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ON THE KONHAUSER SETS OF BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

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Recently, Joseph D. E. Konhauser discussed two polynomial sets $\{Y_n^\alpha(x; k)\}$ and $\{Z_n^\alpha(x; k)\}$, which are biorthogonal with respect to the weight function $x^\alpha e^{-x}$ over the interval $(0, \infty)$, where $\alpha > -1$ and k is a positive integer. For the polynomials $Y_n^\alpha(x; k)$, the following bilateral generating function is derived in this paper:

$$\begin{aligned}\sum_{n=0}^{\infty} Y_n^\alpha(x; k)\zeta_n(y)t^n &= (1-t)^{-(\alpha+1)/k} \exp\{x[1-(1-t)^{-1/k}]\} \\ &\quad \cdot G[x(1-t)^{-1/k},\ \ yt/(1-t)]\,, \end{aligned} \tag{1}$$

where

$$G[x, t] = \sum_{n=0}^{\infty} \lambda_n Y_n^\alpha(x; k)t^n\,, \tag{2}$$

the $\lambda_n \neq 0$ are arbitrary constants, and $\zeta_n(y)$ is a polynomial of degree n in y given by

$$\zeta_n(y) = \sum_{r=0}^{n} \binom{n}{r} \lambda_r y^r\,. \tag{3}$$

It is also shown that the polynomials $Z_n^\alpha(x; k)$ can be expressed as a finite sum of $Z_n^\alpha(y; k)$ in the form

$$Z_n^\alpha(x; k) = \left(\frac{x}{y}\right)^{kn} \sum_{r=0}^{n} \binom{\alpha+kn}{kr} \frac{(kr)!}{r!} [(y/x)^k - 1]^r Z_{n-r}^\alpha(y; k)\,. \tag{4}$$

For $k = 2$, formulas (1) and (4) yield corresponding properties for the polynomials introduced earlier by Preiser [4]. Moreover, when $k=1$, both (1) and (4) would reduce to similar results involving the generalized Laguerre polynomials $L_n^\alpha(x)$. For results analogous to (1) and (4), involving certain classes of functions, the reader may be referred to our papers [5] and [6], respectively.

2. The following results will be required in our analysis.

(i) The generating function [3, p. 803]:

$$\begin{aligned}&\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^\alpha(x; k)t^n \\ &= (1-t)^{-(\alpha+mk+1)/k} \exp\{x[1-(1-t)^{-1/k}]\} Y_m^\alpha(x(1-t)^{-1/k}; k)\,,\end{aligned} \tag{5}$$

where m is any integer $\geqq 0$.

(ii) The explicit expression for $Z_n^\alpha(x; k)$:

$$(6) \qquad Z_n^\alpha(x; k) = \frac{\Gamma(\alpha + kn + 1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(\alpha + kj + 1)} ,$$

which is equation (5), p. 304 of Konhauser [2].

From (6) it follows fairly easily that

$$(7) \qquad \begin{aligned} &\sum_{n=0}^{\infty} Z_n^\alpha(x; k) \frac{t^n}{(\alpha + 1)_{kn}} \\ &= e^t {}_0F_k[-; (\alpha + 1)/k, \cdots, (\alpha + k)/k; -(x/k)^k t] , \end{aligned}$$

since k is a positive integer.

3. Proof of the bilateral generating function (1). Substituting for the coefficients $\zeta_n(y)$ from (3) on the left-hand side of (1), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} Y_n^\alpha(x; k)\zeta_n(y)t^n &= \sum_{n=0}^{\infty} Y_n^\alpha(x; k)t^n \sum_{r=0}^{n} \binom{n}{r} \lambda_r y^r \\ &= \sum_{r=0}^{\infty} \lambda_r (yt)^r \sum_{n=0}^{\infty} \binom{n+r}{r} Y_{n+r}^\alpha(x; k)t^n \\ &= (1 - t)^{-(\alpha+1)/k} \exp\{x[1 - (1 - t)^{-1/k}]\} \\ &\quad \cdot \sum_{r=0}^{\infty} \lambda_r Y_r^\alpha(x(1 - t)^{-1/k}; k)(yt/(1 - t))^r , \end{aligned}$$

by applying (5), and formula (1) would follow if we interpret this last expression by means of (2).

4. Proof of the summation formula (4). In the generating function (7), if we set $t = (y/k)^k z$, we shall get

$$(8) \qquad \sum_{n=0}^{\infty} Z_n^\alpha(x; k) \frac{(y/k)^{kn} z^n}{(\alpha + 1)_{kn}} = \exp\{(y/k)^k z\} {}_0F_k[-(xy/k^2)^k z] ,$$

which, on interchanging x and y, gives us

$$(9) \qquad \sum_{n=0}^{\infty} Z_n^\alpha(y; k) \frac{(x/k)^{kn} z^n}{(\alpha + 1)_{kn}} = \exp\{(x/k)^k z\} {}_0F_k[-(xy/k^2)^k z] ,$$

where, for convenience,

$$(10) \qquad {}_0F_k[\xi] \equiv {}_0F_k[-; (\alpha + 1)/k, \cdots, (\alpha + k)/k; \xi] .$$

From (8) and (9) it follows at once that

$$(11) \qquad \begin{aligned} &\sum_{n=0}^{\infty} Z_n^\alpha(x; k) \frac{(y/k)^{kn} z^n}{(\alpha + 1)_{kn}} \\ &= \exp\{z[(y/k)^k - (x/k)^k]\} \sum_{n=0}^{\infty} Z_n^\alpha(y; k) \frac{(x/k)^{kn} z^n}{(\alpha + 1)_{kn}} , \end{aligned}$$

and on equating coefficients of z^n in (11), we shall be led to our summation formula (4).

5. Applications. First of all we notice that formula (4) may be rewritten as

$$Z_n^\alpha(\mu x; k) = \sum_{r=0}^{n} \binom{\alpha + kn}{kr} \frac{(kr)!}{r!} \mu^{k(n-r)}(1 - \mu^k)^r Z_{n-r}^\alpha(x; k) , \tag{12}$$

which provides us with a multiplication formula for the polynomials $Z_n^\alpha(x; k)$.

On the other hand, by assigning suitable values to the arbitrary coefficients λ_n it is fairly straightforward to obtain, from our formula (1), a large variety of bilateral generating functions for the polynomials $Y_n^\alpha(x; k)$. For instance, if we let

$$\lambda_n = \frac{(-1)^n}{\Gamma(\beta + ln + 1)} , \quad n = 0, 1, 2, \cdots; l = 1, 2, 3, \cdots; \tag{13}$$

and make use of the definition (6), we shall readily arrive at the bilateral generating function

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{n!}{\Gamma(\beta + ln + 1)} Y_n^\alpha(x; k) Z_n^\beta(y; l) t^n \\ &= (1-t)^{-(\alpha+1)/k} \exp\{x[1-(1-t)^{-1/k}]\} H[x(1-t)^{-1/k}, -y^l t/(1-t)] , \end{aligned} \tag{14}$$

where, for convenience,

$$H[x, t] = \sum_{n=0}^{\infty} Y_n^\alpha(x; k) \frac{t^n}{\Gamma(\beta + ln + 1)} . \tag{15}$$

For $k = l = 1$ and $\alpha = \beta$, the generating relation (14) would evidently reduce to the well-known Hille-Hardy formula for the Laguerre polynomials.

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