HOMOMORPHISMS OF MATRIX RINGS
INTO MATRIX RINGS

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Let $V_n(R_n)$ be the universal ring with respect to embeddings of the matrix ring R_n into $n \times n$ matrix rings over commutative rings. A construction and a representation is given for this ring. As a main tool in the construction, it is proved that every R homomorphism of R_n, R a commutative ring, is the restriction of an inner automorphism of U_n, for some $U \supseteq R$. Using this, a necessary and sufficient condition for n^2 matrices in R_n to be matrix units is given.

1. **Introduction and notations.** All rings to be considered in this paper, except those denoted specifically as matrix rings, will be commutative rings with unit. All homomorphisms are unitary. The unit of a subring coincides with the unit of its over-ring.

Denote by R_n the ring of $n \times n$ matrices over a ring R. Let $\eta: R \to S$ be a ring homomorphism then η induces a homomorphism $\eta_n: R_n \to S_n$ given by: $\eta_n(r_{ij}) = (\eta(r_{ij}))$. If $A \in R_n$, $(A)_{ij}$ will denote the (i,j)th entry of A. The identity element and the standard matrix units of all matrix rings will be denoted by I and $\{E_{ij}\}$ respectively.

Let A be an R algebra. It was proved by Amitsur ([1], Theorem 2) that there exists a commutative R algebra $V_m^R(A)$, and a map $\rho: A \to (V_m^R(A))_m$ which is universal for homomorphisms of A into $m \times m$ matrix rings over commutative rings, i.e.;

(1) For every $\tau: A \to H_m$, with H a commutative R algebra, there exists a homomorphism $\eta: V_m^R(A) \to H$ such that the following diagram is commutative;

$$
\begin{array}{ccc}
A & \xrightarrow{\rho} & (V_m^R(A))_m \\
& \tau \searrow & \downarrow \eta_m \\
& & H_m
\end{array}
$$

(2) $V_m^R(A)$ is generated over R by the entries $\{[\rho(a)]_{ij} \mid a \in A\}$.

Properties (1) and (2) determine $V_m^R(A)$ up to isomorphism and ρ up to a multiple by an isomorphism of $V_m^R(A)$.

In this paper we will give an explicit construction for the ring $V_m^R(R_n)$. The case $n = m$ will be treated separately. We start with investigating the nature of R-homomorphisms of R_n into itself.

2. On the automorphisms of matrix rings.

LEMMA 1. *Let C be a subdirectly irreducible commutative ring. C can be embedded in a local ring Q which is the complete ring of quotients of C. For noetherian C we may take $C = Q$.*

Proof. Let $\mathfrak{M}$ be the set zero divisors in C, then, by [6] $\mathfrak{M}$ is a maximal ideal in C. Let Q be the local ring $C_{\mathfrak{m}}$ and let f be the canonical homomorphism $f\colon C \to C_{\mathfrak{m}}$.

The elements of $C - \mathfrak{M}$ are not zero divisors in C, hence f is an injection. Furthermore, the elements of $C - \mathfrak{M}$ are exactly the regular elements of C, and so Q is the complete ring of quotients of C.

Now, if C is noetherian we have, by [3], $\mathfrak{M} = \mathfrak{N}(C)$-the nil radical of C, and therefore, $J(C) \subseteqq \mathfrak{M} = \mathfrak{N}(C) \subseteqq J(C)$ where $J(C)$ denotes the Jacobson radical of C. Hence $\mathfrak{M} = J(C)$ and being maximal it is the unique maximal ideal of C. Consequently C is local and $C = C_{\mathfrak{m}} = Q$.

THEOREM 2. *Let C and Q be as in the lemma; $\{E_{ij} \mid 1 \leqq i, j \leqq n\}$ be the set of the standard matrix units in C_n, and $\{F_{ij} \mid i \leqq i, j \leqq n\}$ another set of matrix units in C_n, then there exists a matrix $A \in C_n$, invertible in Q_n such that:*

$$1 \leqq i, j \leqq n \qquad E_{ij}A = AF_{ij} \, .$$

If C is noetherian A is invertible in C_n.

Proof. By definition $\sum_{\nu=1}^{n} F_{\nu\nu} = I$ hence $\sum_{\nu=1}^{n} (F_{\nu\nu})_{11} = (\sum_{\nu=1}^{n} F_{\nu\nu})_{11} = 1$. Now C is subdirectly irreducible, so that the zero divisors in C form an ideal. Consequently not all of the $(F_{\nu\nu})_{11}$ are zero divisors and we have some $1 \leqq \nu \leqq n$ such that $(F_{\nu\nu})_{11}$ is regular. Without loss of generality we may assume that $\alpha = (F_{11})_{11}$ is regular. Put now

$$A = \sum_{\nu=1}^{n} E_{\nu 1}F_{1\nu} \qquad B' = \sum_{\nu=1}^{n} F_{\nu 1}E_{1\nu}$$

than for all i and j we have $E_{ij}A = E_{i1}F_{1j} = AF_{ij}$ and also

$$AB' = \Big(\sum_{\nu=1}^{n} E_{\nu 1}F_{1\nu}\Big)\Big(\sum_{\mu=1}^{n} F_{\mu 1}E_{1\mu}\Big) = \sum_{\nu=1}^{n} E_{\nu 1}F_{11}E_{1\nu} = \alpha I \, .$$

Now α is regular in C and hence invertible in Q, thus $B = \alpha^{-1}B'$ is the inverse of A in Q_n. If C is noetherian then by Lemma 1 $C = Q$ and $B = A^{-1} \in C_n$.

Note that in a local ring the noninvertible elements form an ideal so that the proof of Theorem 2 can be easily modified to give

an elementary proof of the following well-known (e.g. [2]) theorem: "Let C be a local ring then every C-homomorphism of C_n is an inner automorphism".

THEOREM 3.   *Let C be a commutative ring with unit, E_{ij}, $F_{ij} \in C_n$ as above, then there exists a commutative ring U containing C and an invertible matrix $A \in U_n$ such that $E_{ij}A = AF_{ij}$ for all $1 \leqq i, j \leqq n$.*

Proof. C may be represented as a subdirect product of subdirectly irreducible rings ([4], Theorem 1, p. 219). There exists therefore a set of subdirectly irreducible rings with unit, $\{C^\gamma \mid \gamma \in \Gamma\}$ such that $C \subseteqq \prod_{\gamma \in \Gamma} C^\gamma \subseteqq \prod_{\gamma \in \Gamma} Q^\gamma = U$ where Q^γ is the complete ring of quotients of C^γ. Hence $C_n \subseteqq (\Pi C^\gamma)_n \subseteqq (\Pi Q^\gamma)_n = U_n$. Let $\pi^\gamma \colon \Pi Q^\gamma \to Q^\gamma$ be the canonical projection. Put $E_{ij}^\gamma = \pi_n^\gamma (E_{ij})$, $F_{ij}^\gamma = \pi_n^\gamma (F_{ij})$ then, by definition E_{ij}^γ are the standard matrix units in C^γ, F_{ij}^γ are another set of matrix units. By Theorem 2 it follows that there are invertible matrices $A^\gamma \in Q_n^\gamma$ such that $E_{ij}^\gamma A^\gamma = A^\gamma F_{ij}^\gamma$. Let $A \in (\Pi Q^\gamma)_n = U_n$ be defined by $((A)_{\nu\mu})_{(\gamma)} = (A^\gamma)_{\nu\mu}$, namely for every $\gamma \in \Gamma$ $\pi_n^\gamma(A) = A^\gamma$. Clearly A is invertible in U_m, its inverse being given by $((A^{-1})_{\nu\mu})_{(\gamma)} = ((A^\gamma)^{-1})_{\nu\mu}$. Clearly A satisfies $E_{ij}A = AF_{ij}$ for all i and j.

COROLLARY 4.   (1)   *For a given ring C there exists a ring $U \supseteqq C$ such that every C-homomorphism $\eta \colon C_n \to C_n$ can be extended to an inner automorphism of U_n.*

(2)   *Given η, the ring U of (1) may be chosen so that the inner automorphism will be given by a matrix of determinant 1.*

Proof.   (1)   follows immediately from Theorem 3 by taking $F_{ij} = \eta(E_{ij})$ then, by the theorem we have a ring U and a matrix $A \in U_n$ such that $A^{-1}BA = \eta(B)$, $B \in C_n$.

(2)   For a fixed η we adjoin to U the nth root of $a^{-1} = \det(A^{-1})$ and replace A by $a^{-1/n}A$.

REMARK.   The ring U of Theorem 3 is not uniquely determined. For example one may take $U = \Pi C_m$ where the product is taken over all maximal ideals in C. This ring will have the same property.

COROLLARY 5.   *Let C be a commutative ring with unit, $\{F_{ij} \mid 1 \leqq i, j \leqq n\}$ a set of matrix units in C_n then $\mathrm{Cent}_{C_n}(F_{ij})$-the centralizer of all the F_{ij} in C_n — is C, and every element in C_n may be written in a unique way as $\sum_{ij} c_{ij}F_{ij}$, $c_{ij} \in C$.*

Proof. By Theorem 3 $F_{ij} = AE_{ij}A^{-1}$ where A is in some larger matrix ring $U_n \supseteq C_n$. Hence; $\mathrm{Cent}_{C_n}(F_{ij}) = \mathrm{Cent}_{U_n}(F_{ij}) \cap C_n = A \mathrm{Cent}_{U_n}(E_{ij})A^{-1} \cap C_n = U \cap C_n = C$. The proof of the second part of the corollary is classical (e.g. [4], Proposition 6, p. 52).

3. **The ring $V_n^R(R_n)$.** Let R be a fixed ring with unit. All rings henceforth will be R-algebras and all homomorphisms R-homomorphisms. We shall write $V_n(A)$ for $V_n^R(A)$. We now proceed to give an explicit construction for the ring $V_n(R_n)$.

Let $\{x_{ij} \mid 1 \leq i, j \leq n\}$ be n^2 commutative indeterminates over R, let $R[x_{ij}]$ denote the ring of polynominals in the x_{ij} over R. Denote by $D \subseteq R[x_{ij}]$ the ideal generated in $R[x_{ij}]$ by the polynomial $\det(x_{ij}) - 1$, and put $K = R[x_{ij}]/D$. Clearly we may take $R \subseteq K$. Put $\xi_{ij} = x_{ij} + D \in K$, and $\Xi = (\xi_{ij}) \in K_n$. The matrix Ξ is invertible in K_n, and its inverse is given by $\Xi^{-1} = \mathrm{adj}\Xi = (\hat{\Xi}_{ji})$ where $\hat{\Xi}_{ij}$ is the algebraic complement of ξ_{ij} in Ξ.

Let S be the subalgebra of K generated over R by the n^4 elements $\{\xi_{ij}\hat{\Xi}_{kl} \mid 1 \leq i, j, k, l \leq n\}$. ($S$ contains the unit element of K for $\sum_{i=1}^n \xi_{1i}\hat{\Xi}_{1i} = \det(\xi_{ij}) = 1$.)

Define a map $\rho: R_n \to K_n$ by $\rho(B) = \Xi B \Xi^{-1}$. ρ is clearly a unitary R homomorphism, furthermore we have:

$$[\rho(E_{ij})]_{kl} = (\Xi E_{ij}\Xi^{-1})_{kl} = \sum_\mu \sum_\nu (\Xi)_{k\mu}(E_{ij})_{\mu\nu}(\Xi^{-1})_{\nu l}$$

$$= (\Xi)_{ki}(\Xi^{-1})_{jl} = \xi_{ki}\hat{\Xi}_{lj} \in S$$

so that $\rho(R_n) \subseteq S_n$ and we may regard ρ as a map from R_n to S_n and note that the entries $\{[\rho(A)]_{ij} \mid A \in R_n\}$ generate S. For this ring S and the homomorphism ρ we prove;

THEOREM 6. *S is the universal ring $V_n(R_n)$ and ρ is the canonical embedding of R_n in $S_n = (V_n(R_n))_n$.*

Proof. We have seen that S is generated by the appropriate elements so that all that remains to be shown is that every homomorphism $\tau: R_n \to C_n$ factors through ρ.

As C is an R algebra, we have the natural homomorphism $i: R \to C$. Denote $i(r) = r'$ $i_n(B) = B'$ for all $r \in R$, $B \in R_n$.

Let E_{ij} be the standard matrix units of R_n than E'_{ij} are the standard matrix units of C_n and $\tau(E_{ij}) = F_{ij}$ are a set of matrix units in C_n. By Theorem 3 there exists an R algebra $U \supseteq C$ and an invertible matrix $(a_{ij}) = A \in U_n$ such that $F_{ij} = AE'_{ij}A^{-1}$; furthermore, by Corollary 4 (2) we may suppose that $\det(A) = 1$. We clearly have for all $B \in R_n$ $\tau(B) = AB'A^{-1}$.

Define $\eta^\circ: R[x_{ij}] \to U$ by $\eta^\circ(x_{ij}) = a_{ij}$, then $\eta^\circ(\det(x_{ij}) - 1) = \det(\eta^\circ(x_{ij})) - 1 = \det(A) - 1 = 0$, and therefore η° induces a homomorphism $\bar{\eta}: K = R[x_{ij}]/D \to U$ such that $\bar{\eta}(\xi_{ij}) = a_{ij}$, and we have the map $\bar{\eta}_n: K_n \to U_n$ for which $\bar{\eta}_n(\varXi) = A$, $\bar{\eta}_n(\varXi^{-1}) = A^{-1}$ and $\bar{\eta}_n(B) = B'$ for $B \in R_n$. For all $B \in R_n$ we have:

$$\bar{\eta}_n\rho(B) = \bar{\eta}_n(\varXi B\varXi^{-1}) = \bar{\eta}_n(\varXi)\bar{\eta}_n(B)\bar{\eta}_n(\varXi^{-1}) = AB'A^{-1} = \tau(B) \text{ so that}$$
$\bar{\eta}_n\rho = \tau$. Let η be the restriction of $\bar{\eta}$ to S, then we have: $\rho(R_n) \subseteqq S_n$ and $\eta_n(S_n) \subseteqq C_n$. The last inclusion follows from the fact that for $B \in R_n$ $\eta([\rho(B)]_{ij} = [\eta_n\rho(B)]_{ij} = [\tau(B)]_{ij} \in C$ and since S is generated by the elements $[\rho(B)]_{ij}, \eta(S) \subseteqq C$. Consequently the following diagram is well defined and commutative.

$$
\begin{array}{ccc}
R_n & \xrightarrow{\rho} & S_n \\
& {}_{\tau}\searrow & \downarrow{}_{\eta_a} \\
& & C_n
\end{array}
$$
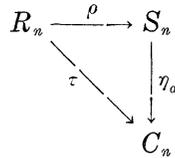
which completes the proof.

COROLLARY 7. (1) *If R is an integral domain (in particular $R = F$ a field) then so is $V_n(R_n)$.*

(2) *If R is noetherian so is $V_n(R_n)$.*

Proof. (1) If R is a domain then so is $R[x_{ij}]$ and the polynomial $\det(x_{ij}) - 1$ is prime in $R[x_{ij}]$. Hence, D is a prime ideal and $K = R[x_{ij}]/D$ is a domain. $V_n(R_n) = S \subseteqq K$ is hence also a domain.

(2) $S = V_n(R_n)$ is finitely generated over R (see also [1]).

4. **An alternative representation of $V_n(R_n)$.** $V_n(R_n)$ was shown to be generated by n^4 elements $\xi_{ki}\widehat{\varXi}_{lj}$. We aim now to describe the ring in terms of these generators and their relations. To this end we begin with a ring $R[z_{kl}^{ij}]$ with z_{kl}^{ij} n^4 commutative indeterminates over R. The elements z_{kl}^{ij} are to represent the generators $\xi_{ki}\widehat{\varXi}_{lj}$ and so they must satisfy the relations arising from the commutativity of the ξ_{ij} and from the fact that $\varXi\varXi^{-1} = \varXi^{-1}\varXi = I$, namely;

(1) $z_{kl}^{ij}z_{tr}^{sq} = z_{kr}^{iq}z_{tl}^{sj}$ for all i, j, k, l, s, q, t, r
(2) $\sum_{i=1}^n z_{kl}^{ii} = \delta_{kl}$ for all k, l
(3) $\sum_{k=1}^n z_{kk}^{ij} = \delta_{ij}$ for all i, j.

In fact we will show that the generators of $V_n(R_n)$ satisfy no other relations except these. This will be done by showing that $R[z_{kl}^{ij}]$ modulo those relations is again the universal ring $V_n(R_n)$.

We begin with conditions for matrices in a matrix ring to be matrix units.

THEOREM 8. *Let* $\{F^{ij} \mid 1 \leq i, j \leq n\}$ *be a set of* n^2 *matrices in a matrix ring* C_n. *The* $\{F^{ij}\}$ *are a set of matrix units in* C_n *if and only if they satisfy the following conditions:*

(1) $(F^{ij})_{kl}(F^{sq})_{tr} = (F^{iq})_{kr}(F^{sj})_{tl}$ for all i, j, k, l, s, q, t, r

(2) $\sum_{i=1}^{n} (F^{ii})_{kl} = \delta_{kl}$ for all k, l

(3) $\sum_{k=1}^{n} (F^{ij})_{kk} = \delta_{ij}$ for all i, j.

Proof. Suppose the F^{ij} are matrix units in C_n, then, by Theorem 3 we have $F^{ij} = AE_{ij}A^{-1}$ where $A \in U_n$, for some $U \cong C$. Put $A = (a_{ij})$ and $A^{-1} = (a'_{ij})$ and evaluate the left side of (1);

$$(F^{ij})_{kl}(F^{sq})_{tr} = (AE_{ij}A^{-1})_{kl}(AE_{sq}A^{-1})_{tr}$$

$$= \left(\sum_{\nu\mu} a_{k\nu}(E_{ij})_{\nu\mu}a'_{\mu l}\right)\left(\sum_{\pi\sigma} a_{t\pi}(E_{sq})_{\pi\sigma}a'_{\sigma r}\right)$$

$$= (a_{ki}a'_{jl})(a_{ts}a'_{qr}) \ .$$

While the right side of (1) gives:

$$(F^{iq})_{kr}(F^{sj})_{tl} = (AE_{iq}A^{-1})_{kr}(AE_{sj}A^{-1})_{tl}$$

$$= \left(\sum_{\nu\mu} a_{k\nu}(E_{iq})_{\nu\mu}a'_{\mu r}\right)\left(\sum_{\pi\sigma} a_{t\pi}(E_{sj})_{\pi\sigma}a'_{\sigma l}\right)$$

$$= (a_{ki}a'_{qr})(a_{ts}a'_{jl}) \ .$$

Which, by the commutativity in U proves (1). To prove (2) we have only to notice that $\sum_{i=1}^{n} (F^{ii})_{kl} = (\sum_{i=1}^{n} F^{ii})_{kl} = (I)_{kl} = \delta_{kl}$. Condition (3) states that $tr(F^{ij}) = \delta_{ij}$, now for $i \neq j$ we have

$$tr(F^{ij}) = tr(F^{ii}F^{ij}) = tr(F^{ij}F^{ii}) = tr(0) = 0 = \delta_{ij}$$

while for $i = j$ we have, using (1) and (2)

$$tr(F^{ii}) = 1 \cdot \sum_{k=1}^{n} (F^{ii})_{kk} = \left(\sum_{l=1}^{n} (F^{ll})_{ii}\right)\left(\sum_{k=1}^{n} (F^{ii})_{kk}\right)$$

$$= \sum_{l,k} (F^{li})_{ik}(F^{il})_{ki} = \sum_{l} (F^{ll})_{ii} = 1 = \delta_{ii} \ .$$

Conversely, suppose the F^{ij} satisfy conditions 1, 2, 3 than

$$[(F^{\nu\mu})(F^{\pi\sigma})]_{kl} = \sum_{t=1}^{n} (F^{\nu\mu})_{kt}(F^{\pi\sigma})_{tl} = \sum_{t=1}^{n} (F^{\nu\sigma})_{kl}(F^{\pi\mu})_{tt}$$

$$= (F^{\nu\sigma})_{kl}\sum_{t=1}^{n} (F^{\pi\mu})_{tt} = (F^{\nu\sigma})_{kl}\delta_{\pi\mu} = (\delta_{\pi\mu}F^{\nu\sigma})_{kl} \ .$$

This being true for all k, l we have $F^{\nu\mu}F^{\pi\sigma} = \delta_{\pi\mu}F^{\nu\sigma}$. We also have $(\sum_{i=1}^{n} F^{ii})_{kl} = \sum_{i=1}^{n} (F^{ii})_{kl} = \delta_{kl} = (I)_{kl}$ hence $\sum_{i=1}^{n} F^{ii} = I$ which concludes the proof that the F^{ij} are indeed matrix units. (A similar result on orthogonal idempotents was obtained in [7].)

We are finally in the position to present $V_n(R_n)$ in terms of

generators and relations. We start as before with the ring $R[z_{kl}^{ij}]$ and let J be the ideal generated by all the polynomials of the forms: $z_{kl}^{ij}z_{tr}^{sq} - z_{kr}^{iq}z_{tl}^{sj}$, $\sum_{i=1}^{n} z_{kl}^{ii} - \delta_{kl}$, $\sum_{k=1}^{n} z_{kk}^{ij} - \delta_{ij}$ where the indices range over all possible combinations. We denote by ζ_{kl}^{ij} the class $z_{kl}^{ij} + J$ in the quotient ring $S' = R[x_{kl}^{ij}]/J = R[\zeta_{kl}^{ij}]$. We define now a homomorphism $\rho': R_n \to S'_n$ by $[\rho'(A)]_{kl} = \sum_{i=n}^{n} \sum_{j=1}^{n} a_{ij}\zeta_{kl}^{ij}$.

The relations imposed on the ζ_{kl}^{ij} and Theorem 8 imply that the matrices $\rho'(E_{ij}) = (\zeta_{kl}^{ij})$ are clearly a set of matrix units in S' so that evidently ρ' is indeed an R-homomorphism.

THEOREM 9. *With these notations* $S' = V_n(R_n)$ *and* ρ' *is the canonical embedding of* R_n *in* $S'_n = (V_n(R_n))_n$.

Proof. We note that $[\rho'(E_{ij})]_{kl} = \zeta_{kl}^{ij}$ so that S' is generated by the appropriate elements. All that remains to be shown is the universal property of $(S'\rho')$. Let $\tau: R_n \to C_n$ be a homomophism, $F^{ij} = \tau(E_{ij})$ are then a set of matrix units in C_n. Define $\tilde{\eta}: R[z_{kl}^{ij}] \to C_n$ by $\tilde{\eta}(z_{kl}^{ij}) = [\tau(E_{ij})]_{kl} = (F^{ij})_{kl}$, by Theorem 8 $J \subseteq \operatorname{Ker} \tilde{\eta}$ and so $\tilde{\eta}$ induces a map $\eta: S' \to C$ for which $\eta(\zeta_{kl}^{ij}) = [\tau(E_{ij})]_{kl}$. It remains to show that $\eta_n\rho' = \tau$, and clearly it is enough to demonstrate the equality on the generators E_{ij} of R_n, indeed $[\eta_n\rho'(E_{ij})]_{kl} = \eta[[\rho'(E_{ij})]_{kl}] = \eta(\zeta_{kl}^{ij}) = [\tau(E_{ij})]_{kl}$. This being true for all k, l and for all i, j we have $\eta_n\rho' = \tau$ as required.

REMARK. By the uniqueness of $V_n(R_n)$ S and S' should be isomorphic. The isomorphism is given by the correspondence $\theta: \zeta_{kl}^{ij} \to \xi_{ki}\widehat{E}_{lj}$ and $\rho = \theta\rho'$.

5. **Embedding in matrix rings of different order.** In this section we investigate the homomorphisms of a matrix ring into matrix rings of higher orders. In particular we give a description of the ring $V_m(R_n)$ for all n and m.

If n and m are integers such that n/m we have an injection $\delta: C_n \to C_m$ which places an $n \times n$ matrix m/n-times along the diagonal of an $m \times m$ matrix. The combined map $R_n \xrightarrow{i_n} C_n \xrightarrow{\delta} C_m$ will be denoted by δ'.

Our first result is elementary:

LEMMA 10. *There exists a unitary R homomorphism* $\tau: R_n \to C_m$ *if and only if* n/m.

Proof. If n/m we have exhibited such an homomorphism, namely δ'. Conversely, suppose there exists a $\tau: R_n \to C_m$. Let $\mathfrak{M}$ be a

maximal ideal in C. We have the induced R homomorphism $\bar{\tau}: R_n \xrightarrow{\tau}$ $C_m \xrightarrow{\pi_m} (C/\mathfrak{M})_m = K_m$, therefore, without loss of generality we may assume $C = K$ a field. Let E_{ij} be the standard matrix units of R_n, then $\tau(E_{ij}) = f_{ij}$ are n^2 matrix units in K_m. We have $K_n \cong \sum_{ij} f_{ij} K \subseteqq K_m$ and $K_m \cong K_n \bigotimes_K \operatorname{Cent}_{K_m}(K_n)$ by taking dimension over k we clearly have n/m.

Consequently we can assert that $V_m(R_n) \neq \{0\}$ if and only if m/n, which we shall assume henceforth. We would like now to generalize Theorem 3 to the case of n^2 matrix units in a matrix ring of order m. Turning to subdirectly irreducible components does not seem to be very helpful and so we localize. Our next result is again not new. It was proved for example by Knus [5] in a more general setting. Our proof is rather elementary except for the use of the classical Skolem-Noether theorem.

THEOREM 11. *Let $(C, \mathfrak{M})$ be a local ring. $\{e_{ij}\}$ and $\{f_{ij}\}$ two sets of n^2 matrix units in C_m (n/m), there exists an invertible matrix $A \in C_m$ such that $f_{ij} = Ae_{ij}A^{-1}$ for $1 \leq i, j \leq n$.*

Proof. $\{\pi_m(e_{ij})\}$ and $\{\pi_m(f_{ij})\}$ are two sets of n^2 matrix units in a matrix ring $(C/\mathfrak{M})_m$ over a field. By the Skolem-Noether theorem there exists an invertible matrix $\bar{y} \in (C/\mathfrak{M})_m$ such that $\pi_m(f_{ij}) = \bar{y}\pi_m(e_{ij})\bar{y}^{-1}$. Let y be a matrix in C_m with $\pi_m(y) = \bar{y}$. $0 \neq \det(\pi_m(y)) = \pi(\det(y)) \in C/\mathfrak{M}$, therefore, $\det(y) \notin \mathfrak{M}$ is invertible in C and y is an invertible matrix in C_m. Put $A = \sum_{\nu=1}^{n} f_{\nu 1} y e_{1\nu}$ then we have

$$(1) \qquad 1 \leq i, j \leq n \qquad f_{ij}A = f_{i1}ye_{1j} = Ae_{ij}$$

furthermore:

$$\pi_m(A)\bar{y}^{-1} = \sum_{\nu=1}^{n} \pi_m(f_{\nu 1})\, \bar{y}\pi_m(e_{1\nu})\bar{y}^{-1} = \sum_{\nu=1}^{n} \pi_m(f_{\nu 1})\pi_m(f_{1\nu})$$

$$= \sum_{\nu=1}^{n} \pi_m(f_{\nu\nu}) = \pi_m(I) = I,$$

so $\pi_m(A)$ is invertible in $(C/\mathfrak{M})_m$ and hence, as above, A is invertible in C_m which, by (1), completes the proof of the theorem.

The next theorem can be proved with the help of Theorem 11 in the same way that we proved Theorem 3.

THEOREM 12. *Let C be a commutative ring with unit. $\{e_{ij}\}$ and $\{f_{ij}\}$ two sets of n^2 matrix units in C_m (n/m). There exists a commutative ring $U \supseteqq C$ and an invertible matrix $A \in U_m$ such that $e_{ij} = Af_{ij}A^{-1}$ for all $1 \leq i, j \leq n$. The ring U is independent of*

the $\{e_{ij}\}$ and $\{f_{ij}\}$. For fixed $\{e_{ij}\}$ we may choose the ring U and the matrix A in such a way that $\det(A) = 1$.

We can now show that $V_m(R_n)$ is a subalgebra of $V_m(R_m)$, more precisely:

THEOREM 13. Let $\rho: R_m \rightarrow (V_m^R(R_m))_m$ and $\delta: R_n \rightarrow R_m$ be the canonical maps. Then $V_m^R(R_n)$ is the subalgebra S, generated in $V_m^R(R_m)$ by the entries $\{[\rho\delta(B)_{ij} \mid B \in R_n\}$ and $\rho\delta$ is the corresponding canonical embedding.

Proof. The condition on the generators of S is fulfilled by definition so we have only to show the factoring property for maps. Let $\tau: R_n \rightarrow C_m$ be a homomorphism, by the usual reasoning we have a ring $U \supseteq C$ and a matrix $A \in U_m$ such that for $B \in R_n$ $\tau(B) = A\delta'(B)A^{-1}$ where δ' is as in Lemma 10. Define $\tau': R_m \rightarrow U_m$ by $\tau'(D) = AD'A^{-1}$, we have the following commutative diagram:

$$
\begin{array}{ccc}
R_n \xrightarrow{\ \partial\ } R_m \xrightarrow{\ \rho\ } (V_m(R_m))_m \\
\tau \big\downarrow \qquad \tau' \big\downarrow \qquad \bar{\eta}_m \\
C_m \lhook\joinrel\longrightarrow U_m
\end{array}
$$

The square on the left is commutative by the definitions, while $\bar{\eta}$ and the commutativity of the triangle are given by the universality of $V_m(R_m)$.

Define η to be the restriction of $\bar{\eta}$ to S. Then, for the generators of S we have $\eta([\rho\delta(B)]_{ij}) = [\bar{\eta}_m\rho\delta(B)]_{ij} = [\tau(B)]_{ij} \in C$. Therefore, $\eta(S) \subseteq C$ and the diagram

$$
\begin{array}{ccc}
R_n \xrightarrow{\ \rho\delta\ } S_m \\
\quad {\scriptstyle\tau} \searrow \quad \big\downarrow {\scriptstyle\eta_m} \\
C_m
\end{array}
$$

is well defined and commutative, which concludes the proof of the theorem.

REMARK. Obviously Corollary 7 may now be formulated for the ring $V_m^R(R_n)$.

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