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CONVERGENCE OF BAIRE MEASURES

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Assume that there are no measurable cardinals. Then E. Granirer has proved that if a net $\{m_i\}$ of finite Baire measures on a completely regular Hausdorff space converges weakly to a finite Baire measure m, then $\{m_i\}$ converges to m uniformly on each uniformly bounded, equicontinuous subset of C^b, the space of bounded continuous functions. In this paper a relatively simple proof of Granirer's theorem is given based on a recent result of the author. The same method is used to prove the following analogue of Granirer's theorem. Let $\{m_i\}$ be a net of Baire measures on X each having compact support in the realcompactification of the underlying space X, and assume that $\int_X fdm_i \to \int_X fdm$ for every continuous function f on X where m is a Baire measure having compact support in the realcompactification of X. Then $\{m_i\}$ converges to m uniformly on each pointwise bounded, equicontinuous subset of C, the space of continuous functions on X. (The situation in the presence of measurable cardinals is also treated.)

In what follows, X will denote a completely regular Hausdorff space, C will denote the linear space of all continuous real-valued functions on X and C^b will denote the subspace of C consisting of all the uniformly bounded functions in C. The *Baire algebra* is the smallest σ-algebra on X with respect to which each of the functions in C is measurable. (Equivalently, it is the σ-algebra generated by the zero sets in X.) The linear space of all signed Baire measures on X with finite variation is denoted by M_σ, and the set of nonnegative elements in M_σ (i.e., the set of finite Baire measures) is denoted by M_σ^+. The space M_σ and C^b may be paired in the sense of Bourbaki by the bilinear form $\langle m, f\rangle = \int_X fdm = \int_X fdm^+ - \int_X fdm^-$ for all $m \in M_\sigma$ and all $f \in C^b$. By the *weak topology* on M_σ, will we mean the topology $\sigma(M_\sigma, C^b)$.

Let νX denote the realcompactification of X. (See [2], p. 116.) A Baire measure m on X is said to have *compact support in the realcompactification of* X if there is a compact set $G \subset \nu X$ such that for every zero set Z in νX with $G \subset Z$, it follows that $m(X \cap Z) = m(X)$. Let M_c denote the subspace of M_σ consisting of those elements whose total variations have compact support in the realcompactification of X. The set of nonnegative elements of M_c is denoted by M_c^+. It is not hard to verify that if $m \in M_c^+$, then $C \subset L^1(m)$. Hence the spaces M_c and C may be paired in the sense of Bourbaki by the bilinear form $\langle m, f\rangle = \int_X f dm = \int_X f dm^+ - \int_X f dm^-$ for all $m \in M_c$ and all $f \in C$. By the *weak topology* on M_c, we will mean the topology $\sigma(M_c, C)$.

Let B be a subset of C. Then B is *pointwise bounded* if for every $x \in X$, $\sup\{|f(x)| : f \in B\} < \infty$. It is said to be *uniformly bounded*, if $\sup\{\|f\|_X : f \in B\} < \infty$ where $\|f\|_X = \sup\{|f(x)| : x \in X\}$. (Of course, if B is uniformly bounded, then $B \subset C^b$.) The set B is *equicontinuous* (or *locally equicontinuous*) if for every $x \in X$ and for every positive number ε, there is a neighborhood U of x such that for all $y \in U$ and all $f \in B$, $|f(x) - f(y)| \leqq \varepsilon$. Let $\mathscr{E}$ denote the family of all pointwise bounded, equicontinuous subsets of C; and let $\mathscr{E}^b$ denote the family of all uniformly bounded, equicontinuous subsets of C^b. It is clear that if $B \in \mathscr{E}^b$, then B is a $\sigma(C^b, M_\sigma)$-bounded and that $C^b = \bigcup\{B: B \in \mathscr{E}^b\}$. Hence it follows that the topology e^b of uniform convergence on the sets in $\mathscr{E}^b$ is a locally convex topology on M_σ which is compatible with the pair (M_σ, C^b). (See [7], p. 255.) It is also the case that if $B \in \mathscr{E}$, then B is a $\sigma(C, M_c)$-bounded subset of C. (This fact is proved in Proposition 2.2 below.) Since $C = \bigcup\{B: B \in \mathscr{E}\}$, it follows that the topology e of uniform convergence on the sets in $\mathscr{E}$ is a locally convex topology on M_c compatible with the pair (M_c, C).

Recall that a set Y has a *measurable cardinal* if there is a probability measure defined on the algebra of all subsets of Y which is zero on all singleton sets. Otherwise, Y is said to have a *nonmeasurable cardinal*. It is consistent with the standard axiomatic treatments of set theory to assume that all sets have nonmeasurable cardinals. It is also known that if the continuum hypothesis holds, then the continuum has a nonmeasurable cardinal. It is not known whether or not the statement that there are no measurable cardinals is independent of the axioms of set theory.

The completely regular Hausdorff space X is a *D-space* if whenever d is a continuous pseudometric on X and Y is a d-discrete subset of X, then Y has a nonmeasurable cardinal. The concept of a D-space was introduced by Granirer in [3]. From the remarks made above about measurable cardinals, it is clearly consistent with the usual axioms of set theory to assume that every completely regular Hausdorff space is a D-space. The following result is proved by Granirer. (See [3], Theorem 2.)

THEOREM A. *Let X be a completely regular Hausdorff space. Then X is a D-space if and only if whenever $\{m_i\}$ is a net in M_σ^+ which converges weakly to $m \in M_\sigma$, then $\{m_i\}$ converges to m for the topology e^b.*

We will present a relatively simple proof of this theorem based on Theorem 1.1 below which was recently obtained by the author. (In fact, our main result, Theorem 1.5, is somewhat stronger than Theorem A.) The advantage of our method is that it allows the analysis to be carried out for nets of measures with finite support and reduces the measure theory needed to a minimum. The same method yields a proof of the following result which we believe to be new.

THEOREM B. *Let X be completely regular Hausdorff. Then the following hold.*

1. *If X is a D-space, then whenever $\{m_i\}$ is a net in M_c^+ which converges weakly to m in M_c, it follows that $\{m_i\}$ converges to m for the topology e.*

2. *Assume the continuum hypothesis. If X is not a D-space, then there is a net $\{m_i\}$ in M_c^+ which converges weakly to some m in M_c such that $\{m_i\}$ is not convergent for the topology e.*

1. Weak convergence in M_σ. Let L denote the subspace of M_σ consisting of those elements whose total variations have finite support. Hence $m \in L$ if and only if there is a finite set $A \subset X$ such that $m(B) = 0$ for every Baire set B disjoint from A. Every element $m \in L$ has a unique extension to a finite signed measure $\bar{m}$ on the algebra of subsets of X. For each $m \in L$, let ξ be the real-valued function defined by $\xi(x) = \bar{m}(\{x\})$ for all $x \in X$. In this way the space L may be identified with the space of all real-valued functions on X which vanish on the complement of a finite subset of X. We will use this representation of L throughout the paper. For notational purposes, we will use ξ to denote a generic element of L. The restriction to L of the bilinear form pairing M_σ and C^b is given by $\langle \xi, f \rangle = \Sigma\{\xi(x)f(x): x \in X\}$ for all $\xi \in L$ and all $f \in C^b$. The set of non-negative functions in L will be denoted by L^+.

A Baire measure m on X is said to be *separable* if for every continuous pseudometric d on X, there is a d-closed set $Z \subset X$ such that $m(X - Z) = 0$ and such that Z is d-separable. (Since every d-closed set is a zero set in X, it follows that $m(X - Z)$ is defined.) An arbitrary element of M is separable if its total-variation is separable. Let M_s denote the subspace of M_σ consisting of the separable elements of M_σ. The space M_s was first introduced by Dudley in [1]. It can be shown that X is a D-space if and only if $M_s = M_\sigma$. (Indeed, if X is a D-space, then $M_s = M_\sigma$ is a consequence of Theorem III, p. 137 in [8]. On the other hand, if X is not a D-space, then there is a continuous pseudometric d on X and a d-discrete set $Y \subset X$ such that Y has a measurable cardinal. If μ is a nontrivial measure on the subsets of Y, let m be defined by $m(B) = \mu(B \cap Y)$ for every Baire set B in X. It is clear that $m \in M_\sigma$. However, m is not d-separable as can easily be seen so that $m \notin M_s$.) Hence, again it is consistent with the axioms of set theory to assume that $M_s = M_\sigma$ for all completely regular Hausdorff spaces. The following result was proved by the author in [6].

THEOREM 1.1. *Let X be a completely regular Hausdorff space and let M_s be equipped with the topology e^b of uniform convergence on the uniformly bounded, equicontinuous subsets of C^b. Then the following hold.*

1. *M_s is complete.*
2. *L is dense in M_s.*
3. *The dual space of M_s is C^b.*

We will require several results from the theory of measures on a topological space which we will now review briefly. (The reader is referred to [9] for further details.) Recall that a Baire measure m is *τ-additive* if whenever $\{Z_i: i \in I\}$ is a downward directed system of zero sets in X with $\bigcap \{Z_i: i \in I\} = \emptyset$, then $m(Z_i) \to 0$. (The family $\{Z_i: i \in I\}$ is *downward directed* if for each pair $i_1, i_2 \in I$, there is $i_3 \in I$ such that $Z_{i_3} \subset Z_{i_1} \cap Z_{i_2}$.) Equivalently, m is net-additive if for each upward directed system $\{U_i: i \in I\}$ of cozero sets (complements of zero sets) in X with $\bigcup \{U_i: i \in I\} = X$, then $m(U_i) \to m(X)$. The *support* of a Baire measure m is the set $\operatorname{supp} m = \bigcap \{Z: Z$ is a zero set in X and $m(X) = m(Z)\}$. If $\operatorname{supp} m = \emptyset$, then m is said to be *entirely without support*. The following result is proved in [5].

THEOREM 1.2. *Let m be a Baire measure on X. If m is not net-additive, then there is a Baire measure m' on X such that $0 < m' \leqq m$ and such that m' is entirely without support.*

If d is a continuous pseudometric on X, define an equivalence relation on X by $x \equiv y$ if $d(x, y) = 0$; and let X^* denote the set of equivalence classes. For $\bar{x}, \bar{y} \in X^*$, define $d^*(\bar{x}, \bar{y}) = d(x, y)$. Then (X^*, d^*) is a metric space which we will call the *metric space associated with d*. Let $Q: X \to X^*$ be the quotient map. Since Q is continuous, it follows that $Q^{-1}[B]$ is a Baire set in X whenever B is a Baire set in X^*. If m is a Baire measure on X, define $\bar{m}(B) = m(Q^{-1}[B])$ for every Baire set in X^*. Then $\bar{m}$ is a Baire measure on X^*. The following lemma is a consequence of Theorem 28 and Remark 4, p. 175 of Varadarajan in [9]. However, since the proof given below is essentially different, we will include it for the sake of completeness.

LEMMA 1.3. *Let d be a continuous pseudometric on X, and let*

m be a separable Baire measure on X. If $\{U_i: i \in I\}$ is a cover of X by d-open sets and if ε is an arbitrary positive number, then there is a finite set $\{i_1, \cdots, i_n\} \subset I$ such that $m(X - \bigcup_{k=1}^n U_{i_k}) \leqq \varepsilon$.

Proof. Let (X^*, d^*) be the metric space associated with d, and let $\bar{m}$ be the Baire measure on X^* corresponding to m. It will be sufficient to prove that $\bar{m}$ is net-additive on X^*. Indeed, assume that $\bar{m}$ is net-additive. Since U_i is d-open, $\bar{U}_i = Q[U_i]$ is open in X^*; and hence it is a cozero set in X^*. The family of all finite unions of the sets in $\{\bar{U}_i: i \in I\}$ is then an upward directed family of cozero sets whose union is X^*. But then there is a finite set $\{i_1, \cdots, i_n\} \subset I$ such that $m(X - \bigcup_{k=1}^n U_{i_k}) = \bar{m}(X^* - \bigcup_{k=1}^n \bar{U}_{i_k}) \leqq \varepsilon$ since $\bar{m}$ is net-additive.

We will now show that $\bar{m}$ is net-additive. If this is not the case, then by Theorem 1.2 there is a Baire measure μ on X^* such that $0 < \mu \leqq \bar{m}$ and such that μ is entirely without support. Then there is a separable Baire measure m_0 on X such that $m_0 \leqq m$ and such that $\bar{m}_0 = \mu$. Indeed, let $E = \{f \in C^b: f = f^* \circ Q$ for some $f^* \in C^b(X^*)\}$; and define $\varphi(f) = \int_{X^*} f^* d\mu$ for each $f \in E$ where $f = f^* \circ Q$. Then φ^* is a linear functional on the linear space E. Furthermore, φ^* is majorized on E by the subadditive functional p defined on C^b by $p(f) = \int_X f^+ dm$ for all $f \in C^b$. Hence by the Hahn-Banach theorem, there is a linear functional φ on C^b which extends φ^* and which is majorized by p on C^b. It is not difficult to verify that φ is nonnegative and satisfies the integral property. (A nonnegative functional φ on C^b satisfies the *integral property* if for every decreasing sequence $\{f_n\} \subset C^b$ such that $f_n \downarrow 0$ pointwise, it follows that $\varphi(f_n) \downarrow 0$.) It follows by the Alexandrov representation theorem (see Theorems 1.2 and 1.5 in [5]) that there is a Baire measure m_0 on X such that $\varphi(f) = \int_X f dm_0$ for all $f \in C^b$. It is clear that $m_0 \leqq m$ and that $\bar{m}_0 = \mu$ as claimed. (Note that since m is separable and since $m_0 \leqq m$, it follows that m_0 is also separable.)

Since $\bar{m}_0$ is entirely without support in X^*, there is for each $\bar{x} \in X^*$ an open set $U_{\bar{x}}$ in X^* with $\bar{m}_0(U_{\bar{x}}) = 0$. Since $\{U_{\bar{x}}: \bar{x} \in X^*\}$ is an open cover of X^* and since X^* is paracompact (being a metric space), there is a partition of unity $\{f_j^*: j \in J\}$ subordinate to the cover $\{U_{\bar{x}}: x \in X^*\}$. For each finite set $\tau \subset J$, define $f_\tau = \sum \{f_j^* \circ Q: j \in \tau\}$. Then $\{f_\tau\}$ is easily seen to be uniformly bounded and equicontinuous. Since the net $\{f_\tau\}$ converges to 1 pointwise, hence by Proposition 9.2 in [6], $\int_X f_\tau dm_0 \rightarrow m_0(X)$. On the other hand, since f_j^* has its support in $U_{\bar{x}}$ for some $\bar{x} \in X^*$, it follows that $\int_X f_j^* \circ Q dm_0 = \int_{X^*} f_j^* d\bar{m}_0 = 0$.

Thus $\int_X f_\tau dm_0 = 0$ for all τ. Thus $m_0(X) = \lim \int_X f_\tau dm_0 = 0$. This contradicts the fact that $m_0(X) = \bar{m}_0(X^*) = \mu(X^*) > 0$. The proof is complete.

We remark here that Lemma 1.3 is the only result from the theory of measures in a topological space which will be required in proof of Theorem 1.5 (the main result in this section). This theorem is somewhat stronger than Theorem A. A proof of Theorem A itself can be based on a result of Marczewski and Sikorski ([8], Theorem III) without reference to Lemma 1.3. (This result of Marczewski and Sikorski is also used by Granirer in his proof of Theorem B.)

For $\xi \in L$ and $W \subset X$, define the element $(\xi)_W \in L$ by $(\xi)_W(x) = \xi(x)$ for $x \in W$ and $(\xi)_W(x) = 0$ for $x \in X - W$. (That is, $(\xi)_W = \xi \cdot \mathscr{X}_W$ where $\mathscr{X}_W$ is the characteristic function of the set W. We can now prove the following.

PROPOSITION 1.4. *Let X be a completely regular Hausdorff space, and let $\{\xi_i : i \in I\}$ be a net in L^+. Assume that $\{\xi_i\}$ converges to $m \in M_s$ in the $\sigma(M_s, C^b)$-sense. Then $\{\xi_i\}$ converges to m in the e^b-sense.*

Proof. We will show that $\{\xi_i\}$ is an e^b-Cauchy net. The result will then be immediate from Theorem 1.1. Assume without loss of generality that $m \neq 0$. Fix a set $B \in \mathscr{E}^b$ and a positive number ε. For $x, y \in X$, define $d(x, y) = \sup\{|f(x) - f(y)| : f \subset B\}$. Then it is easily verified that since $B \in \mathscr{E}^b$, d is a continuous pseudometric on X. Since the net $\{\langle \xi_i, 1\rangle\}$ converges to $\langle m, 1\rangle$, we may assume without loss of generality that $P = \sup\{|\langle \xi_i, 1\rangle| : i \in I\}$ is finite. Let $M = \sup\{\|f\|_X : f \in B\}$ which is also finite since B is uniformly bounded.

Since d is continuous, there is for each $x \in X$ a d-open set U_x such that $d(x, y) \leqq \varepsilon P^{-1}$ for all $y \in U_x$. In particular, we then have,

$$(1) \qquad |f(x) - f(y)| \leqq \varepsilon P^{-1} \quad \text{for all} \quad y \in U_x, f \in B\,.$$

By Lemma 1.3 there is a finite set $\{x_1, \cdots, x_n\} \subset X$ such that $m(X - U) \leqq \varepsilon M^{-1}$ where $U = \bigcup_{k=1}^n U_{x_k}$. (Note that each U_x is a cozero set in X so that U is also a cozero set.) Since m is regular, there is a zero set Z in X such that $Z \subset U$ and such that $m(U - Z) \leqq \varepsilon M^{-1}$. Let $f_0 \in C^b$ be such that $0 \leqq f_0 \leqq 1, f_0 = 1$ on $X - U$ and $f_0 = 0$ on Z. Since $\{\xi_i\}$ converges to m weakly, there is an $i_1 \in I$ such that if $i \leqq i_1$, then $|\langle \xi_i - m, f_0\rangle| \leqq \varepsilon M^{-1}$. Since $\xi_i \geqq 0$, we have for $i \geqq i_1$,

$$\begin{aligned} 0 &\leqq \langle (\xi_i)_{X-U}, 1\rangle \leqq \langle \xi_i, f_0\rangle \\ &\leqq \langle \xi_i - m, f_0\rangle + \int_{X-Z} f_0 dm \leqq \varepsilon M^{-1} + m(X - Z) \\ &\leqq \varepsilon M^{-1} + m(X - U) + m(X - Z) \leqq 3\varepsilon M^{-1}\,. \end{aligned}$$

Thus we have demonstrated the following inequality which we note for future reference.

$$(2)\qquad \langle(\xi_i)_{X-U}, 1\rangle \leqq 3\varepsilon M^{-1}, \quad \text{for all} \quad i \geqq i_1 .$$

The set of vectors $K = \{(f(x_1), \cdots, f(x_n)) : f \in B\}$ is a totally-bounded set in R^n. Hence there is a finite set $A \subset B$ such that the set $K_A = \{(f(x_1), \cdots, f(x_n)) : f \in A\}$ is an εP^{-1}-net for K. Since $\{\xi_i\}$ is weakly convergent and since A is finite, there is an $i_2 \in I$ such that if $i, j \geqq i_2$, then

$$(3)\qquad |\langle \xi_i - \xi_j, f\rangle| \leqq \varepsilon, \quad \text{for all} \quad f \in A .$$

Let $i_0 \in I$ be greater than both i_1, i_2. Fix $i, j \geqq i_0$ and let $f \in B$. Choose $f^* \in A$ such that $|f(x_k) - f^*(x_k)| \leqq \varepsilon P^{-1}$ for all $k = 1, \cdots, n$. We then have by (2) and (3) that

$$\begin{aligned}
\langle \xi_i - \xi_j, f\rangle &= \langle \xi_i - \xi_j, f^*\rangle + \langle \xi_i - \xi_j, f - f^*\rangle \\
&\leqq \varepsilon + \langle(\xi_i - \xi_j)_{X-U}, |f - f^*|\rangle + \langle(\xi_i - \xi_j)_U, |f - f^*|\rangle \\
&\leqq \varepsilon + \langle(\xi_i)_{X-U}, |f| + |f^*|\rangle + \langle(\xi_j)_{X-U}, |f| + |f^*|\rangle \\
&\qquad + \langle(\xi_i - \xi_j)_U, f - f^*\rangle \\
&\leqq \varepsilon + 2M\langle(\xi_i)_{X-U}, 1\rangle + 2M\langle(\xi_j)_{X-U}, 1\rangle \\
&\qquad + \langle(\xi_i - \xi_j)_U, f - f^*\rangle \\
&\leqq \varepsilon + 2M(3\varepsilon M^{-1}) + 2M(3\varepsilon M^{-1}) + \langle(\xi_i - \xi_j)_U, f - f^*\rangle \\
&\leqq 13\varepsilon + \langle(\xi_i - \xi_j)_U, f - f^*\rangle .
\end{aligned}$$

Hence we have shown that,

$$(4)\qquad \langle \xi_i - \xi_j, f\rangle \leqq 13\varepsilon + \langle(\xi_i - \xi_j)_U, f - f^*\rangle, \quad \text{for all} \quad i, j \geqq i_0 .$$

Let $U_k = U_{x_k}$ for $k = 1, \cdots, n$ and let $U_0 = \varnothing$. By (1) and the fact that $|f(x_k) - f^*(x_k)| \leqq \varepsilon P^{-1}$ for all $k = 1, \cdots, n$, we have for $i \geqq i_0$ that

$$\begin{aligned}
&\langle(\xi_i)_U, |f - f^*|\rangle \\
&= \sum_{k=1}^{n} \langle(\xi_i)_{U_k - U_{k-1}}, |f - f^*|\rangle \\
&\leqq \sum_{k=1}^{n} \sum_{x \in U_k - U_{k-1}} \xi_i(x)\, |f(x) - f^*(x)| \\
&\leqq \sum_{k=1}^{n} \sum_{x \in U_k - U_{k-1}} \xi_i(x)\{|f(x) - f(x_k)| + |f(x_k) - f^*(x_k)| \\
&\quad + |f^*(x_k) - f^*(x)|\} \\
&\leqq 3\varepsilon P^{-1} \sum_{k=1}^{n} \sum_{x \in U_k - U_{k-1}} \xi_i(x) \leqq 3\varepsilon P^{-1}\langle \xi_i, 1\rangle \leqq 3\varepsilon .
\end{aligned}$$

That is, we have

$$(5) \qquad \langle (\xi_i)_U, f - f^* \rangle \leqq 3\varepsilon, \quad \text{for all} \quad i \geqq i_0 .$$

Combining (4) and (5), we obtain that for all $i, j \geqq i_0$ and all $f \in B$,

$$\langle \xi_i - \xi_j, f \rangle \leqq 13\varepsilon + \langle (\xi_i)_U, |f - f^*| \rangle + \langle (\xi_j)_U, |f - f^*| \rangle \leqq 19\varepsilon .$$

Since ε and B were arbitrary, it now follows that $\{\xi_i\}$ is an e^b-Cauchy net. The proof is complete.

THEOREM 1.5. *Let X be completely regular Hausdorff. Then the weak topology and the e^b-topology are identical on M_s^+.*

Proof. It is sufficient to show that if G is an e^b-closed set in M_s^+, then G is weakly closed in M_s^+. But this is immediate from Proposition 1.4 and the fact that L^+ is weakly dense in M_s^+. The proof is complete.

Theorem A now follows easily. Indeed, if X is a D-space, then $M_s = M_\sigma$ as noted above; and Theorem A reduces to Theorem 1.5. If X is not a D-space, then there is a Baire measure m with $m \in M_\sigma - M_s$. Since L^+ is weakly dense in M_σ, there is a net $\{\xi_i\}$ in L^+ which converges weakly to m. However, $\{\xi_i\}$ will not converge in the e^b-sense since otherwise $\{\xi_i\}$ would be an e^b-Cauchy net which would imply by Theorem 1.1 that $m \in M_s$.

In [3] Granirer proves the following as an application of Theorem A. It is also an immediate consequence of Theorem 1.1.

THEOREM 1.6. *Let X be a completely regular Hausdorff space. Then X is a D-space if and only if every uniformly bounded, equicontinuous subset of C^b is relatively $\sigma(M_\sigma, C^b)$-compact.*

Proof. If X is a D-space, then $M_\sigma = M_s$. Since by Theorem 1.1 the dual of M_σ with the topology e^b is C^b, it follows by the Banach-Alaoglu theorem that $B^{\circ\circ}$ is $\sigma(C^b, M_\sigma)$-compact whenever $B \in \mathscr{E}^b$. (Of course, $B^{\circ\circ}$ denotes the bipolar of B for the pair.) On the other hand, if X is not a D-space, then by Theorem 1.1, M_s is a proper closed subspace of M_σ for the topology e^b. Hence by the Hahn-Banach theorem, the dual space of M_σ for this topology is strictly larger than C^b. This implies by the Mackey-Arens theorem that there is a $B \in \mathscr{E}^b$ such that $B^{\circ\circ}$ is not $\sigma(C^b, M_\sigma)$-compact. But, as is easily verified, $B^{\circ\circ} \in \mathscr{E}^b$. This completes the proof.

2. Weak convergence in M_c. The following is proved in [6], Theorem 4.4. (The essence of the theorem is due to Hewitt in [4].)

THEOREM 2.1. *The order dual of C is isomorphic as a Riesz* *space to* M_c. *The isomorphism is given by* $\varphi \leftrightarrow m$ *where* $\varphi(f) = \int_X f dm$ *for all* $f \in C$. *In particular,* $C \subset L^1(m)$ *for all* $m \in M_c^+$.

We now prove the following as promised in the introduction.

PROPOSITION 2.2. *If* $B \in \mathscr{E}$, *then* B *is a* $\sigma(C, M_c)$*-bounded subset of* C.

Proof. Fix $m \in M_c^+$. It is sufficient to show that $\left\{\int_X |f| dm : f \in B\right\}$ is bounded. If this is not so, then there is a sequence $\{f_n\} \in B$ such that $\int_X |f_n| dm \to +\infty$. For each $n \in N$, define $g_n = \sup\{|f_k| : k = 1, \cdots, n\}$ and $g = \sup\{|f_k| : k \in N\}$. Then g is a real-valued, continuous function. Indeed, it is clear that g is real-valued since B is pointwise bounded. In order to see that g is continuous, fix $x \in X$ and $\varepsilon > 0$. Let U be a neighborhood of x such that $|f(x) - f(y)| \leqq \varepsilon/3$ for all $y \in U$ and all $f \in B$. We now claim that $|g(x) - g(y)| \leqq \varepsilon$ for all $y \in U$. Indeed, fix $y \in U$, and choose $k \in N$ so large that $|g(x) - g_k(x)| \leqq \varepsilon/3$ and $|g(y) - g_k(y)| \leqq \varepsilon/3$. Then there are $i, j \in \{1, \cdots, k\}$ such that $g_k(x) = |f_i(x)|$ and $g_k(y) = |f_j(y)|$. Hence we have that,

$$\begin{aligned} &|g(x) - g(y)| \\ &\quad \leqq |g(x) - g_k(x)| + |g_k(x) - g_k(y)| + |g_k(y) - g(y)| \\ &\quad \leqq 2\varepsilon/3 + ||f_i|(x) - |f_j|(y)| \\ &\quad \leqq 2\varepsilon/3 + \max\{||f_i|(x) - |f_i|(y)|, ||f_j|(x) - |f_j|(y)|\} \leqq \varepsilon . \end{aligned}$$

The proof is complete.

Define $M_{sc} = M_s \cap M_c$. If X is a D-space, then $M_{sc} = M_c$. On the other hand, if X is not a D-space, then for some continuous pseudometric d on X, there is a d-closed subset $Z \subset X$ with a measurable cardinal. It is known that if the continuum hypothesis holds and if Z has a measurable cardinal, then there is a probability measure on the algebra of all subsets of Z which is zero on all singletons and which assumes only the values 0 or 1. From this it follows that there is a point in νX such that the valuation functional on C corresponding to this point is represented (according to Theorem 2.1) by a nonseparable element of M_c. That is, M_{sc} is a proper subspace of M_c. In summary then, it follows that if the continuum hypothesis holds, then X is a D-space if and only if $M_{sc} = M_c$. The following result is proved in [6].

THEOREM 2.3. *Let* X *be completely regular Hausdorff, and let* M_{sc} *be equipped with the topology* e *of uniform convergence on the pointwise* *bounded, equicontinuous subsets of C. Then the following hold.*

1. *M_{sc} is complete.*
2. *The dual space of M_{sc} is C.*
3. *L is dense in M_{sc}.*

If X itself is realcompact, then obviously $M_{sc} = M_c$. Hence we have the following.

PROPOSITION 2.4. *Let X be realcompact, and let $\{\xi_i: i \in I\}$ be a net in L^+. If $\{\xi_i\}$ converges to $m \in M_c$ for the topology $\sigma(M_c, C)$, then $\{\xi_i\}$ converges to m for the topology e.*

Proof. We will show that $\{\xi_i\}$ is an e-Cauchy net. The result will then follow immediately from Theorem 2.3. Assume without loss of generality that $m \neq 0$. Fix a set $B \subset \mathscr{E}$ and a positive number ε. For $x, y \in X$, define $d(x, y) = \sup\{|f(x) - f(y)|: f \in B\}$. Since $B \in \mathscr{E}$, it follows that d is a continuous pseudometric on X. Let G be the support of m which is a compact subset of X by assumption. Let $M = \sup\{\|f\|_G: f \in B\}$ which is finite since $B \in \mathscr{E}$. For all $x \in X$, define $h(x) = d(x, G) = \inf\{d(x, y): y \in G\}$. Then h is an element of C. Since $\{\xi_i\}$ converges weakly to m, there is an $i_1 \in I$ such that $|\langle \xi_i - m, h\rangle| \leqq \varepsilon$ for all $i \geqq i_1$. But $h = 0$ on G so that $\langle m, h > = 0$. Hence we have that,

$$(1) \qquad |\langle \xi_i, h\rangle| \leqq \varepsilon, \quad \text{for all} \quad i \geqq i_1 .$$

Since the net $\{\langle \xi_i, 1\rangle\}$ converges to $\langle m, 1\rangle$, we may assume that $P = \sup\{|\langle \xi_i, 1\rangle|: i \in I\}$ is finite.

For each $x \in G$, let U_x be a cozero set neighborhood of x such that $|f(x) - f(y)| \leqq \varepsilon P^{-1}$ for all $y \in U_x$ and all $f \in B$. Since G is compact, there is a finite cover $\{U_{x_1}, \cdots, U_{x_n}\}$ of G. Define $U = U_{x_1} \cup \cdots \cup U_{x_n}$. The set of vectors $K = \{(f(x_1), \cdots, f(x_n)): f \in B\}$ is a totolly bounded subset of R^n. Let A be a finite subset of B such that the set $K_A = \{(f(x_1), \cdots, f(x_n)): f \in A\}$ is an εP^{-1}-net for K.

Since $\{\xi_i\}$ is weakly convergent and since A is finite, there is an $i_2 \in I$ such that,

$$(2) \qquad |\langle \xi_i - \xi_j, f\rangle| \leqq \varepsilon, \quad \text{for all} \quad i, j \geqq i_2 \quad \text{and all} \quad f \in A .$$

Finally, as in the proof of Proposition 1.4, there is an $i_3 \in I$ such that,

$$(3) \qquad \langle (\xi_i)_{X-U}, 1\rangle \leqq \varepsilon M^{-1}, \quad \text{for all} \quad i \geqq i_3 .$$

Now let $i_0 \in I$ be greater than i_1, i_2, and i_3. Fix $i, j \geqq i_0$ and let $f \in B$. Choose $f^* \in A$ such that $|f(x_k) - f^*(x_k)| \leqq \varepsilon P^{-1}$ for $k = 1, \cdots, n$. We then have from (2) that,

$$(4)\quad \begin{aligned}\langle \xi_i - \xi_j, f\rangle &= \langle \xi_i - \xi_j, f^*\rangle + \langle \xi_i - \xi_j, f - f^*\rangle \\ &\leqq \varepsilon + \langle (\xi_i - \xi_j)_{X-U}, |f - f^*|\rangle + \langle (\xi_i - \xi_j)_U, |f - f^*|\rangle .\end{aligned}$$

However, for $i \geqq i_0$, letting $U_k = U_{x_k}$ for $k = 1, \cdots, n$ and $U_0 = \varnothing$,

$$\begin{aligned}&\langle (\xi_i)_U, |f - f^*|\rangle \\ &= \sum_{k=1}^{n} \sum_{x \in U_k - U_{k-1}} \xi_i(x)\, |f(x) - f^*(x)| \\ &\leqq \sum_{k=1}^{n} \sum_{x \in U_k - U_{k-1}} \xi_i(x)\{|f(x) - f(x_k)| + |f(x_k) - f^*(x_k)| + |f^*(x_k) - f^*(x)|\} \\ &\leqq 3\varepsilon P^{-1} \sum_{k=1}^{n} \sum_{x \in U_k - U_{k-1}} \xi_i(x) \leqq 3\varepsilon P^{-1}\langle \xi_i, 1\rangle \leqq 3\varepsilon .\end{aligned}$$

Thus we have shown that,

$$(5)\qquad \langle (\xi_i)_U, |f - f^*|\rangle \leqq 3\varepsilon, \quad \text{for all} \quad i \geqq i_0 .$$

Note that if $f \in B$, then $|f| \leqq h + M$. Hence for $i \geqq i$, we have from (1) and (3), that

$$\begin{aligned}&\langle (\xi_i)_{X-U}, |f - f^*|\rangle \\ &\quad \leqq 2\langle (\xi_i)_{X-U}, h + M\rangle \leqq 2\{\langle \xi_i, h\rangle + M\langle (\xi_i)_{X-U}, 1\rangle\} \\ &\quad \leqq 2\{\varepsilon + M\varepsilon M^{-1}\} \leqq 4\varepsilon .\end{aligned}$$

Thus we have shown that,

$$(6)\qquad \langle (\xi_i)_{X-U}, |f - f^*| > \leqq 4\varepsilon, \quad \text{for all } i \geqq i_0 .$$

Combining (4), (5), and (6), we have for $i, j \geqq i_0$ that,

$$\begin{aligned}\langle \xi_i - \xi_j, f\rangle &\leqq \varepsilon + \langle (\xi_i)_{X-U}, |f - f^*|\rangle + \langle (\xi_j)_{X-U}, |f - f^*|\rangle \\ &\quad + \langle (\xi_i)_U, |f - f^*|\rangle + |\langle (\xi_j)_{X-U}, |f - f^*|\rangle \leqq 15\varepsilon .\end{aligned}$$

It follows that $\{\xi_i\}$ is an e-Cauchy net as claimed. The proof is complete.

PROPOSITION 2.5. *Let X be a D-space. Then every continuous pseudometric on X has a (unique) extension to a continuous pseudometric on νX.*

Proof. Let $\tilde{X}$ denote the completion of X for the finest uniform structure on X compatible with the topology on X. Denote this structure by $\mathscr{U}^f$. Then every continuous pseudometric on X has a unique extension to $\tilde{X}$ since the set of all such pseudometrics is a gauge for this uniformity. The proof will be complete if we show that $\tilde{X} = \nu X$. But since every continuous real-valued function on X is $\mathscr{U}^f$-uniformly continuous, it follows that X is C-embedded in $\tilde{X}$.

Hence $\nu\tilde{X} = \nu X$ (by [2], Theorem 8.6). If we can show that $\tilde{X}$ is a D-space, then by Shirota's theorem ([2], p. 229), it will follow that $\tilde{X} = \nu\tilde{X}$; and the proof will be complete.

Assume that $\tilde{X}$ is not a D-space. Then there is a continuous pseudometric $\tilde{d}$ on $\tilde{X}$ and a $\tilde{d}$-closed, discrete subset $\tilde{Z}$ of $\tilde{X}$ which has a measurable cardinal. Let d denote the restriction of $\tilde{d}$ to X. For each $x \in \tilde{Z}$, define $0 < \alpha(x) = \inf\{\tilde{d}(x, y): y \in \tilde{Z} \text{ and } x \neq y\}$. Since X is dense in $\tilde{X}$, there is for each point $x \in \tilde{Z}$ a point $\psi(x) \in X$ such that $\tilde{d}(x, \psi(x)) \leqq \alpha(x)/3$. (Such a function exists by the axiom of choice.) Then the set $Z = \{\psi(x): x \in \tilde{Z}\}$ is a d-discrete subset of X. Since ψ is clearly one-to-one, Z also has a measurable cardinal. But this contradicts the assumption that X is a D-space. The proof is complete.

We note that the fact $\tilde{X}$ is a D-space in the above proof is a special case of Remark 2, p. 11 in [3]. For $f \in C$, let $\bar{f}$ denote the unique continuous extension of f to νX. If B is a subset of C, let $\bar{B} = \{\bar{f}: f \in B\}$. We then have the following.

PROPOSITION 2.6. *Let X be a D-space. If B is a pointwise bounded and equicontinuous subset of $C(X)$, then $\bar{B}$ is a pointwise bounded and equicontinuous subset of $C(\nu X)$.*

Proof. For each pair $x, y \in X$, define $d(x, y) = \sup\{|f(x) - f(y)|: f \in B\}$. Since B is pointwise bounded and equicontinuous on X, it follows that d is a continuous pseudometric on X. By Proposition 2.5 there is an unique continuous extension $\tilde{d}$ of d to νX. It then follows that for all $x, y \in \nu X$ and for all $f \in B$, $|\bar{f}(x) - \bar{f}(y)| \leqq \tilde{d}(x, y)$. But this implies that $\bar{B}$ is equicontinuous and pointwise bounded on νX. The proof is complete.

THEOREM 2.7. *Let X be a D-space, and let $\{m_i\}$ be a net in M_c^+. If $\{m_i\}$ converges weakly to $m \in M_c$, then $\{m_i\}$ converges to m for the topology e.*

Proof. Since L^+ is weakly dense in M_c, it is sufficient to show that if $\{\xi_i\}$ is a net in L^+ which converges weakly to $m \in M_c$, then $\{\xi_i\}$ converges for the topology e. Hence fix $B \in \mathscr{E}$. For $f \in C$, let $\bar{f}$ be its extension to νX; and let $\bar{B} = \{\bar{f}: f \in B\}$ as above. For each $m \in M_c(X)$ and for each $f \in C$, define $\varphi(\bar{f}) = \int_X f dm$. Then by Theorem 2.1, there is an $\bar{m} \in M_c(\nu X)$ such that $\bar{\varphi}(\bar{f}) = \int_{\nu X} \bar{f} d\bar{m}$ for all $f \in C(X)$. Since $\{\xi_i\}$ converges to m for the $\sigma(M_c(X), C(X))$ topology, it follows that $\{\bar{\xi}_i\}$ converges to $\bar{m}$ for the $\sigma(M_c(\nu X), C(\nu X))$ topology. Since B is pointwise bounded and equicontinuous on X, it follows by Proposition 2.6 that $\bar{B}$ is pointwise bounded and equicontinuous on νX. Since νX is realcompact, it follows from Proposition 2.4 that $\{\bar{\xi}_i\}$ converges to $\bar{m}$ uniformly over $\bar{B}$. But it is then immediate that $\{\xi_i\}$ converges to m uniformly over B. The proof is complete.

Theorem B now follows easily. Indeed, if X is a D-space, then it reduces to Theorem 2.7. On the other hand, assume that X is not a D-space. As we have noted above, if the continuum hypothesis holds, it follows that M_{sc} is a proper subspace of M_c. Let $m \in M_c^+ - M_{sc}^+$. Since L^+ is weakly dense in M_c^+, there is a net $\{\xi_i\}$ in L^+ which converges weakly to m. However, by Theorem 2.3, M_{sc} is complete for the topology e so that $\{\xi_i\}$ does not converge for the topology e. The proof is complete.

We can also prove the following analogue of Theorem 1.6 (Granirer's Theorem 1).

THEOREM 2.8. *Let X be completely regular Hausdorff. Then the following hold.*

1. *If X is a D-space, then every pointwise bounded, equicontinuous subset of C is relatively $\sigma(C, M_c)$-compact.*

2. *Assume the continuum hypothesis. If X is not a D-space, then there is a pointwise bounded, equicontinuous subset of C which is not relatively $\sigma(C, M_c)$-compact.*

Proof. 1. If X is a D-space, $M_{sc} = M_c$. Hence $B^{\circ\circ}$ is $\sigma(C, M_c)$-compact for every $B \in \mathscr{E}$ by Theorem 2.3 and the Banach-Alaoglu theorem.

2. If X is not a D-space, then the continuum hypothesis implies that M_{sc} is a proper subspace of M_c. By Theorem 2.3, M_{sc} is a closed subspace for the topology e. It follows by the Hahn-Banach theorem that the dual space of M_c for the topology e is then strictly larger than C. Hence by the Mackey-Arens theorem, there is a $B \in \mathscr{E}$ for which $B^{\circ\circ}$ is not $\sigma(C, M_c)$-compact. But, as is easily verified, if $B \in \mathscr{E}$, then $B^{\circ\circ} \in \mathscr{E}$. The proof is complete.

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