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ON Σ-INVERSE SEMIGROUPS

S. SRIBALA

In this paper, the Preston-Vagner theorem on representation of inverse semigroups is extended to a class of uniform inverse semigroups. In this connection the notion of Σ-uniformity on an inverse simigroup is introduced which is a modification of the congruence uniformity defined by a set of idempotent separating congruences on the inverse semigroup. Such an inverse semigroup is called a Σ-inverse semigroup. First, it is proved that a Σ-inverse semigroup is complete if and only if all its maximal subgroups are complete and it is compact if and only if the set of its idempotents is finite and all its maximal subgroups are compact. Next, the symmetric Σ-inverse semigroup of bi-Lipchitzian maps between U-open subsets of an uniform space is defined and finally, it is shown that any Σ-inverse semigroup can be embedded isomorphically into a symmetric Σ-inverse semigroup.

1. Σ-inverse semigroups. We refer to [1] for information on semigroups and to [2] for uniform spaces. We shall always consider symmetric Hausdorff uniformities. Let $(X, \mathfrak{U})$ be a uniform space where $\mathfrak{U} = \{U_k; k \in K\}$. A subset Y of X is said to be U-open if $x \in Y \Rightarrow U_k(x) \subset Y$ for all $k \in K$. A U-open subset is both open and closed. The set of all U-open subsets of X is closed for the operations of union and intersection and contains the null set ϕ and X. A mapping α of X into itself is called Lipchitzian if $(x, y) \in U_k \Rightarrow (x\alpha, y\alpha) \in U_k$. If α is a Lipchitzian map and if α^{-1} exists and is also Lipchitzian, then α is called a bi-Lipchitzian map.

DEFINITION 1. Let S be an inverse semigroup. A symmetric Hausdorff uniformity $\mathfrak{U} = \{U_k; k \in K\}$ is called a Σ-uniformity on S if the following conditions hold:

(Σ 1) $U_k \subseteqq \mathscr{H}$ for each $k \in K$.

(Σ 2) The maps $\lambda_a: x \rightarrow ax$ and $\rho_a: x \rightarrow xa$ of S are Lipchitzian maps.

(Σ 3) The map $a \rightarrow a^{-1}$ of S is Lipchitzian.

If $\mathfrak{U}$ is a Σ-uniformity on S, then $(S, \mathfrak{U})$ is called a Σ-inverse semigroup.

In the sequel, $(S, \mathfrak{U})$ denotes a Σ-inverse semigroup.

PROPOSITION 2. *Multiplication in a Σ-inverse semigroup $(S, \mathfrak{U})$ is uniformly continuous.*

Proof. Given $U_k \in \mathfrak{U}$ there exists $U_{k_1} \in \mathfrak{U}$ such that $U_{k_1} \circ U_{k_1} \subseteqq U_k$.

Then, $(x, x') \in U_{k_1}$ $(y, y') \in U_{k_1} \Rightarrow (xy, x'y) \in U_{k_1}$, $(x'y, x'y') \in U_{k_1}$ (by Σ 2) $\Rightarrow$ $(xy, x'y') \in U_{k_1} \circ U_{k_1} \subseteqq U_k$.

PROPOSITION 3. *The set E of idempotent of $(S, \mathfrak{U})$ is a closed, discrete subset of S.*

Proof. E is discrete by $\Sigma 1$. If $x \in \bar{E}$, then $U_k(x) \cap E \neq \emptyset$ for every $k \in K$. Let $U_{k_1} \in \mathfrak{U}$ be such that $U_{k_1} \circ U_{k_1} \circ U_{k_1} \subseteqq U_k$. Since $U_{k_1}(x) \cap E \neq \emptyset$, there is $e_{k_1} \in E$ such that $(x, e_{k_1}) \in U_{k_1}$. Then, $(xe_{k_1}, e_{k_1}) \in U_{k_1}$, $(x^2, xe_{k_1}) \in U_{k_1}$ by Σ 2 and so $(x^2, x) \in U_{k_1} \circ U_{k_1} \cdot U_{k_1} \subseteqq U_k$. This is true for all $k \in K$. Hence $x = x^2 \in E$ and E is closed.

PROPOSITION 4. *Let T be an inverse subsemigroup of a Σ-inverse semigroup $(S, \mathfrak{U})$. Then T with the relative uniformity is a Σ-inverse semigroup.*

Proof. The relative uniformity for T is given by $\mathfrak{U}_T = \{U_k \cap T \times T, k \in K\}$ which satisfies the conditions Σ 2 and Σ 3 of Definition 1. To show that Σ 1 is satisfied, it is enough to observe that for $a, b \in T$, $a\mathscr{H}b$ in T if and only if $a\mathscr{H}b$ in S.

The following can easily be proved.

PROPOSITION 5. *The maximal subgroups $H_e(e \in E)$ of S are closed. They are topological groups for the relative topology. Further, if $e\mathscr{D}f$, then H_e and H_f are homeomorphic.*

We now give a necessary and sufficient condition for the completeness and compactness of a Σ-inverse semigroup $(S, \mathfrak{U})$.

THEOREM 6. *A Σ-inverse semigroup $(S, \mathfrak{U})$ is complete if and only if all its maximal subgroups are complete.*

Proof. If S is complete, then the subgroup H_e are all complete, being closed subsets of S. Conversely suppose that each H_e is complete. Let $\{x_k; k \in K\}$ be a Cauchy K-net in S. Then, given $k \in K$, there exists $k_1 \in K$ such that $(x_{k'}, x_{k''}) \in U_k$ for all $k', k'' \geqq k_1$. Hence we can assume without any loss in generality that any given Cauchy K-net is contained in a single $\mathscr{H}$ class. Suppose that the Cauchy K-net $\{x_k\}$ is contained in the $\mathscr{H}$ class $R_e \cap L_f$, $(e, f \in E)$. Let z be any element of $R_e \cap L_f$. Then $\{x_k z^{-1}\}$ is a Cauchy K-net in H_e and so converges to some point $y \in H_e$. Then, we have $\lim_{k \in K} x_k = yz$. For, given U_{k_0}, there is a $k_1 \in K$ such that $(x_k z^{-1}, y) \in U_{k_0}$ for all $k \geqq k_1$ and so $(x_k z^{-1} z, yz) = (x_k, yz) \in U_{k_0}$(by Σ 2) for all $k \geqq k_1$ and $\lim_{k \in K} x_k = yz$. Thus S is complete.

THEOREM 7. *A Σ-inverse semigroup $(S, \mathfrak{U})$ is compact if and only if E is finite and each H_e is compact.*

Proof. If S is compact, it follows that H_e is compact and E is finite. Conversely, suppose that E is finite and each H_e is compact. We first show that for any $a \in S$, H_a is compact. Let $e, f \in E$ be such that $a \in R_e \cap L_f$. Then $H_a = H_e a$ and $x \to xa$ is uniformly continuous and so H_a is compact. The distinct $\mathscr{H}$-classes of S are the nonempty sets $R_e \cap L_f (e, f \in E)$ and these are only finite in number because E is finite. Thus, S is a union of finite number of compact sets $H_a (a \in S)$ and so is compact.

A natural example of a Σ-uniformity on an inverse semigroup S is given by the congruence uniformity defined by a set $\{\theta_k; k \in K)$ of idempotent separating congruences on S such that given $k_1, k_2 \in K$ there exists $k_3 \in K$ such that $\theta_{k_3} \subset \theta_{k_1} \cap \theta_{k_2}$ and $\bigcap_{k \in K} \theta_k = \tau$, where the uniformity is given by the sets $U_k = \{(x, y), x, y \in S/x\theta_k y\}$. In fact, we have

PROPOSITION 8. *If $(S, \mathfrak{U})$ is a Σ-inverse semigroup with idempotent surroundings* (i.e., $U_k \circ U_k \subset U_k$) *then each U_k is an idempotent separating congruence on S.*

2. Symmetric Σ-inverse semigroup. Let $(X, \mathfrak{U})$ be a uniform space. Let $\mathscr{I}(X)$ be the symmetric inverse semigroup of all partial $(1-1)$ transformations on X. Let $\Omega(X)$ be the subset of $\mathscr{I}(X)$ consisting of all partial bi-Lipchitzian maps between U-open subsets of X. $\Omega(X)$ is not empty as it contains the null map and identity map of X.

PROPOSITION 9. *$\Omega(X)$ is an inverse subsemigroup of $\mathscr{I}(X)$.*

Proof. If $\alpha \in \Omega(X)$, then α^{-1} also belongs to $\Omega(X)$. Thus it is enough to show that $\Omega(X)$ is a subsemigroup of $\mathscr{I}(X)$. Let $\alpha, \beta \in \Omega(X)$ and

$$A = \nabla(\alpha) \cap \Delta(\beta) .$$

(Note: $\Delta(\alpha)$ denotes the domain and $\nabla(\alpha)$ the range of the partial map α). Then A is U-open. If $A = \varnothing$ then $\alpha\beta = 0 \in \Omega(X)$. If $A \neq \varnothing$, let $A_1 = A\alpha^{-1}$, $A_2 = A\beta$. A_1 is U-open, since, $x \in A_1$, $(x, y) \in U_k \Rightarrow y \in \Delta(\alpha)$, $(x\alpha, y\alpha) \in U_k \Rightarrow y\alpha \in U_k(x\alpha) \subset A \Rightarrow y \in A_1$. Similarly, A_2 is also U-open. It is clear that $\alpha\beta$ is a $(1-1)$ Lipchitzian map of A onto B whose inverse $\beta^{-1}\alpha^{-1}$ is also Lipchitzian. Thus $\alpha\beta \in \Omega(X)$. Hence $\Omega(X)$ is an inverse subsemigroup of $\mathscr{I}(X)$.

The uniformity on X induces in a natural way a uniformity on $\Omega(X)$ which is defined as follows.

DEFINITION 10. For each $k \in K$ let

$$U_k^* = (\alpha, \beta), \alpha, \beta \in \Omega(X)/\alpha \mathscr{H} \beta, (x\alpha, x\beta) \in U_k \text{ for all } x \in \Delta(\alpha), (y\alpha^{-1}, y\beta^{-1}) \in U_k \text{ for all } y \in \nabla(\alpha)\} .$$

$$\text{Let } \mathfrak{U}^* = \{U_k^*; k \in K\} .$$

It is easily verified that $\mathfrak{U}^*$ defines a Hausdorff uniformity on $\Omega(X)$ such that $U_k^* \subset \mathscr{H}$ for all $k \in K$.

PROPOSITION 11. *Left and right multiplication in $\Omega(X)$ are Lipchitzian maps.*

Proof. Let $\alpha, \beta, \gamma \in \Omega(X)$, $(\alpha, \beta) \in U_k^*$. Then $\alpha \mathscr{H} \beta$, $(x\alpha, x\beta) \in U_k$ for all $x \in \Delta(\alpha)$ and $(y\alpha^{-1}, y\beta^{-1}) \in U_k$ for all $y \in \nabla(\alpha)$. Let $C = \nabla(\alpha) \cap \Delta(\gamma) = \nabla(\beta) \cap \Delta(\gamma)$, $A_1 = C\alpha^{-1}$, $A_2 = C\beta^{-1}$ and $B = C\gamma$. All these subsets are U-open. Now, $x \in A_1 \Rightarrow x \in \Delta(\alpha) \Rightarrow (x\alpha, x\beta) \in U_k \Rightarrow x\beta \in U_k(x\alpha) \subseteqq C \Rightarrow x \in A_2 \Rightarrow A_1 \subseteqq A_2$. Similarly $A_2 \subseteqq A_1$ and so $A_1 = A_2$. We have $\alpha\gamma \mathscr{H} \beta\gamma$ since $\Delta(\alpha\gamma) = A_1 = A_2 = \Delta(\beta\gamma)$ and $\nabla(\alpha\gamma) = B = \nabla(\beta\gamma)$. Since α, β, γ are bi-Lipchitzian maps, it follows that $(x\alpha\gamma, x\beta\gamma) \in U_k$ for all $x \in \Delta(\alpha\gamma)$ and $(y\gamma^{-1}\alpha^{-1}, y\gamma^{-1}\beta^{-1}) \in U_k$ for all $y \in \nabla(\alpha\gamma)$. Thus $(\alpha\gamma, \beta\gamma) \in U_k^*$ and multiplication on the right by elements of $\Omega(X)$ is a Lipchitzian map. Similarly, we can show that the left multiplication is also a Lipchitzian map.

PROPOSITION 12. *The map $\alpha \rightarrow \alpha^{-1}$ of $\Omega(X)$ is Lipchitzian.*

Proof. $(\alpha, \beta) \in U_k^* \Leftrightarrow \alpha \mathscr{H} \beta$, $(x\alpha, x\beta) \in U_k$ for all $x \in \Delta(\alpha)$ and $(y\alpha^{-1}, y\beta^{-1}) \in U_k$ for all $y \in \nabla(\alpha) \Leftrightarrow \alpha^{-1} \mathscr{H} \beta^{-1}$, $(y\alpha^{-1}, y\beta^{-1}) \in U_k$ for all $y \in \Delta(\alpha^{-1}) = \nabla(\alpha)$ and $(x\alpha, x\beta) \in U_k$ for all $x \in \nabla(\alpha^{-1}) = \Delta(\alpha) \Leftrightarrow (\alpha^{-1}, \beta^{-1}) \in U_k^*$.

From the definition of the uniformity $\mathfrak{U}^*$ and of Propositions 11 and 12 it follows immediately that $(\Omega(X), \mathfrak{U}^*)$ satisfies the conditions $\Sigma 1$–$\Sigma 3$ of Definition 1 and thus we have

THEOREM 13. *$(\Omega(X), \mathfrak{U}^*)$ is a Σ-inverse semigroup.*

DEFINITION 14. $(\Omega(X), \mathfrak{U}^*)$ is called the symmetric Σ-inverse semigroup of partial bi-Lipchitzian maps on $(X, \mathfrak{U})$ or shortly, the symmetric Σ-inverse semigroup on $(X, \mathfrak{U})$.

THEOREM 15. *Let $(X, \mathfrak{U})$ be a complete uniform space with idempotent surroundings. Then $(\Omega(X), \mathfrak{U}^*)$ is complete.*

Proof. Let $\{\alpha_k, k \in K\}$ be a Cauchy K-net in $\Omega(X)$. Without loss in generality we can assume that the Cauchy K-net is contained in a single $\mathscr{H}$-class. Let A be the common domain of α_k, $k \in K$ and B their range. We now define a map $\alpha: A \rightarrow B$ as follows. Define $x\alpha = \lim_{k \in K} x\alpha_k$, $x \in A$. Since B is closed and $\{x\alpha_k\}$ is a Cauchy K-net in B, we have $x\alpha \in B$. α is a well defined map of A into B. Let $x, y \in A$ with $x\alpha = y\alpha$. Then $\lim_{k \in K} x\alpha_k = \lim_{k \in K} y\alpha_k$. Given U_{k_0} we can find $k_1 \in K$ such that $(x\alpha_k, x\alpha) \in U_{k_0}$, $(y\alpha_k, y\alpha) \in U_{k_0}$ for all $k \geqq k_1$. Thus $(x\alpha_k, yx_k) \in U_{k_0} \cdot U_{k_0} = U_{k_0}$ for all $k \geqq k_1$. Since α_k^{-1} is Lipchitzian, $(x\alpha_k\alpha_k^{-1}, y\alpha_k\alpha_k^{-1}) \in U_{k_0}$ for all $k \geqq k_1$ i.e., $(x, y) \in U_{k_0}$. Since U_{k_0} is arbitrary, we have $x = y$ and so α is $(1-1)$. We next show that α is onto. If $y \in B$, then $\{y\alpha_k^{-1}\}$ is a Cauchy K-net in A and hence converges to a point $x \in A$. Then $x\alpha = y$. For, given $U_{k_0} \in \mathfrak{U}$, there exists $k_1 \in K$ such that $(x, y\alpha_k^{-1}) \in U_{k_0}$ for all $k \geqq k_1$ and so $(x\alpha_k, y) \in U_{k_0}$ for all $k \geqq k_1$. Thus $y = \lim_{k \in K} \cdot x\alpha_k = x\alpha$ and α is onto. The maps α and α^{-1} are Lipchitzian. For, let $(x, y) \in U_{k_0}$, $x, y \in A$, $k_0 \in K$. Then $(x\alpha_k, y\alpha_k) \in U_{k_0}$ for all $k \in K$. Since $x\alpha = \lim_{k \in K} x\alpha_k$, $y\alpha = \lim_{k \in K} y\alpha_k$ we can find $k_1 \in K$ such that $(x\alpha, x\alpha_k) \in U_{k_0}$, $(y\alpha, y\alpha_k) \in U_{k_0}$ for all $k \geqq k_1$. Hence $(x\alpha, y\alpha) \in U_{k_0} \circ U_{k_0} \cdot U_{k_0} \subseteqq U_{k_0}$. Thus α is Lipchitzian. Similarly we can show that α^{-1} is Lipchitzian and so $\alpha \in \Omega(X)$. It now remains only to show that $\alpha = \lim_{k \in K} \alpha_k$ in $\Omega(X)$. Since $\{\alpha_k, k \in K\}$ is a Cauchy net, given $U_{k_0}^*$ we can find $k_1 \in K$ such that $(\alpha_k, \alpha_{k'}) \in U_{k_0}^*$ for all $k, k' \geqq k_1$ and so $(x\alpha_k, x\alpha_{k'}) \in U_{k_0}$ for all $k, k' \geqq k_1$. Since $x\alpha = \lim_{k \in K} x\alpha_k$ we can find $k_2 \in K$ such that $(x\alpha, x\alpha_k) \in U_{k_0}$ for all $k \geqq k_2$. Let $k_3 \in K$ be such that $k_3 \geqq k_1, k_2$. Then $(x\alpha, x\alpha_{k_3}) \in U_{k_0}$, $(x\alpha_k, x\alpha_{k_3}) \in U_{k_0}$, $(x\alpha, x\alpha_{k_3}) \in U_{k_0}$ for all $k \geqq k_3$. Similarly, we can show that if $y \in B$, then $(y\alpha^{-1}, y\alpha_{k_3}^{-1}) \in U_{k_0}$, $(y\alpha^{-1}, y\alpha_k^{-1}) \in U_{k_0}$, $(y\alpha_k^{-1}, y\alpha_{k_3}^{-1}) \in U_{k_0}$ for all $k \geqq k_3$ and so $(\alpha, \alpha_k) \in U_{k_0}^*$ for all $k \geqq k_3$. Thus $\alpha = \lim_{k \in K} \alpha_k$ and so $\Omega(X)$ is complete.

3. Representation of Σ-inverse semigroups. We now consider the representation of a Σ-inverse semigroup by partial bi-Lipchitzian maps. Let $(S, \mathfrak{U})$ be a Σ-inverse semigroup and $(\Omega(X), \mathfrak{U}^*)$ the symmetric Σ-inverse semigroup on $(S, \mathfrak{U})$. Let ρ be the right regular representation of S in $\mathscr{I}(S)$. We now have

PROPOSITION 16. *$S\rho$ is a closed inverse subsemigroup of $(\Omega(X), \mathfrak{U}^*)$.*

Proof. The set $Sa (a \in S)$ is U-open, for, if $x \in Sa$ and if $(x, b) \in U_k$ for some $k \in K$, then $x \mathscr{H} b$ and $Sb = Sx \subseteqq Sa$ and so $b \in Sa$. The map $\rho_a: S_a^{-1} \rightarrow Sa$ is $(1-1)$ Lipchitzian between U-open sets whose inverse ρ_a^{-1} is also Lipchitzian and $\rho_a \in \Omega(X)$. Thus $S\rho \subset \Omega(S)$ and $S\rho$ is an inverse subsemigroup of $\Omega(S)$. Now, let $\eta \in \overline{S\rho}$. Then $U_k^*(\eta) \cap S\rho \neq \phi$ for every $k \in K$. Let $j \in K$ be such that $U_j \circ U_j \subseteqq U_k$. Then we can find $a_j \in S$ such that $(\eta, \rho_{a_j}) \in U_j^*$. Then $\Delta(\eta) = Sa_j^{-1}$, $\nabla(\eta) = Sa_j$. Let $e_j = a_j a_j^{-1}$ and $b = e_j\eta$. Then $(\eta, \rho_{a_j}) \in U_j^* \Rightarrow (e_j\eta, e_j\rho_{a_j}) = (b, a_j) \in U_j \Rightarrow b \mathscr{H} a_j$. Hence $Sb^{-1} = Sa_j^{-1} = \Delta(\eta)$ and $Sb = Sa_j = \nabla(\eta)$ and so $\eta \mathscr{H} \rho_b$.

Further, if $x \in \Delta(\eta)$, then $(x\eta, xa_j) \in U_j$, $(xb, xa_j) \in U_j$ and so $(x\eta, xb) \in U_j \circ U_j \subseteqq U_k$. This is true for all $k \in K$ and so $x\eta = xb$ for all $x \in \Delta(\eta)$. That is $\eta = \rho_b$ and $S\rho$ is closed.

We now have

THEOREM 17. *A Σ-inverse semigroup $(S, \mathfrak{U})$ can be embedded isomorphically in a symmetric Σ-inverse semigroup.*

Proof. The map $\rho: S \to \Omega(S)$ given by $a \to \rho_a$ is clearly an algebraic isomorphism of S onto $S\rho \subseteqq \Omega(S)$. To prove that it is a uniform isomorphism we will show that for $a, b \in S$, $(a, b) \in U_k \Leftrightarrow (\rho_a, \rho_b) \in U_k^*$. Now $(a, b) \in U_k \Rightarrow a\mathscr{H}b \Rightarrow \rho_a\mathscr{H}\rho_b$. If $x \in Sa^{-1}$, $y \in Sa$, then $(a, b) \in U_k \Rightarrow (xa, xb) \in U_k$ and $(ya^{-1}, yb^{-1}) \in U_k$ and thus $(\rho_a, \rho_b) \in U_k^*$. Conversely $(\rho_a, \rho_b) \in U_k^* \Rightarrow Sa = Sb$, $Sa^{-1} = Sb^{-1} \Rightarrow a\mathscr{H}b$. So, if $a, b \in R_e \cap L_f$, then $(a, b) = (e\rho_a, e\rho_b) \in U_k$. Thus ρ is a uniform isomorphism of $(S, \mathfrak{U})$ onto a closed Σ-inverse subsemigroup of $(\Omega(S), \mathfrak{U}^*)$.

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THE RAMANUJAN INSTITUTE FOR ADVANCED STUDY IN MATHEMATICS
UNIVERSITY OF MADRAS
MADRAS-5, INDIA.