PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 2, 1973

ON THE LATTICE OF PROXIMITIES OF ČECH COMPATIBLE WITH A GIVEN CLOSURE SPACE

W. J. THRON AND R. H. WARREN

Let (X, c) be a Čech closure space. By $\mathfrak{M}$ we denote the family of all proximities of Čech on X which induce c. $\mathfrak{M}$ is known to be a complete lattice under set inclusion as ordering. The analogue of the R_0 separation axiom as defined for topological spaces is introduced into closure spaces. R_0-closure spaces are exactly those spaces for which $\mathfrak{M} \neq \phi$. Other characterizations for R_0-closure spaces are presented. The most interesting one is: every R_0-closure space is a subspace of a product of a certain number of copies of a fixed R_0-closure space. A number of techniques for constructing elements of $\mathfrak{M}$ are developed. By means of one of these constructions, all covers of any member of $\mathfrak{M}$ can be obtained. Using these constructions the following structural properties of $\mathfrak{M}$ are derived: $\mathfrak{M}$ is strongly atomic, $\mathfrak{M}$ is distributive, $\mathfrak{M}$ has no antiatoms, $|\mathfrak{M}| = 0, 1$ or $|\mathfrak{M}| \geqq 2^{2^{\aleph_0}}$.

1. Introduction.

E. Čech in [2] has studied a basic proximity structure (see Definition 1.3). The closure operator induced by such a structure is in general not a Kuratowski closure operator, since it may fail to satisfy the condition $c(c(A)) \subset c(A)$, however it satisfies the other three conditions and thus (X, c) is a closure space (Definition 1.1). Since Čech called his basic proximity just a "proximity" and since this term is commonly used to denote a proximity of Efremovič, we shall refer to the basic proximities of Čech as $\check{C}$-proximities. We did not wish to use the name "Čech proximity" because this term already has another meaning in the literature [2, p. 447].

This paper is primarily concerned with a study of the order structure of the family $\mathfrak{M}$ of all $\check{C}$-proximities which induce the same closure operator on a given set. Čech [2] proved that $\mathfrak{M}$ is a complete lattice. He characterized least upper bounds in $\mathfrak{M}$, the least and greatest elements in $\mathfrak{M}$, and those closure spaces for which $\mathfrak{M} \neq \phi$.

The symbol $\mathscr{P}(X)$ denotes the power set of X, $|A|$ indicates the cardinal number of the set A, the triple bar $\equiv$ is reserved for definitions and $\square$ signals the end of a proof.

DEFINITION 1.1. [2, p. 237] Let X be a set. A function $c: \mathscr{P}(X) \to \mathscr{P}(X)$ is called a *Čech closure operator* on X iff it satisfies the following three axioms:

C1: $\bar{\phi} = \phi$;

C2: for every $A \subset X$, $A \subset \bar{A}$;

C3: for all $A, B \subset X$, $\bar{A} \cup \bar{B} = \overline{A \cup B}$.

In stating these axioms, we have denoted $c(A)$ by $\bar{A}$. We shall also use this notation in the following material, since from the context one can determine whether $\bar{A}$ denotes a topological closure or a Čech closure. The pair (X, c) is called a *closure space*. We note that C1, C2, C3 are three of the four Kuratowski closure axioms. The fourth is: for every $A \subset X$, $\bar{\bar{A}} \subset \bar{A}$.

DEFINITION 1.2. [2, p. 270] Let (X, c), (Y, d) be closure spaces and let $f: X \to Y$. Then f is *continuous* iff, given $A \subset X$, it follows that $f(\bar{A}) \subset \overline{f(A)}$.

DEFINITION 1.3. [2, p. 439] A relation $\mathscr{P}$ on $\mathscr{P}(X)$ is said to define a *Č-proximity* on a set X iff it satisfies the conditions:

P1: $(A, B) \in \mathscr{P}$ implies $(B, A) \in \mathscr{P}$;

P2: $(A, B \cup C) \in \mathscr{P}$ iff $(A, B) \in \mathscr{P}$ or $(A, C) \in \mathscr{P}$;

P3: $(\phi, A) \notin \mathscr{P}$ for every $A \subset X$;

P4: $A \cap B \neq \phi$ implies $(A, B) \in \mathscr{P}$.

We now list a number of basic results about Č-proximities which were established by Čech. Let $\mathscr{P}$ be a Č-proximity on X. The function $c = c(\mathscr{P}): \mathscr{P}(X) \to \mathscr{P}(X)$ defined by $x \in \bar{A} = c(A)$ iff $([x], A) \in \mathscr{P}$ is a Čech closure operator which satisfies: $x \in [\bar{y}]$ implies $y \in [\bar{x}]$. We say that $\mathscr{P}$ induces c or that $\mathscr{P}$ is compatible with c. More generally, for a relation $\mathscr{S}$ on $\mathscr{P}(X)$, we say that $\mathscr{S}$ induces c if for each $A \subset X$, $c(A) = [x: ([x], A) \in \mathscr{S}]$. If (X, c) is a closure space satisfying $x \in [\bar{y}]$ implies $y \in [\bar{x}]$, then

$$\mathscr{R} \equiv [(A, B): (\bar{A} \cap B) \cup (A \cap \bar{B}) \neq \phi]$$

is a Č-proximity on X compatible with c. Let $\mathfrak{M} = \mathfrak{M}(X, c)$ be the family of all Č-proximities on X which induce c. If $[\mathscr{P}_i: i \in I] \subset \mathfrak{M}$, then $\bigcup [\mathscr{P}_i: i \in I] \in \mathfrak{M}$. Let $\mathfrak{M}$ be partially ordered by set inclusion. Then $\mathfrak{M}$ has a least element $\mathscr{R}$ (defined above) and a greatest element

$$\mathscr{W} \equiv \mathscr{R} \cup [(A, B): A \text{ and } B \text{ are infinite subsets of } X] .$$

It then follows [4, pp. 7–10] that $\mathfrak{M}$ is a complete lattice with the operator $\vee = \cup$.

The following definitions will be useful in describing some of our results in this paper.

DEFINITION 1.4. Let $(L, \leqq)$ be a partially ordered set. If $a, b \in L$, we say *a covers b* or *b is covered by a* when $a > b$ and $a > c > b$ is not satisfied for any $c \in L$. Moreover, $(L, \leqq)$ is said to be *covered* iff, given $x \in L$ such that there is $y \in L$ satisfying $y > x$, then there is $z \in L$ which covers x and satisfies $z \leqq y$. Also $(L, \leqq)$ is said to be *anticovered* iff the dual of $(L, \leqq)$ is covered.

DEFINITION 1.5. Let $(L, \leqq)$ be a partially ordered set. If $(L, \leqq)$ has a least element $\mathscr{o}$, then $a \in L$ is an *atom* iff a covers $\mathscr{o}$. Also $c \in L$ is an *antiatom* iff c is an atom in the dual of $(L, \leqq)$. Furthermore, $(L, \leqq)$ is called *atomic* when each $x \in L$, x not the least element, is the least upper bound of the atoms $\leqq x$. Moreover, $(L, \leqq)$ is called *strongly atomic* iff, given $a \in L$, the partially ordered set $[b: a \leqq b \in L]$ is atomic. Also $(L, \leqq)$ is *antiatomic* iff the dual of $(L, \leqq)$ is atomic.

We note that if $(L, \leqq)$ is strongly atomic and has a least element, then $(L, \leqq)$ is atomic. Also if $(L, \leqq)$ is strongly atomic, then $(L, \leqq)$ is covered. However, if $(L, \leqq)$ is covered, then $(L, \leqq)$ may not be atomic or strongly atomic. To verify the last statement, let N be the set of natural numbers and define $a \leqq b$ iff a divides b. To see that $(N, \leqq)$ is not atomic, we observe that the only atom $\leqq 4$ is 2. To see that $(N, \leqq)$ is covered, let a properly divide b. Then there is prime p such that ap divides b. Thus ap covers a.

DEFINITION 1.6. A lattice $(L, \vee, \wedge)$ is *infinitely meet distributive* iff, given nonempty $B \subset L$ and $a \in L$, then $a \wedge (\bigvee B) = \bigvee [a \wedge b: b \in B]$.

DEFINITION 1.7. A lattice $(L, \vee, \wedge)$ with least element $\mathscr{o}$ and greatest element $\mathscr{h}$ is said to be *complemented* iff, for each $x \in L$, there is $y \in L$ such that $x \vee y = \mathscr{h}$ and $x \wedge y = \mathscr{o}$.

2. R_0-closure spaces.

Since a closure space (X, c) has a compatible $\check{C}$-proximity iff $x \in [\bar{y}]$ implies $y \in [\bar{x}]$, it seems appropriate to give this condition a name. Moreover, a topological space is R_0 iff this condition is satisfied [3, p. 106].

DEFINITION 2.1. Let (X, c) be a closure space. We say that (X, c) is R_0 iff, given x, y in X such that $x \in [\bar{y}]$, then $y \in [\bar{x}]$.

Clearly, every R_0-topological space is an R_0-closure space. The following example of an R_0-closure space, which is not a topological space, will be useful in the sequel.

EXAMPLE 2.1. Let $S = [r, s, t]$ and let $d: \mathscr{P}(S) \rightarrow \mathscr{P}(S)$ be defined by:

$d(\phi) = (\phi)$,
$d([r]) = d(S) = d([r, s]) = d([r, t]) = d([s, t]) = S$,
$d([s]) = [r, s]$ and
$d([t]) = [r, t]$.

THEOREM 2.1. *Let (X, c) be a closure space. Then the following are equivalent*:

(a) (X, c) *is* R_0.

(b) *There is a* $\check{C}$*-proximity on* X *which induces* c, i.e., $\mathfrak{M} \neq \phi$.

(c) *There is a semi-uniformity on* X *which induces* c.

(d) *Given* $A \subset X$ *and* $x \notin \bar{A}$, *then* $[\bar{x}] \cap A = \phi$; i.e., *each subset of* X *is separated from the points excluded from its closure.*

(e) *Given* $A \subset X$ *and* $x \in (X - \bar{A})$, *then* $[\bar{x}] \subset (X - A)$; i.e., *each subset of* X *contains the closure of the points in its interior.*

(f) (X, c) *is homeomorphic to a subspace of a product of spaces* (S, d) *given in Example* 2.1.

Proof. In [2] it is shown that (a), (b), and (c) are equivalent, although the name R_0 is not used. The proof that (a), (d), and (e) are equivalent is straightforward and therefore is omitted.

(a) $\Rightarrow$ (f). By Theorem 17 C.17 in [2], it suffices to show that there is a family $[f_\alpha: X \to S]$ such that:

(1) Each f_α is continuous.

(2) The family distinguishes points.

(3) If $x \in X$ and $A \subset X$ such that $x \notin \bar{A}$, then there is an α such that $f_\alpha(x) \notin \overline{f_\alpha(A)}$.

To form such a family, if $A, B \subset X$ and $(\bar{A} \cap B) \cup (A \cap \bar{B}) = \phi$, then we define $g: X \to S$ by

$$g(x) = \begin{cases} r & \text{if} \quad x \in X - (A \cup B) \\ s & \text{if} \quad x \in A \\ t & \text{if} \quad x \in B\,. \end{cases}$$

To verify that g is continuous, it suffices to show that if $C \subset X$ and $\overline{g(C)} \neq S$, then $g(\bar{C}) \subset \overline{g(C)}$. If $g(C) = [s]$, then $C \subset A$ and $\bar{C} \subset \bar{A}$. Since $\bar{A} \cap B = \phi$, it follows that $g(\bar{C}) \subset [r, s] = \overline{g(C)}$. Similarly, if $g(C) = [t]$, then $g(\bar{C}) \subset \overline{g(C)}$.

If $y, z \in X$, $y \neq z$ and $y \in [\bar{z}]$, then we define $h: X \to S$ by

$$h(x) = \begin{cases} r & \text{if} \quad x \neq z \\ s & \text{if} \quad x = z\,. \end{cases}$$

To see that h is continuous, it suffices to show that if $C \subset X$ and $\overline{h(C)} \neq S$, then $h(\bar{C}) \subset \overline{h(C)}$. So we consider $h(C) = [s]$. Then $C = [z]$ and $h(\bar{C}) = [r, s] = \overline{h(C)}$.

We define the family $[f_\alpha]$ to consist of all those maps g, h which we have specified above.

To verify (2), let $y, z \in X$ and $y \neq z$. If $y \in [\bar{z}]$, then there is a map h which distinguishes y and z. If $y \notin [\bar{z}]$, then, since (X, c) is R_0, $z \notin [\bar{y}]$ and there is a map g which distinguishes y and z.

To verify (3), let $x \notin \bar{A}$. Given $a \in A$, then $[\bar{a}] \subset \bar{A}$ and $x \notin [\bar{a}]$. Since (X, c) is R_0, $a \notin [\bar{x}]$. Therefore, $[\bar{x}] \cap A = \phi$ and there is a map g such that $g(x) \notin \overline{g(A)}$.

(f) $\Rightarrow$ (a). Since products (Theorem 23 D.11 in [2]), subspaces and homeomorphic images of R_0-closure spaces are R_0-closure spaces, the result follows. □

It is well known that in a topological space $(X, \mathscr{T})$ the R_0-axiom is equivalent to each of the following statements: given $x, y \in X$, then $[\bar{x}] = [\bar{y}]$ or $[\bar{x}] \cap [\bar{y}] = \phi$; or, given $x \in G \in \mathscr{T}$, then $[\bar{x}] \subset G$. However, these statements are not equivalent to the R_0-axiom for closure spaces.

THEOREM 2.2. *Let (X, c) be a closure space. If $[\bar{x}] = [\bar{y}]$ or $[\bar{x}] \cap [\bar{y}] = \phi$ for all x, y in X, then (X, c) is R_0; but the converse is false.*

Proof. The proof of the positive assertion is straightforward and therefore is omitted. The converse fails in the R_0-closure space given in Example 2.1.

THEOREM 2.3. *Let (X, c) be a closure space. If (X, c) is R_0, then each open set contains the closure of each of its points; but the converse is false.*

Proof. The positive assertion is easily established. To see that the converse is false, consider the following example: Let $X = [a, b, c]$ and let $c: \mathscr{P}(X) \to \mathscr{P}(X)$ be defined by

$$c(A) = \begin{cases} \phi & \text{if } A = \phi \\ [a, c] & \text{if } A = [c] \\ X & \text{otherwise.} \end{cases} \quad \square$$

Similarly, one shows that if a closure space is R_0, then closed sets are separated from the points they exclude; but the converse is false.

3. Construction of proximites of Čech.

In this section we characterize the least member of $\mathfrak{M}$ in three ways, describe several techniques for constructing members of $\mathfrak{M}$ and derive some properties of these constructions.

THEOREM 3.1. *Let (X, c) be an R_0-closure space, and let $\mathscr{S}$ be a relation on $\mathscr{P}(X)$. If $\mathscr{R} \subset \mathscr{S} \subset \mathscr{W}$, then $\mathscr{S}$ induces c and satisfies* P3 *and* P4.

Proof. Obvious.

THEOREM 3.2. *Let (X, c) be an R_0-closure space. Let $\mathscr{D} \equiv [([x], A): x \in \bar{A}, A \subset X]$ and let $\mathscr{E} \equiv [(C, D): \exists([x], A) \in \mathscr{D}$ such that $(x \in C$ and $A \subset D)$ or $(x \in D$ and $A \subset C)]$. Then $\mathscr{E} = \mathscr{R}$.*

Proof. The proof is an easy verification. □

In order to analyze Theorem 3.2, let $\mathscr{P} \in \mathfrak{M}(X, c)$. Since c is compatible with $\mathscr{P}$, it is necessary that $\mathscr{D} \subset \mathscr{P}$. Also from that part of P2 which insures that $C \supset B$ and $A\delta B$ implies $A\delta C$ and from P1, it follows that $\mathscr{D} \subset \mathscr{E} \subset \mathscr{P}$. What is surprising is that no further alteration of $\mathscr{E}$, to accommodate the second part of P2 as well as P3 and P4, is necessary to obtain $\mathscr{R}$.

THEOREM 3.3. *Let (X, c) be an R_0-closure space and let (S, d) be the closure space in Example 2.1. Then the least $\check{C}$-proximity $\mathscr{R}$ in $\mathfrak{M}(X, c)$ is defined by $(A, B) \notin \mathscr{R}$ iff there is a continuous function $g: (X, c) \rightarrow (S, d)$ such that $g(A) \subset [s]$ and $g(B) \subset [t]$.*

Proof. Assume $(A, B) \notin \mathscr{R}$. Then $(\bar{A} \cap B) \cup (A \cup \bar{B}) = \phi$ and the existence of a suitable function g was shown in the proof of Theorem 2.1.

Conversely, assume there is a continuous function g such that $g(A) \subset [s]$ and $g(B) \subset [t]$. Then $g(\bar{A}) \subset \overline{g(A)} \subset [r, s]$, and thus $\bar{A} \cap B = \phi$. Similarly $A \cap \bar{B} = \phi$. Hence $(A, B) \notin \mathscr{R}$.

DEFINITION 3.1. [2, 25 A.7] A mapping f from a $\check{C}$-proximity space $(X, \mathscr{P})$ to a $\check{C}$-proximity space $(Y, \mathscr{P}^*)$ is said to be *p-continuous* iff $(A, B) \in \mathscr{P}$ implies $(f(A), f(B)) \in \mathscr{P}^*$.

An equivalent formulation of this definition is: f is p-continuous iff for all $(C, D) \notin \mathscr{P}^*$ with $C, D \subset Y$, it is true that $(f^{-1}(C), f^{-1}(D)) \notin \mathscr{P}$.

It is known [2, 25 A. 10] that every p-continuous function is a continuous function with respect to the induced closure operators. It is easily verified that there is only one-proximity $\mathscr{R}^d$ on S compatible with d (the space (S, d) is defined in Example 2.1) and that

$$\sim\mathscr{R}^d = [(\phi, A): A \subset S] \cup [(B, \phi): B \subset S] \cup [([s], [t])] .$$

In this context the following theorem may be of interest.

THEOREM 3.4. *Let $\mathscr{P} \in \mathfrak{M}(X, c)$. Then $\mathscr{P} = \mathscr{R}$ iff all functions which are continuous from (X, c) to (S, d) are p-continuous from*

$(X, \mathscr{P})$ to $(S, \mathscr{R}^d)$. Here $\mathscr{R}$ is the least Č-proximity in $\mathfrak{M}(X, c)$, (S, d) is the space in Example 2.1 and $\mathscr{R}^d$ is the unique Č-proximity in $\mathfrak{M}(S, d)$.

Proof. Assume $\mathscr{P} = \mathscr{R}$. Let $f\colon (X, c) \to (S, d)$ be continuous and let $(A, B) \in \mathscr{P}$. Then $\bar{A} \cap B \neq \phi$ or $A \cap \bar{B} \neq \phi$; say $A \cap \bar{B} \neq \phi$. Choose a in $A \cup \bar{B}$. Since f is continuous, $f(a) \in f(\bar{B}) \subset \overline{f(B)}$. Thus $f(A) \cap \overline{f(B)} \neq \phi$ and $(f(A), f(B)) \in \mathscr{R}^d$. Therefore, f is p-continuous.

Conversely, assume $\mathscr{P} \neq \mathscr{R}$. Then there is $(A, B) \in \mathscr{P} - \mathscr{R}$. So $(\bar{A} \cap B) \cup (A \cap \bar{B}) = \phi$. We define $g\colon X \to S$ by

$$g(x) = \begin{cases} r & \text{if } x \in X - (A \cup B) \\ s & \text{if } x \in A \\ t & \text{if } x \in B\,. \end{cases}$$

In the proof of Theorem 2.1 we verified that g is continuous. However, g is not p-continuous since $(g(A), g(B)) \notin \mathscr{R}^d$.

THEOREM 3.5 *Let (X, c) be an R_0-closure space and let $\mathscr{R}_1 \equiv [(A, B)\colon \bar{A} \cap \bar{B} \neq \phi]$. Then $\mathscr{R}_1 \in \mathfrak{M}$ iff, given $A \subset X$ and $x \in X$ such that $\bar{A} \cap [\bar{x}] \neq \phi$, it follows that $x \in \bar{A}$.*

Proof. The proof is a straightforward verification. □

It is easily shown that if (X, c) is a closure space, then $\mathscr{R}_1 = [(A, B)\colon A, B \subset X$ and if $f\colon (X, c) \to (X, c)$ is continuous, then $\overline{f(A)} \cap \overline{f(B)} \neq \phi]$.

DEFINITION 3.2. Let (X, c) be an R_0-closure space, let $E \subset X$, let m be an infinite cardinal number and let $\mathscr{P} \in \mathfrak{M}$. We introduce the following notations.

(i) $\mathscr{P}(E, m) \equiv \mathscr{W} \cap (\mathscr{P} \cup [(A, B)\colon |A \cap E| \geqq m$ or $|B \cap E| \geqq m])$.

(ii) $\mathscr{P}\{E, m\} \equiv \mathscr{P} \cup [(A, B)\colon |A \cap E| \geqq m$ and $|B \cap E| \geqq m]$.

THEOREM 3.6. *The relation $\mathscr{P}\{E, m\}$ is in $\mathfrak{M}$ and has the following properties:*

(i) *If $E \subset F$, $m \leqq m_1$ and $\mathscr{P}' \subset \mathscr{P}$, then $\mathscr{P}'\{E, m_1\} \subset \mathscr{P}\{F, m\}$.*

(ii) *If $|F| < m$, then $\mathscr{P}\{E, m\} = \mathscr{P}\{E \cup F, m\} = \mathscr{P}\{E - F, m\}$.*

(iii) *$\mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\} = \mathscr{P}\{E \cap F, m\}$.*

(iv) *$\mathscr{P}\{E, m\} \cup \mathscr{P}\{F, m\} \subset \mathscr{P}\{E \cup F, m\}$ and, in general, equality does not hold.*

(v) *If $m \leqq m_1$ and $|F - E| \leqq m_1$, then $(\mathscr{P}\{E, m\})\{F, m_1\} = \mathscr{P}\{E, m\}$.*

(vi) *If $m \geqq m_1$ and $|E - F| < m$, then $(\mathscr{P}\{E, m\})\{F, m_1\} = \mathscr{P}\{F, m_1\}$.*

(vii) *If* $\mathscr{P}, \mathscr{P}' \in \mathfrak{M}$ *and* $\mathscr{P} \cup \mathscr{P}' \subsetneqq \mathscr{P}\{E, m\}$, *then there exists* $\mathscr{P}^* \in \mathfrak{M}$ *such that* $\mathscr{P}' \subsetneqq \mathscr{P}^* \subsetneqq \mathscr{P}\{E, m\}$.

Proof. Clearly $\mathscr{P}\{E, m\} \subset \mathscr{W}$. Since $\mathscr{R} \subset \mathscr{P} \subset \mathscr{P}\{E, m\}$, it follows from Theorem 3.1 that $\mathscr{P}\{E, m\}$ induces c and satisfies P3, P4. Clearly P1 holds in $\mathscr{P}\{E, m\}$. The verification of P2 is straightforward.

(i) Let $(A, B) \in \mathscr{P}'\{E, m_1\}$. If $(A, B) \in \mathscr{P}'$, then $(A, B) \in \mathscr{P} \subset \mathscr{P}\{F, m\}$. If $(A, B) \notin \mathscr{P}'$, then $|A \cap E| \geqq m_1$ and $|B \cap E| \geqq m_1$. Since $E \subset F$ and $m \leqq m_1$, it follows that $|A \cap F| \geqq m$ and $|B \cap F| \geqq m$. Thus $(A, B) \in \mathscr{P}\{F, m\}$.

(ii) Since $|F| < m$ and m is an infinite cardinal number, it is known from set theory that $|A \cap E| \geqq m$ iff $|A \cap (E \cup F)| \geqq m$ iff $|A \cap (E - F)| \geqq m$.

(iii) Let $(A, B) \in \mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\}$. If $(A, B) \in \mathscr{P}$, then $(A, B) \in \mathscr{P}\{E \cap F, m\}$. So we assume that $(A, B) \notin \mathscr{P}$. Suppose that $|B \cap E \cap F| < m$. Since $(A, B) \notin \mathscr{P}$, P2 implies that $(A, B \cap E \cap F) \notin \mathscr{P}$. Thus $(A, B \cap E \cap F) \notin \mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\}$. Because $B = (B \cap E \cap F) \cup (B - (E \cap F))$, it follows from P2 that $(A, B - (E \cap F)) \in \mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\}$. In as much as $B - (E \cap F) = (B - E) \cup (B - F)$, P2 implies that:

Case 1. $(A, B - E) \in \mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\}$. Then $(A, B - E) \in \mathscr{P}\{E, m\}$. Again, since $(A, B) \notin \mathscr{P}$, P2 implies that $(A, B - E) \notin \mathscr{P}$. Therefore, $|(B - E) \cap E| \geqq m$ which is a contradiction.

Case 2. $(A, B - F) \in \mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\}$. An argument similar to Case 1 leads to the contradiction $|(B - F) \cap F| \geqq m$.

As a result of the contradictions, $|B \cap E \cap F| \geqq m$. Similarly, $|A \cap E \cap F| \geqq m$. Therefore, $(A, B) \in \mathscr{P}\{E \cap F, m\}$, and $\mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\} \subset \mathscr{P}\{E \cap F, m\}$.

By (i), $\mathscr{P}\{E \cap F, m\} \subset \mathscr{P}\{E, m\} \cap \mathscr{P}\{E, m\}$. Since $\mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\}$ is the union of all members of $\mathfrak{M}$ contained in $\mathscr{P}\{E, m\} \cap \mathscr{P}\{F, m\}$, it follows that $\mathscr{P}\{E \cap F, m\} \subset \mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\}$.

(iv) By (i), $\mathscr{P}\{E, m\} \subset \mathscr{P}\{E \cup F, m\}$ and $\mathscr{P}\{F, m\} \subset \mathscr{P}\{E \cup F, m\}$. Therefore, $\mathscr{P}\{E, m\} \cup \mathscr{P}\{F, m\} \subset \mathscr{P}\{E \cup F, m\}$.

Next we give an example where $\mathscr{P}\{E \cup F, m\} \not\subset \mathscr{P}\{E, m\} \cup \mathscr{P}\{F, m\}$. Let X be the set of real numbers, $\mathscr{T}$ the usual topology on X, $m = |X|$, $\mathscr{P} = \mathscr{R}$, $E = [x \in X: 0 \leqq x \leqq 1]$ and $F = [x \in X: 2 \leqq x \leqq 3]$. Then $(E, F) \in \mathscr{P}\{E \cup F, m\}$ but $(E, F) \notin \mathscr{P}\{E, m\} \cup \mathscr{P}\{F, m\}$.

(v) Clearly $\mathscr{P}\{E, m\} \subset \mathscr{P}\{E, m\} \cup [(A, B): |A \cap F| \geqq m_1$ and $|B \cap F| \geqq m_1] = (\mathscr{P}\{E, m\})\{F, m_1\}$.

Let $|A \cap F| \geqq m_1$. We write $A \cap F = (A \cap (F - E)) \cup (A \cap E \cap F)$. Since $|A \cap (F - E)| \leqq |F - E| < m_1$, it follows that $|A \cap E \cap F| \geqq m_1$.

So $|A \cap E| \geqq m_1 \geqq m$. Therefore, we have shown that $[(A, B); |A \cap F| \geqq m_1$ and $|B \cap F| \geqq m_1] \subset [(A, B): |A \cap E| \geqq m$ and $|B \cap E| \geqq m]$. Thus $[(A, B): |A \cap E| \geqq m$ and $|B \cap E| \geqq m] \subset \mathscr{P}\{E, m\}$ implies $(\mathscr{P}\{E, m\})\{F, m_1\} \subset \mathscr{P}\{E, m\}$.

(vi) The proof is similar to (v) and therefore is omitted.

(vii) Let $(C, D) \in \mathscr{P}\{E, m\} - (\mathscr{P} \cup \mathscr{P}')$. Thus $|C \cap E| \geqq m$ and $|D \cap E| \geqq m$. We partition D into two disjoint sets D_1, D_2 such that $|D_1 \cap E| = |D_2 \cap E| = |D \cap E|$. Then $D_1 \cap E = D_1 \cap (E - D_2)$. So $|D_1 \cap (E - D_2)| \geqq m$.

We write $C \cap E = (C \cap E \cap D_1) \cup (C \cap E \cap D_2) \cup ((C \cap E) - (D_1 \cap D_2))$. Since $|C \cap E| \geqq m$, it follows that $|(C \cap E \cap D_1) \cup ((C \cap E) - (D_1 \cap D_2))| \geqq m$ or $|(C \cap E \cap D_2) \cup ((C \cap E) - (D_1 \cap D_2))| \geqq m$; say the former is true. Let $F = C - (E \cap D_2)$. Then $F \cap (E - D_2) = (C \cap E \cap D_1) \cup ((C \cap E) - (D_1 \cup D_2))$ and $|F \cap (E - D_2)| \geqq m$.

Let $\mathscr{P}^* = \mathscr{P}' \{E - D_2, m\}$. Then $(F, D_1) \in \mathscr{P}^*$ by the above work. Since $(C, D) \notin \mathscr{P}'$, $F \subset C$ and $D_1 \subset D$, P1 and P2 imply that $(F, D_1) \notin \mathscr{P}'$. Thus $\mathscr{P}' \subsetneqq \mathscr{P}^*$.

Clearly $(C, D_2) \in \mathscr{P}\{E, m\}$ and $\mathscr{P}^* \subset \mathscr{P}\{E, m\}$. Since $(C, D_2) \in \mathscr{P}'$ implies by P2 that $(C, D) \in \mathscr{P}'$, contrary to assumption, we must have $(C, D_2) \notin \mathscr{P}'$. Also $D_2 \cap (E - D_2) = \phi$ implies that $(C, D_2) \notin \mathscr{P}^*$. Thus $\mathscr{P}^* \subsetneqq \mathscr{P}\{E, m\}$.

THEOREM 3.7. *The relation $\mathscr{P}(E, m)$ is in $\mathfrak{M}$ and has the following properties*:

(i) *If $E \subset F$, $m \leqq m_1$ and $\mathscr{P}' \subset \mathscr{P}$, then $\mathscr{P}'(E, m_1) \subset \mathscr{P}(F, m)$.*

(ii) *If $|F| < m$, then $\mathscr{P}(E, m) = \mathscr{P}(E \cup F, m) = \mathscr{P}(E - F, m)$.*

(iii) *$\mathscr{P}(E, m) \wedge \mathscr{P}(F, m) \supset \mathscr{P}(E \cap F, m)$ and, in general, equality does not hold.*

(iv) *$\mathscr{P}(E, m) \cup \mathscr{P}(F, m) = \mathscr{P}(E \cup F, m)$.*

(v) *If $m \leqq m_1$ and $|F - E| < m_1$, then $(\mathscr{P}(E, m))(F, m_1) = \mathscr{P}(E, m)$.*

(vi) *If $m \geqq m_1$ and $|E - F| < m$, then $(\mathscr{P}(E, m))(F, m_1) = \mathscr{P}(F, m_1)$.*

(vii) *If $\mathscr{P}, \mathscr{P}' \in \mathfrak{M}$ and $\mathscr{P} \cup \mathscr{P}' \subsetneqq \mathscr{P}(E, m)$, then there exists $\mathscr{P}^* \in \mathfrak{M}$ such that $\mathscr{P}' \subsetneqq \mathscr{P}^* \subsetneqq \mathscr{P}(E, m)$.*

Proof. The proof is similar to Theorem 3.6 and therefore is omitted.

THEOREM 3.8.

(i) *$\mathscr{P}\{E, m\} \subset \mathscr{P}(E, m)$.*

(ii) *In general, $[\mathscr{R}(E, m): E \subset X, m$ an infinite cardinal$]$ neither contains nor is contained in $[\mathscr{R}\{E, m\}: E \subset X, m$ an infinite cardinal$]$.*

(iii) *In general, $\mathfrak{M} \neq [\mathscr{R}(E, m), \mathscr{R}\{E, m\}: E \subset X, m$ an infinite cardinal$]$.*

Proof. (i) The result is immediate by comparing the definitions of $\mathscr{P}\{E, m\}$ and $\mathscr{P}(E, m)$.

(ii) Let X be the set of real numbers, $\mathscr{T}$ the usual topology on X, $F = [x \in X\colon 0 \leqq x \leqq 4]$ and $t = |X|$. Then $\mathscr{R}\{F, t\} \notin [\mathscr{R}(E, m)\colon E \subset X, m$ an infinite cardinal] and $\mathscr{R}(F, t) \notin [\mathscr{R}\{E, m\}\colon E \subset X, m$ an infinite cardinal].

(iii) Let X be the set of real numbers, let $\mathscr{T}$ be the usual topology on X and let $\mathscr{R}_1 = [(A, B)\colon \bar{A} \cap \bar{B} \neq \phi]$. Then $\mathscr{R}_1 \in \mathfrak{M}$ by Theorem 3.5. However, $\mathscr{R}_1 \notin [\mathscr{R}(E, m), \mathscr{R}\{E, m\}\colon E \subset X, m$ an infinite cardinal].

THEOREM 3.9. *If $\mathscr{R} \neq \mathscr{R}\{E, m\}$, then $\mathscr{R}\{E, m\}$ covers no element of $\mathfrak{M}$. If $\mathscr{R} \neq \mathscr{R}(E, m)$, then $\mathscr{R}(E, m)$ covers no element of $\mathfrak{M}$.*

Proof. Suppose $\mathscr{P} \in \mathfrak{M}$ and $\mathscr{P} \subsetneqq \mathscr{R}\{E, m\}$. Since $\mathscr{R} \cup \mathscr{P} = \mathscr{P}$, we appeal to Theorem 3.6 (vii) to see that $\mathscr{R}\{E, m\}$ does not cover $\mathscr{P}$. The second statement follows from Theorem 3.7 (vii).

THEOREM 3.10. *Let (X, c) be a closure space such that $|\mathfrak{M}| > 1$. In addition, let $\mathscr{P} \in \mathfrak{M}$ and $\mathscr{P} \neq \mathscr{W}$. Let $(C, D) \in \mathscr{W} - \mathscr{P}$ and let $\mathscr{F}, \mathscr{G}$ be nonprincipal ultrafilters on X containing C, D respectively. Then $\mathscr{P}' = \mathscr{P} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F})$ is in $\mathfrak{M}$ and $\mathscr{P}'$ covers $\mathscr{P}$.*

Proof. Clearly $\mathscr{R} \subset \mathscr{P} \subset \mathscr{P}'$. Since each member of a nonprincipal ultrafilter is an infinite set, $\mathscr{F} \times \mathscr{G} \subset \mathscr{W}$. Hence $\mathscr{P}' \subset \mathscr{W}$. By Theorem 3.1, $\mathscr{P}'$ induces c and satisfies P3, P4.

Clearly $\mathscr{P}'$ satisfies P1. To verify P2, let $(A, B \cup C) \in \mathscr{F} \times \mathscr{G}$. Hence $B \cup C \in \mathscr{G}$. Since $\mathscr{G}$ is an ultrafilter, it is known [4, p. 84] that $B \in \mathscr{G}$ or $C \in \mathscr{G}$. Thus $(A, B) \in \mathscr{F} \times \mathscr{G}$ or $(A, C) \in \mathscr{F} \times \mathscr{G}$.

Conversely, let $(A, B) \in \mathscr{F} \times \mathscr{G}$ and $C \subset X$. Since $\mathscr{G}$ is a filter and $B \in \mathscr{G}$, $B \cup C$ is in $\mathscr{G}$. Thus $(A, B \cup C) \in \mathscr{F} \times \mathscr{G}$.

We have shown that $\mathscr{P}' \in \mathfrak{M}$.

Suppose there exists $\mathscr{P}^* \in \mathfrak{M}$ and $\mathscr{P} \subsetneqq \mathscr{P}^* \subset \mathscr{P}'$. We will show that $\mathscr{P}^* = \mathscr{P}'$. Let (F, G) belong to $\mathscr{P}^* - \mathscr{P}$. Then (F, G) is in $\mathscr{P}' - \mathscr{P}$ and (F, G) belongs to $\mathscr{F} \times \mathscr{G}$ or $\mathscr{G} \times \mathscr{F}$, say $\mathscr{F} \times \mathscr{G}$. To verify that $\mathscr{F} \times \mathscr{G} \subset \mathscr{P}^*$, we let (A, B) be in $\mathscr{F} \times \mathscr{G}$. Then $G = (G \cap B) \cup (G - B)$. By P2, $(F, G \cap B) \in \mathscr{P}^*$ or $(F, G - B) \in \mathscr{P}^*$. Since the assumption that $(F, G - B) \in \mathscr{P}^*$ leads to a contradiction, we conclude that $(F, G \cap B) \in \mathscr{P}^*$.

Because $\mathscr{F}$ and $\mathscr{G}$ are filters, a similar argument shows $(F \cap A, G \cap B) \in \mathscr{P}^*$. P1 and P2 imply that $(A, B) \in \mathscr{P}^*$. Therefore, $\mathscr{F} \times \mathscr{G} \subset \mathscr{P}^*$. P1 implies that $\mathscr{G} \times \mathscr{F} \subset \mathscr{P}^*$. Hence $\mathscr{P}' \subset \mathscr{P}^*$ and $\mathscr{P}^* = \mathscr{P}'$.

THEOREM 3.11. *Let (X, c) be a closure space such that $|\mathfrak{M}| > 1$. Let $\mathscr{P} \in \mathfrak{M}$, let $\mathscr{P} \neq \mathscr{R}$ and let (C, D) belong to $\mathscr{P} - \mathscr{R}$. Then there exist nonprincipal ultrafilters $\mathscr{F}, \mathscr{G}$ on X such that $(C, D) \in \mathscr{F} \times \mathscr{G} \subset \mathscr{P}$.*

Proof. Fix (C, D) in $\mathscr{P} - \mathscr{R}$. Let $\mathscr{H} = [E: E \subset D$ and $(C, E) \in \mathscr{P} - \mathscr{R}]$. Let $\mathfrak{S}$ be the family of all subsets $\mathscr{H}^*$ of $\mathscr{H}$ having the property: $A, B \in \mathscr{H}^*$ implies $A \cap B \in \mathscr{H}^*$. We partially order $\mathfrak{S}$ by set inclusion. By Zorn's lemma, $\mathfrak{S}$ has a maximal element $\mathscr{G}_1$.

$\mathscr{G}_1$ is a filter base on X due to the formation of $\mathscr{H}$ and $\mathfrak{S}$. Hence, there exists an ultrafilter $\mathscr{G}$ on X containing $\mathscr{G}_1$ [4, pp. 78, 79, 83]. Furthermore, $D \in \mathscr{G}_1$ and $\mathscr{G}$ is a nonprincipal ultrafilter. Also, if $G \in \mathscr{G}$, then $(C, G \cap D) \in \mathscr{P} - \mathscr{R}$.

Let $\mathscr{L} = [L: L \subset C$ and $(L, G \cap D) \in \mathscr{P} - \mathscr{R}$ for all $G \in \mathscr{G}]$. Let $\mathfrak{F}$ be the family of all subsets $\mathscr{L}^*$ of $\mathscr{L}$ having the property: $S, T \in \mathscr{L}^*$ implies $S \cap T \in \mathscr{L}^*$. We partially order $\mathfrak{F}$ by set inclusion. By Zorn's lemma, $\mathfrak{F}$ has a maximal element $\mathscr{F}_1$ which is a filter base on X. Hence there exists an ultrafilter $\mathscr{F}$ on X containing $\mathscr{F}_1$.

Moreover, $C \in \mathscr{F}_1$ and $\mathscr{F}$ is a nonprincipal ultrafilter. Finally, if (F, G) is in $\mathscr{F} \times \mathscr{G}$, then $(F \cap C, G \cap D) \in \mathscr{P}$. Thus P1, P2 imply that $(F, G) \in \mathscr{P}$. So $\mathscr{F} \times \mathscr{G} \subset \mathscr{P}$.

THEOREM 3.12. *Let (X, c) be a closure space such that $\mathscr{P}, \mathscr{P}' \in \mathfrak{M}$. Then $\mathscr{P}'$ covers $\mathscr{P}$ iff, given (C, D) in $\mathscr{P}' - \mathscr{P}$, there exist nonprincipal ultrafilters $\mathscr{F}, \mathscr{G}$ on X containing C, D respectively such that $\mathscr{P}' = \mathscr{P} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F})$.*

Proof. Assume $\mathscr{P}'$ covers $\mathscr{P}$. Let (C, D) belong to $\mathscr{P}' - \mathscr{P}$. Since $\mathscr{R} \subset \mathscr{P}$, (C, D) is in $\mathscr{P}' - \mathscr{R}$. By Theorem 3.11, there are nonprincipal ultrafilters $\mathscr{F}, \mathscr{G}$ on X such that $(C, D) \in \mathscr{F} \times \mathscr{G} \subset \mathscr{P}'$. P1 implies $\mathscr{G} \times \mathscr{F} \subset \mathscr{P}'$. Thus $\mathscr{P} \subsetneqq \mathscr{P} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F}) \subset \mathscr{P}'$. By Theorem 3.10, $\mathscr{P} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F})$ is in $\mathfrak{M}$. Since $\mathscr{P}'$ covers $\mathscr{P}$, $\mathscr{P}' = \mathscr{P} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F})$.

The converse is a direct application of Theorem 3.10.

4. The structure of the lattice of Č-proximities compatible with a given R_0-closure space. In this section we first characterize greatest lower bound in $\mathfrak{M}$. Then it is shown that $\mathfrak{M}$ is strongly atomic and distributive. Finally, we prove that $\mathfrak{M}$ has no antiatoms and that if $|\mathfrak{M}| > 1$, then $|\mathfrak{M}| \geqq 2^{2^{\aleph_0}}$.

LEMMA 4.1. [2, p. 441] *Let $\mathscr{P}$ be a Č-proximity on X and let*

$(A, B) \in \mathscr{P}$. If $A = \bigcup [A_i: 1 \leqq i \leqq n]$ and $B = \bigcup [B_j: 1 \leqq j \leqq m]$ where n and m are integers, then there exists i, j such that $(A_i, B_j) \in \mathscr{P}$.

THEOREM 4.1. *Let (X, c) be an R_0-closure space. Let K be a nonempty index set and $\mathscr{P} = \bigwedge [\mathscr{P}_\alpha: \mathscr{P}_\alpha \in \mathfrak{M}$ and $\alpha \in K]$. Then $(A, B) \in \mathscr{P}$ iff, given $A = \bigcup [A_i: i \in I]$ and $B = \bigcup [B_j: j \in J]$ where I and J are finite sets, it follows that there exists i, j such that $(A_i, B_j) \in \mathscr{P}_\alpha$ for each $\alpha \in K$.*

Proof. Let $(A, B) \in \mathscr{P}$, let $A = \bigcup [A_i: i \in I]$ and let $B = \bigcup [B_j: j \in J]$ where I, J are finite sets. We appeal to Lemma 4.1 to obtain i, j such that $(A_i, B_j) \in \mathscr{P}$. Since $\mathscr{P} \subset \mathscr{P}_\alpha$ for each $\alpha \in K$, $(A_i, B_j) \in \mathscr{P}_\alpha$.

Conversely, let $\mathscr{P}' = [(A, B)$: if $A = \bigcup [A_i: i \in I]$ and $B = \bigcup [B_j: j \in J]$ where I, J are finite sets, then $\exists i, j$ such that $(A_i, B_j) \in \mathscr{P}_\alpha$ for each $\alpha \in K]$. Čech has proved [2, p. 470] that $\mathscr{P}'$ is a $\check{C}$-proximity on X, $\mathscr{P}' \subset \mathscr{P}_\alpha$ for each $\alpha \in K$ and if $\mathscr{P}^*$ is any $\check{C}$-proximity on X such that $\mathscr{P}^* \subset \mathscr{P}_\alpha$ for each $\alpha \in K$, then $\mathscr{P}^* \subset \mathscr{P}'$. We shall prove that $\mathscr{P}'$ induces c. It then follows that $\mathscr{P} \in \mathfrak{M}$ and $\mathscr{P}' = \bigwedge [\mathscr{P}_\alpha: \mathscr{P}_\alpha \in \mathfrak{M}$ and $\alpha \in K]$.

Let $(C, D) \in \mathscr{R}$ and suppose $C = \bigcup [C_i: i \in I]$ and $D = \bigcup [D_j: j \in J]$ where I, J are finite sets. By Lemma 4.1 there exist i and j such that $(C_i, D_j) \in \mathscr{R}$. Since $\mathscr{R} \subset \mathscr{P}_\alpha$ for each $\alpha \in K$, $(C_i, D_j) \in \mathscr{P}_\alpha$. Consequently $(C, D) \in \mathscr{P}'$. Thus $\mathscr{R} \subset \mathscr{P}'$. Since $\mathscr{P}' \subset \mathscr{P}_\alpha \subset \mathscr{W}$, we have $\mathscr{R} \subset \mathscr{P}' \subset \mathscr{W}$. By Theorem 3.1, $\mathscr{P}'$ induces c. □

We observe that the operation of meet in $\mathfrak{M}(X, c)$ is the restriction of the operation of meet in the family of all $\check{C}$-proximities on X (no compatibility requirement). This follows from Theorem 4.1 and [2, p. 470]. Čech has established the analogous conclusion for the operation of join in these two lattices [2, p. 448]. Therefore, $\mathfrak{M}(X, c)$ is a sublattice of the lattice of all $\check{C}$-proximities on X.

THEOREM 4.2. *If $\mathscr{P} \in \mathfrak{M}$ and $\mathscr{P} \neq \mathscr{W}$, then there exists $\mathscr{P}^* \in \mathfrak{M}$ such that $\mathscr{P} \subsetneqq \mathscr{P}^* \subsetneqq \mathscr{W}$. Therefore, the lattice $\mathfrak{M}$ has no antiatoms. Also, $\mathfrak{M}$ is not antiatomic and is not anticovered iff $|\mathfrak{M}| > 1$.*

Proof. Since $\mathscr{W} = \mathscr{R}\{X, \aleph_0\}$ and $\mathscr{R} \cup \mathscr{P} = \mathscr{P} \subsetneqq \mathscr{W}$, we appeal to Theorem 3.6 (vii) to obtain $\mathscr{P}^*$ satisfying the theorem. The last two statements follow from the appropriate definitions.

COROLLARY 4.1. *If $|\mathfrak{M}| > 1$, then $\mathfrak{M}$ is not lattice isomorphic to a power set lattice.*

Proof. Every power set lattice with more than one element has antiatoms.

THEOREM 4.3. *The lattice $\mathfrak{M}$ is strongly atomic and consequently, atomic and covered.*

Proof. Let $\mathscr{P} \in \mathfrak{M}$ and $\mathscr{P} \subset \mathscr{P}^* \in \mathfrak{M}$. If $(C, D) \in \mathscr{P}^* - \mathscr{P}$, then, by Theorem 3.11, there exist nonprincipal ultrafilters $\mathscr{F}$, $\mathscr{G}$ on X such that $(C, D) \in \mathscr{F} \times \mathscr{G} \subset \mathscr{P}^*$. P1 implies $\mathscr{G} \times \mathscr{F} \subset \mathscr{P}^*$. Thus $\mathscr{P}_{C,D} \equiv \mathscr{P} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F}) \subset \mathscr{P}^*$. By Theorem 3.10, $\mathscr{P}_{C,D}$ is an atom in the lattice $([\mathscr{P}' \in \mathfrak{M}: \mathscr{P}' \supset \mathscr{P}], \subset)$. Since $\bigcup [\mathscr{P}_{C,D}: (C, D) \text{ is in } \mathscr{P}^* - \mathscr{P}] = \mathscr{P}^*$, the lattice $([\mathscr{P}' \in \mathfrak{M}: \mathscr{P}' \supset \mathscr{P}], \subset)$ is atomic.

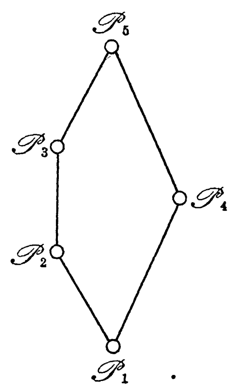
$\mathfrak{M}$ is atomic because every strongly atomic lattice with a least element is atomic. $\mathfrak{M}$ is covered since every strongly atomic lattice is covered.

COROLLARY 4.2. *If $|\mathfrak{M}| > 1$, then $\mathfrak{M}$ is not infinitely meet distributive.*

Proof. Suppose $\mathfrak{M}$ is infinitely meet distributive. Since it is well known that a complete, infinitely meet distributive lattice is a complete Boolean algebra, $\mathfrak{M}$ is a complete, atomic Boolean algebra. Consequently $\mathfrak{M}$ is isomorphic to a power set lattice which contradicts Corollary 4.1.

LEMMA 4.2. *The lattice $\mathfrak{M}$ is modular.*

Proof. It suffices [1, p. 13] to show that $\mathfrak{M}$ does not contain a sublattice of the form:



.

Suppose $\mathfrak{M}$ does contain such a sublattice. Let $\mathfrak{S}_1 =$ [atoms of $\mathfrak{M}$ which are contained in $\mathscr{P}_3$ but are not contained in $\mathscr{P}_2$] and $\mathfrak{S}_2 =$ [atoms of $\mathfrak{M}$ which are contained in $\mathscr{P}_4$ but are not contained in $\mathscr{P}_1$]. If $\mathfrak{S}_1 \cap \mathfrak{S}_2 \neq \phi$, then $\mathscr{P}_1 \subsetneqq \mathscr{P}_1 \cup (\bigcup (\mathfrak{S}_1 \cap \mathfrak{S}_2)) \subset \mathscr{P}_3 \wedge \mathscr{P}_4$. Since $\mathscr{P}_1 \cup$

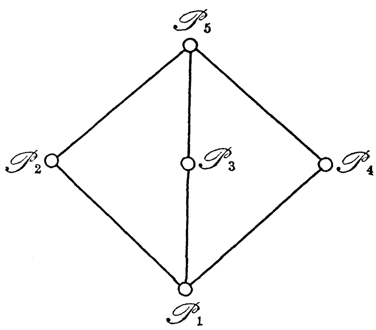
$(\bigcup(\mathfrak{S}_1 \cap \mathfrak{S}_2)) \in \mathfrak{M}$, we have contradicted $\mathscr{P}_1 = \mathscr{P}_3 \wedge \mathscr{P}_4$. Thus $\mathfrak{S}_1 \cap \mathfrak{S}_2 = \phi$.

Since $\mathfrak{M}$ is atomic, there exists $\mathscr{P}' \in \mathfrak{S}_1$. Because $\mathscr{P}' \not\subset \mathscr{P}_1$ and $\mathscr{P}' \notin \mathfrak{S}_2$, $\mathscr{P}' \not\subset \mathscr{P}_4$. Hence there exists $(A, B) \in \mathscr{P}' - \mathscr{P}_4$. Since $\mathscr{P}' \not\subset \mathscr{P}_2$, there exists $(C, D) \in \mathscr{P}' - \mathscr{P}_2$. Because $\mathscr{P}'$ is an atom in $\mathfrak{M}$, we appeal to Theorem 3.12 to obtain nonprincipal ultrafilters $\mathscr{F}$, $\mathscr{G}$ on X containing C, D respectively such that $\mathscr{P}' = \mathscr{R} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F})$. Now (A, B) in $\mathscr{P}' - \mathscr{P}_4$ implies (A, B) belongs to $\mathscr{F} \times \mathscr{G}$ or $\mathscr{G} \times \mathscr{F}$, say $\mathscr{F} \times \mathscr{G}$. Thus $(A \cap C, B \cap D) \in \mathscr{F} \times \mathscr{G} \subset \mathscr{P}' \subset \mathscr{P}_5$.

Since $(C, D) \notin \mathscr{P}_2$, P1 and P2 imply $(A \cap C, B \cap D) \notin \mathscr{P}_2$. Having seen $(A, B) \notin \mathscr{P}_4$, P1 and P2 imply $(A \cap C, B \cap D) \notin \mathscr{P}_4$. Thus $(A \cap C, B \cap D) \notin \mathscr{P}_2 \cup \mathscr{P}_4 = \mathscr{P}_5$, which is a contradiction.

THEOREM 4.4. *The lattice* $\mathfrak{M}$ *is distributive.*

Proof. In view of Lemma 4.2, it suffices [1, p. 39] to show that $\mathfrak{M}$ does not contain a sublattice of the form:



Suppose $\mathfrak{M}$ does contain such a sublattice. Let $\mathfrak{S}_i$ = [atoms of $\mathfrak{M}$ which are contained in $\mathscr{P}_i$ but are not contained in $\mathscr{P}_1$] $(i = 2, 3, 4)$. If $\mathfrak{S}_2 \cap \mathfrak{S}_3 \neq \phi$, then $\mathscr{P}_1 \subsetneqq \mathscr{P}_1 \cup (\bigcup (\mathfrak{S}_2 \cap \mathfrak{S}_3)) \subset \mathscr{P}_2 \wedge \mathscr{P}_3$. Since $\mathscr{P}_1 \cup (\bigcup (\mathfrak{S}_2 \cap \mathfrak{S}_3))$ is in $\mathfrak{M}$, we have contradicted $\mathscr{P}_1 = \mathscr{P}_2 \wedge \mathscr{P}_3$. Thus $\mathfrak{S}_2 \cap \mathfrak{S}_3 = \phi$. Similarly, $\mathfrak{S}_2 \cap \mathfrak{S}_4 = \phi$.

Since $\mathfrak{M}$ is atomic, there exists $\mathscr{P}' \in \mathfrak{S}_2$. Because $\mathscr{P}' \not\subset \mathscr{P}_1$ and $\mathscr{P}' \notin \mathfrak{S}_3$, $\mathscr{P}' \not\subset \mathscr{P}_3$. Hence there exists $(A, B) \in \mathscr{P}' - \mathscr{P}_3$. Similarly, there exists $(C, D) \in \mathscr{P}' - \mathscr{P}_4$. Because $\mathscr{P}'$ is an atom in $\mathfrak{M}$, we appeal to Theorem 3.12 to obtain nonprincipal ultrafilters $\mathscr{F}$, $\mathscr{G}$ on X containing C, D respectively such that $\mathscr{P}' = \mathscr{R} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F})$. $(A, B) \in \mathscr{P}' - \mathscr{P}_3$ implies (A, B) is in $\mathscr{F} \times \mathscr{G}$ or $\mathscr{G} \times \mathscr{F}$, say $\mathscr{F} \times \mathscr{G}$. Therefore, $(A \cap C, B \cap D) \in \mathscr{F} \times \mathscr{G} \subset \mathscr{P}' \subset \mathscr{P}_5$.

Since $(C, D) \notin \mathscr{P}_4$, P1 and P2 imply $(A \cap C, B \cap D) \notin \mathscr{P}_4$. Having seen $(A, B) \notin \mathscr{P}_3$, P1 and P2 imply $(A \cap C, B \cap D) \notin \mathscr{P}_3$. Thus $(A \cap C, B \cap D) \notin \mathscr{P}_3 \cup \mathscr{P}_4 = \mathscr{P}_5$, which is a contradiction.

COROLLARY 4.3. *If* $|\mathfrak{M}| > 1$, *then* $\mathfrak{M}$ *is not complemented.*

Proof. If $\mathfrak{M}$ is complemented, then $\mathfrak{M}$ is a complete Boolean algebra, and thus is infinitely meet distributive [1, p. 118]. This contradicts Corollary 4.2.

THEOREM 4.5. *Let* (X, c) *be an* R_0*-closure space. Then* $|\mathfrak{M}| = 1$ *iff, given two infinite subsets of* X, *at least one of them contains a point in the closure of the other.*

Proof. Assume $|\mathfrak{M}| = 1$. Let A, B be two infinite subsets of X. Then $(A, B) \in \mathscr{W}$. $|\mathfrak{M}| = 1$ implies $\mathscr{R} = \mathscr{W}$. Thus $(\bar{A} \cap B) \cup (A \cap \bar{B}) \neq \phi$.

The converse is true because the assumption says $\mathscr{R} = \mathscr{W}$. Therefore, $|\mathfrak{M}| \leqq 1$. Since (X, c) is R_0, $|\mathfrak{M}| \geqq 1$ by Theorem 2.1. $\square$

We note that $|\mathfrak{M}| = 1$ for each of the following topological spaces: any R_0 topology on a finite set, any set with the indiscrete topology, any set with the minimum T_1 topology and any atom in the lattice of T_1 topologies on a fixed set. We also note that the characterization given in Theorem 4.5 can be expressed as: $|\mathfrak{M}| = 1$ iff any two infinite subsets of X are not separated.

THEOREM 4.6. *Let* (X, c) *be an* R_0*-closure space. Then* $|\mathfrak{M}| = 1$ *or* $2^{2^{\aleph_0}} \leqq |\mathfrak{M}| \leqq 2^{2^{|X|}}$. *Furthermore, if* $|X| \geqq \aleph_0$ *and* m *is a cardinal number such that* $\aleph_0 \leqq m \leqq |X|$, *then there is a* T_1 *topology* $\mathscr{T}$ *on* X *such that* $|\mathfrak{M}(X, \mathscr{T})| = 2^{2^m}$.

Proof. Theorem 2.1 implies $|\mathfrak{M}| \geqq 1$. If $|\mathfrak{M}| > 1$, then there exists $(C, D) \in \mathscr{W} - \mathscr{R}$. Thus C, D are infinite and $C \cap D = \phi$. We appeal to Theorem 3.11 to obtain nonprincipal ultrafilters $\mathscr{U}, \mathscr{V}$ on X containing C, D respectively. Then $\mathscr{R} \cup (\mathscr{U} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{U})$ is in $\mathfrak{M}$ by Theorem 3.10. We note that if $\mathscr{U}, \mathscr{F}$ are distinct nonprincipal ultrafilters on X containing C, then $\mathscr{R} \cup (\mathscr{U} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{U})$ and $\mathscr{R} \cup (\mathscr{F} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{F})$ are distinct. From [2, p. 212] there are $2^{2^{|C|}}$ distinct ultrafilters on X containining C. Since there are $|C|$ principal ultrafilters on X containg C, there are $2^{2^{|C|}}$ distinct nonprincipal ultrafilters on X containing C, and it follows that $2^{2^{\aleph_0}} \leqq 2^{2^{|C|}} \leqq |\mathfrak{M}|$.

On the other hand, since there are $2^{2^{|X|}}$ families of ordered pairs of subsets of X, and since each $\check{C}$-proximity is such a family, $|\mathfrak{M}| \leqq 2^{2^{|X|}}$.

To form $\mathscr{T}$, choose subsets S, T of X such that $S \cap T = \phi$ and $|S| = |T| = m$. Then $[\phi, X - S, X - T, X -$ any finite subset of $X]$ is a subbase for the desired topology $\mathscr{T}$. Since $(S, T) \in \mathscr{W} -$

$\mathscr{R}$, $|\mathfrak{M}(X, \mathscr{T})| > 1$. By the above argument, $|\mathfrak{M}(X, \mathscr{T})| \geqq 2^{2^{|S|}}$.

Let $\mathscr{T}'$ be the relative topology on $S \cup T$. Then $f: \mathfrak{M}(X, \mathscr{T}) \to \mathfrak{M}(S \cup T, \mathscr{T}')$ defined by $f(\mathscr{P}) = \mathscr{P} \cap (\mathscr{P}(S \cup T) \times \mathscr{P}(S \cup T))$ is a $1:1$, onto map. Thus $|\mathfrak{M}(X, \mathscr{T})| = |\mathfrak{M}(S \cup T, \mathscr{T}')|$. By the argument above $|\mathfrak{M}(S \cup T, \mathscr{T}')| \leqq 2^{2^{|S \cup T|}}$, which establishes our result since $|S \cup T| = m$.

THEOREM 4.7. *Let (X, c), (X, d) be R_0-closure spaces. Let $\mathscr{R}_c$, $\mathscr{R}_d$ be the least members of $\mathfrak{M}(X, c)$, $\mathfrak{M}(X, d)$ respectively. If $c(A) \subset d(A)$ for each $A \subset X$, then $\mathscr{R}_c \subset \mathscr{R}_d$.*

Proof. The verification is straightforward.

THEOREM 4.8. *Let (X, c), (X, d) be R_0-closure spaces. Let $\mathscr{R}_c$, $\mathscr{R}_d$ be the least members of $\mathfrak{M}(X, c)$, $\mathfrak{M}(X, d)$ respectively. If $\mathscr{R}_c \subset \mathscr{R}_d$, then $|\mathfrak{M}(X, d)| \leqq |\mathfrak{M}(X, c)|$.*

Proof. Let $\mathfrak{A}$ be the family of atoms of $\mathfrak{M}(X, d)$. By Theorems 3.11 and 3.10, if $\mathscr{S} \in \mathfrak{A}$, then there are nonprincipal ultrafilters $\mathscr{U}$, $\mathscr{V}$ on X such that $\mathscr{S} = \mathscr{R}_d \cup (\mathscr{U} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{U})$. Since $\mathscr{R}_c \subset \mathscr{R}_d$, by Theorem 3.10 $\mathscr{R}_c \cup (\mathscr{U} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{U})$ is an atom in $\mathfrak{M}(X, c)$.

Define $f: \mathfrak{A} \to \mathfrak{M}(X, c)$ by $f(\mathscr{R}_d \cup (\mathscr{U} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{U})) = \mathscr{R}_c \cup (\mathscr{U} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{U})$. Also, define $g: \mathfrak{M}(X, d) \to \mathfrak{M}(X, c)$ by

$$g(\mathscr{P}) = \begin{cases} \bigcup [f(\mathscr{S}): \mathscr{S} \in \mathfrak{A} \text{ and } \mathscr{S} \subset \mathscr{P}] & \text{if } \mathscr{P} \neq \mathscr{R}_d \\ \mathscr{R}_c \quad \text{if } \mathscr{P} = \mathscr{R}_d . \end{cases}$$

To verify that g is $1:1$, let $\mathscr{P}, \mathscr{P}' \in \mathfrak{M}(X, d)$ and $\mathscr{P} \neq \mathscr{P}'$. Since $\mathfrak{M}(X, d)$ is atomic, there exists $\mathscr{S}' \in \mathfrak{A}$ such that ($\mathscr{S}' \subset \mathscr{P}$ and $\mathscr{S}' \not\subset \mathscr{P}'$) or ($\mathscr{S}' \subset \mathscr{P}'$ and $\mathscr{S}' \not\subset \mathscr{P}$); say the former is true. Then $f(\mathscr{S}') \subset g(\mathscr{P})$ by the definition of g. Let (A, B) belong to $\mathscr{S}' - \mathscr{P}'$. Since $\mathscr{R}_d \subset \mathscr{P}'$, $(A, B) \notin \mathscr{R}_d$. By Theorems 3.11 and 3.10 there are nonprincipal ultrafilters $\mathscr{U}$, $\mathscr{V}$ on X such that $(A, B) \in \mathscr{U} \times \mathscr{V}$ and $\mathscr{S}' = \mathscr{R}_d \cup (\mathscr{U} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{U})$. Hence $(A, B) \in f(\mathscr{S}') \subset g(\mathscr{P})$.

On the other hand, $(A, B) \notin \mathscr{P}'$ implies that (A, B) is not a member of any atom contained in $\mathscr{P}'$. Therefore, $(A, B) \notin g(\mathscr{P}')$, and $g(\mathscr{P}) \neq g(\mathscr{P}')$.

REFERENCES

1. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Pub., Vol. 25, Providence, Third Edition, 1967.
2. E. Čech, *Topological Spaces*, Interscience Publishers, John Wiley and Sons, New York, 1966.
3. S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge University Press, London, 1970.

4. W. J. Thron, *Topological Structures*, Holt, Rinehart and Winston, New York, 1966.

Received June 21, 1972.

UNIVERSITY OF COLORADO
AND
AEROSPACE RESEARCH LABORATORIES, WRIGHT-PATTERSON AFB, OHIO