \in \mathscr{F}(R^*)$ and $p^* \cap R \in \mathscr{F}(R)$.

REMARKS 3.2. (1) Possibly the theorem remains true if the set in (3) is replaced by $\{p^* \in \operatorname{Spec} R^*; q \subset p^*$ and $p^* \cap R \in \mathscr{F}(R)\}$. At least the author knows of no case where this last set does not work. Of course, this last set always works in the case that infinitely many of the p^* satisfy: height $p^* =$ height $q + 1$ (2.2).

(2) The proof that (1) $\Rightarrow$ (3) shows that, if $\mathscr{F}(R^*)$ is an infinite set, then R^* has an imbedded prime divisor of zero. The converse is not true, as is seen if altitude $R = 2$.

(3) If altitude $R = 2$ and R^* has an imbedded prime divisor of zero, then $R^{(1)}$ is not a finite R-algebra (3.5)(1) but $R^{(1)} \subsetneqq R_b$, for each nonzero $b \in M$. For an example of this, see [1, Proposition 3.3].

(4) It is easy to see that if altitude $R = 3$, then $\mathscr{F}(R)$ is an infinite set if and only if the Macaulay locus of R is not open. For, the Macaulay locus of R is open if and only if $\mathscr{F}(R)$ is a finite set, when altitude $R = 3$.

To generalize (3.1) to the case where R contains nonzero zero divisors, the following lemma is needed.

LEMMA 3.3. *Let* A *be a Noetherian ring which has no imbedded*

prime divisors of zero, and let altitude $A > 0$. Then $\mathscr{F}(A)$ is a finite set if and only if $A^{(1)} = \bigcap\{A_{(p)}; p \in \operatorname{Spec} A$, height $p = 1$, and p contains a regular element$\} \subsetneqq A_b$, for some regular element $b \in A$, where $A_{(p)} = \{a/c; a \in A$ and c is a regular element in A and not in $p\}$.

Proof. If $\mathscr{F}(A)$ is a finite set, then, since altitude $A > 0$, let b be a regular element in $\bigcap\{P; P \in \mathscr{F}(A)\}$, so $\mathscr{F}(A_b)$ is empty. Hence $A_b = [8,$ Corollary 2.18(2)$]$ $A_b^{(1)} = (A^{(1)})_b \supsetneqq A^{(1)}$. Conversely, if $A^{(1)} \subsetneqq A_b$, then $(A^{(1)})_b = A_b \subsetneqq A_b^{(1)} = (A^{(1)})_b$, so $A_b = A_b^{(1)}$, hence $b \in \bigcap\{P; P \in \mathscr{F}(A)\}$, by $[8,$ Corollary 2.18(2)$]$. Thus $\mathscr{F}(A)$ is a finite set, since b is regular and bA has only finitely many prime divisors.

COROLLARY 3.4. *Let R be a local ring which has no imbedded prime divisors of zero, let altitude $R > 0$, and let R^* be the completion of R. Then the following statements are equivalent:*

(1) $\mathscr{F}(R)$ *is an infinite set.*

(2) *For no regular element $b \in R$ is $R^{(1)} \subsetneqq R_b$.*

(3) *There exists an imbedded prime divisor q^* of zero in R^* such that $q^* = \bigcap\{P \in \mathscr{F}(R^*); q^* \subset P$ and $P \cap R \in \mathscr{F}(R)\}$.*

Proof. (1) $\Leftrightarrow$ (2) is given by (3.3).

$\mathscr{F}(R)$ is an infinite set if and only if $\mathscr{F}(R/q)$ is, for some (minimal) prime divisor q of zero (2.6), and $\mathscr{F}(R/q)$ is an infinite set if and only if there is an imbedded prime divisor $q^{*\prime} = q^*/qR^*$ of zero in R^*/qR^* (equivalently, there exists an imbedded prime divisor q^* of zero in R^* $[5,$ (18.11)$]$) such that $q^{*\prime} = \bigcap\{P' \in \mathscr{F}(R^*/qR^*); q^{*\prime} \subset P'$ and $P' \cap (R/q) \in \mathscr{F}(R/q)\}$ (3.1). By (2.5), there exists $a \in R$, $\notin q$ such that, for all $Q \in \operatorname{Spec} R$ such that $q \subsetneqq Q$ and $a \notin Q$, $Q \in \mathscr{F}(R)$ if and only if $Q/q \in \mathscr{F}(R/q)$. Since $a + q$ is a regular element in R/q, $q^{*\prime} = \bigcap\{P' \in \mathscr{F}(R^*/qR^*); q^{*\prime} \subset P', P' \cap (R/q) \in \mathscr{F}(R/q)$, and $a + q \notin P' \cap (R/q)\}$; so, by (2.5), $\mathscr{F}(R)$ is an infinite set if and only if there is an imbedded prime divisor q^* of zero in R^* such that $q^* = \bigcap\{P; P' = P/qR^* \in \mathscr{F}(R^*/qR^*), q^* \subset P$, and $P \cap R \in \mathscr{F}(R)\}$. Since P' is a prime divisor of $(P' \cap (R/q))(R^*/qR^*) = ((P \cap R)/q)(R^*/qR^*) = ((P \cap R)R^*)/qR^*$ (since $P' \in \mathscr{F}(R^*/qR^*)$ and $P' \cap (R/q) \in \mathscr{F}(R/q))$, P is a prime divisor of $(P \cap R)R^*$, hence $P \in \mathscr{F}(R^*)$ (2.3)(4).

Example 3.6 below shows somewhat more than (3.2)(3). For the example the following information is needed.

REMARKS 3.5. Let (R, M) be a local domain.

(1) $[3,$ IV. (7.2.3)$]$. $R^{(1)}$ is a finite R-algebra if and only if the following condition holds: If $q \subset p$ are prime ideals in the M-adic

completion R^* of R such that q is a prime divisor of zero and height $p \cap R \geqq 2$, then height $p/q \geqq 2$.

(2) [7, Lemma 5.11(1)]. If R is unmixed, then $R^{(1)}$ is a finite R-algebra.

If $R^{(1)}$ is a finite R-algebra, then clearly $R^{(1)} \subseteqq R_b$, for some non-zero b in R. On the other hand, if $R^{(1)}$ is not a finite R-algebra, then there is at least one prime ideal p in R^* such that p contains a prime divisor q of zero such that height $p/q = 1 <$ height $p \cap R$ (3.5) (1), and necessarily $p \cap R \in \mathscr{F}(R)$ (since $p \in \mathscr{F}(R^*)$ (2.2), so p is a prime divisor of bR^*, for each nonzero $b \in p \cap R$, hence $p \cap R$ is a prime divisor of each such bR [5, (18.11)]). Since there is one such p, it might be asked if there are necessarily infinitely many such if depth $q > 1$ (so $p \neq M^*$). The answer is "no". In fact, the example below shows the stronger result that $\mathscr{F}(R)$ can be finite in this case.

EXAMPLE 3.6. *A local domain L such that $L^{(1)}$ is not a finite L-algebra and each imbedded prime divisor of zero in the completion of L has depth greater than one, but $L^{(1)} \subseteqq L_b$, for some nonzero b in L (hence $\mathscr{F}(L)$ is a finite set (2.3)(5)). Further, $\mathscr{F}(L^*)$ is an infinite set.* Let (R, M) be a local domain whose completion (R^*, M^*) has an imbedded prime divisor of zero, and assume $\mathscr{F}(R)$ is a finite set. (For example, let $R = A$ in [1, Proposition 3.3].) Let Q be an M-primary ideal, let $\mathscr{R} = \mathscr{R}(R, Q) = R[tQ, u]$ be the Rees ring of R with respect to Q (t is an indeterminate, $u = 1/t$, and $tQ = \{tm; m \in Q\}$), and let $\mathscr{R}^* = \mathscr{R}(R^*, Q^*)$, where $Q^* = QR^*$. Let $\mathscr{M} = (tQ, M, u)\mathscr{R}$ and $\mathscr{M}^* = (tQ^*, M^*, u)\mathscr{R}^*$ be the maximal homogeneous ideals in $\mathscr{R}$ and $\mathscr{R}^*$, respectively. Then $L = \mathscr{R}_{\mathscr{M}}$ is a dense subspace of $L' = \mathscr{R}^*_{\mathscr{M}^*}$ [7, Lemma 3.2], so, by (3.5)(1), to show that $L^{(1)}$ is not a finite L-algebra, it suffices to prove that there exist prime ideals $q' \subset p'$ in L' such that q' is an imbedded prime divisor of zero and height $p'/q' = 1 <$ height $p' \cap L$ (since this will then be reproduced in the completion L^* of L and L'). Also, since the prime divisors of zero in L' are the ideals $(qR^*[t, u] \cap \mathscr{R}^*)L'$ with q a prime divisor of zero in R^* [9, Theorem 1.5], there are no depth one prime divisors of zero in L' (since $qR^*[t, u] \cap \mathscr{R}^* \subset M^*R^*[t, u] \cap \mathscr{R}^* \subset \mathscr{M}^*$), so there will be none in L^* (since, for each prime ideal P in L', each prime divisor of PL^* has depth equal to depth P, by [5, (36.5) and Exercise p. 135] applied to L'/P (that L'/P satisfies the second chain condition for prime ideals follows from [7, Corollary 2.9], since $R^*/(P \cap R^*)$ does [5, (34.4)])). Let q be an imbedded prime divisor of zero in R^* and let $q^* = qR^*[t, u] \cap \mathscr{R}^*$, so q^* is an imbedded prime divisor of zero in $\mathscr{R}^*$. Let $I = (q^*, u)\mathscr{R}^*$. Then height $q + 1 \leqq$ height $I = $ (say) $h <$ height $\mathscr{M}^*$ (since $\mathscr{R}^*/q^* \cong \mathscr{R}(R^*/q, (Q^* + q)/q)$ [9, Theorem 2.1] and depth $q \geqq 1$, so depth $q^* \geqq 2$), and I is homogeneous, so there is a

height h prime divisor p^* of I which is contained $\mathscr{M}^*$. Since $u\mathscr{R}^* \cap R^* = Q^*$ is M^*-primary, $\mathscr{R}_{p^* \cap \mathscr{R}}$ is a dense subspace of $\mathscr{R}_{p^*}^*$ [7, Lemma 3.2] (so height $p^* \cap \mathscr{R} = $ height $p^* = h > $ height $p^*/q^* = 1$), and $p^* \cap \mathscr{R} \subset \mathscr{M}$. However, $\mathscr{I}(L)$ is a finite set, since $\mathscr{I}(\mathscr{R})$ is, by (2.4). Finally, $\mathscr{I}(L^*)$ is an infinite set, since $\mathscr{I}(L')$ is by (2.2) applied to q^*L'.

REMARK 3.7. One reason for using the ring $L = \mathscr{R}_{\mathscr{M}}$ instead of $P = R[X]_{(M,X)}$ is that for the ring L it *is not* necessary to assume that $R^{(1)}$ is not a finite R-algebra to prove that $L^{(1)}$ is not a finite L-algebra, whereas for P it is necessary to assume this to prove that $P^{(1)}$ is not a finite P-algebra (but this does hold for $R = A$ in [1, Proposition 3.3]). A more important reason is it is an open problem if R must be unmixed when R is integrally closed. Related to this problem, the rings L (as Q varies) show that, if R is integrally closed and not unmixed, then there does not exist an M-primary ideal Q such that $L^{(1)}$ is a finite L-algebra.

4. Unramification of height one prime ideals. To obtain the main result of this section, the following two definitions and lemma are needed.

DEFINITION 4.1. A ring A is *locally factorial* in case A_P is a unique factorization domain, for all prime ideals P in A.

LEMMA 4.2. *Let B be a Noetherian integral domain which contains a locally factorial Noetherian domain A, and let P be a prime ideal in B such that P is a prime divisor of pB, for some height one prime ideal p in A. Then the following statements hold:*

(1) *P is a prime divisor of qB, for all height one prime ideals $q \subsetneqq P \cap A$.*

(2) *P is an imbedded prime divisor of pB if and only if $P \in \mathscr{I}(B)$.*

(3) *P is a minimal prime divisor of pB if and only if height $P = 1$.*

Proof. Since, for each height one prime ideal $q \subsetneqq P \cap A$, $qA_{P \cap A}$ is principal and $A_{P \cap A} \subseteq B_P$, (1) and (2) hold, and (3) also follows from this and the Principal Ideal Theorem.

DEFINITION 4.3. (Cf. [5, pp. 144–145].) Let $A \subset B$ be rings, let P be a prime ideal in B, and let $p = P \cap A$. Then P is *unramified over* A in case $PB_P = pB_P$ and B/P is separably generated over A/p (that is, the quotient field of B/P is a separable extension field of the

quotient field of A/p). If q is a prime ideal in A, then qB is *un-ramified over* A in case, for every prime divisor Q of qB, Q is un-ramified over A.

The following theorem is a considerable generalization of [4, Theorem 5] (see (4.6) below).

THEOREM 4.4. *Let B be a finite separably generated extension domain of a locally factorial Noetherian domain A. Then the following statements are equivalent, where $\mathscr{S} = \{p \in \text{Spec } A;\ \text{height } p = 1\}$:*

(1) *pB is unramified over A, for all except finitely many $p \in \mathscr{S}$.*

(2) *If P is a height one prime ideal in B, then height $P \cap A \leqq 1$, and only finitely many of the ideals pB, where $p \in \mathscr{S}$, have imbedded prime divisors.*

(3) *If P is a prime divisor of pB, where $p \in \mathscr{S}$, then $P \cap A = p$.*

Proof. (1) $\Rightarrow$ (2). By (1), with $p \in \mathscr{S}$, only finitely many pB have imbedded prime divisors. Also, if P is a height one prime ideal in B such that height $P \cap A > 1$, then clearly P is a prime divisor of pB, for all $p \in \mathscr{S}$ such that $p \subset P \cap A$. Since there are infinitely many such p, (1) implies that P is unramified over at least one such p, hence $P \cap A = p$; contradiction. Therefore, height $P \cap A \leqq 1$, as desired.

(2) $\Rightarrow$ (3). Let $p \in \mathscr{S}$ and let P be a prime divisor of pB. If P is an imbedded prime divisor of pB, then $P \cap A = p$, by (2) and (4.2)(1). If P is a minimal prime divisor of pB, the height $P = 1$ (4.2)(3), so (2) implies that $P \cap A = p$.

(3) $\Rightarrow$ (1). Since B is a finite separably generated extension domain of A, there exists a separating transcendence basis $X_1, \cdots, X_n$ in B of B over A and a nonzero element $a \in A$ such that B_a is integrally dependent on $D = A[1/a, X_1, \cdots, X_n]$ [5, (39.11)]. Then D is locally factorial, since A is. Let D' and B' be the integral closures of D and B in the quotient field of B, respectively, so $B'_a = D'$. Since B_a is a finite separable algebraic extension of D, let $b \in B_a$ such that $D[b]$ and B_a have the same quotient field, and let d be the discriminant of the minimum polynomial of b over the quotient field of D (so $d \in D$, since D is integrally closed). Then $D'_d = (D_d)' = D_d[b]$ [5, (10.17) and (10.18)], so every prime ideal in D'_d is unramified over D_d [5, (38.9)]. Let $\mathscr{J}$ be the set of $p \in \mathscr{S}$ such that: (i) $a \in p$; or (ii) $d \in pD$; or (iii) pB has an imbedded prime divisor. That $\mathscr{J}$ is a finite set follows from: (i) A is Noetherian; (ii) D is Noetherian and pD is a height one prime ideal; and (iii) if P is an imbedded prime divisor of pB, then $P \in \mathscr{J}(B)$, by (4.2)(2), and $P \cap A = p$, by (3), hence there are only finitely many such P by (2.9)(1). It will now be shown that if $p \in \mathscr{S}$, $\notin \mathscr{J}$, then pB is unramified over A.

For this, note first that pD is a prime ideal, $pD \cap A = p$, and D/pD is a pure transcendental extension of $(A/p)_{\bar{a}}$, where $\bar{a} = a$ (modulo p). Therefore, since every prime ideal in D'_d is unramified over D_d, pD'_d is unramified over A ($p \in \mathscr{S}$, $\notin \mathscr{F}$).

Since $p \notin \mathscr{F}$, pB has no imbedded prime divisors, so if P is a prime divisor of pB, then height $P = 1$ (4.2)(3). Also, $a \notin P$, since $P \cap A = p$, by (3). Further, $d \notin PB_a$; for B_a is integrally dependent on D, hence height $PB_a \cap D = 1$, since D is integrally closed, and so $pD = PB_a \cap D$ and $d \notin pD$. Therefore, $B_P = (B_{ad})_{PB_{ad}}$ and $D_d[b] \subseteqq B_{ad} \subseteqq D'_d = D_d[b]$. Hence, since pD'_d is unramified over A, PB_P is unramified over A, and so pB is unramified over A.

REMARKS 4.5. (1) In (2) of (4.4), the condition: "Height $P \cap A \leqq 1$, for all height one prime ideals P in B"; is equivalent to: "Height $QB > 1$, for all prime ideals Q in A such that height $Q > 1$. Also, the condition: "Only finitely many of the ideals pB have imbedded prime divisors", is equivalent to: "If P is an imbedded prime divisor of pB, then $P \cap A = p$", by (4.2)(1) and (2) and (2.9)(1).

(2) A regular domain satisfies the conditions on A in (4.4).

(3) The proof of (4.4) shows that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) for an arbitrary Noetherian domain which contains A, and (3) $\Rightarrow$ (1) with $\mathscr{C}$ replacing B in the statements, where either $\mathscr{C} = B'$ or $\mathscr{C}$ is a Noetherian ring such that $B \subseteqq \mathscr{C} \subseteqq B'$. (That $\mathscr{F}(\mathscr{C})$ is finite follows from: $a\mathscr{C}$ and $d\mathscr{C}_a$ have only finitely many imbedded prime divisors and $\mathscr{C}_{ad} = D'_d$.)

COROLLARY 4.6. (Cf. [4, Theorem 5].) *If B is a finite separably generated extension domain of a Dedekind domain A, then (1)–(3) of (4.4) hold.*

Proof. A is a locally factorial Noetherian domain, and (3) in (4.4) is satisfied.

COROLLARY 4.7. *Let A and B be as in (4.4), and assume that A is a regular local domain of altitude two. If $MB = B$, where M is the maximal ideal in A, then (1)–(3) of (4.4) hold.*

Proof. The proof is the same as the proof of (4.6).

COROLLARY 4.8. *Let A and B be as in (4.4), and assume that B is a flat A-algebra. Then (1)–(3) of (4.4) hold.*

Proof. (3) in (4.4) is satisfied.

COROLLARY 4.9. *Let A and B be as in (4.4), and assume (1)–(3)*

hold. *Then B_P is a regular local ring, for all except finitely many height one prime ideals P in B.*

Proof. Let P be a height one prime ideal in B. If $P \cap A \neq (0)$, then height $P \cap A = 1$ (4.4)(3), and only finitely many such P fail to satisfy B_P is a regular local ring (4.4)(1). If $P \cap A = (0)$, then let $S = A \sim (0)$, let $F = A_S$, and let $C = B_S$. Then D_S is a finite separably generated extension domain of the field F, hence it is well known [5, (36.6)] that the integral closure C' of C is a finite C-algebra. Therefore, only finitely many height one prime ideals Q in C fail to satisfy C_Q is a regular local ring (namely those Q which contain $C : C'$). Since PC is a height one prime ideal, for all such P (height $P = 1$ and $P \cap A = (0)$), the conclusion follows.

COROLLARY 4.10. *Let A and B be as in (4.4), let $\mathscr{C}$ be a ring such that either $\mathscr{C} = B'$, the integral closure of B, or $\mathscr{C}$ is Noetherian and $B \subseteq \mathscr{C} \subseteq B'$, and let $\mathscr{H} = \{P \in \mathrm{Spec}\ \mathscr{C};\ \text{height}\ P = 1 \geq \text{height}\ P \cap A\}$. Then $\mathscr{C}_P$ is a regular local ring, for all except finitely many $P \in \mathscr{H}$.*

Proof. Only finitely many $P \in \mathscr{H}$ such that $P \cap A = (0)$ fail to satisfy $\mathscr{C}_P$ is a regular local ring, as in the proof of (4.9). Also, by the proof that (3) $\Rightarrow$ (1) in (4.4), for all $P \in \mathscr{H}$ such that $P \cap A \neq (0)$ and $a \notin P$ and $d \notin P\mathscr{C}_a$, $\mathscr{C}_P$ is a regular local ring. The conclusion follows from this.

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