BOUNDS FOR PRODUCTS OF INTERVAL FUNCTIONS

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Since it is possible for $\Pi(1 + G)$ to exist and not be zero when $G$ is unbounded and $1 + G$ is not bounded away from zero, the conditions under which products of the form $|\Pi(x_{q-1}, x_q)\Pi(1 + G)|$ are bounded or bounded away from zero for suitable subdivisions $\{x_q\}$ of $[a, b]$ are important in many theorems concerning product integrals. Conditions are obtained for such bounds to exist for products of the form $\Pi(1 + FG)$ and $\Pi(1 + F + G)$, where $F'$ and $G$ are functions from $R \times R$ to $R$. Further, these results are used to obtain an existence theorem for product integrals.

All integrals and definitions are of the subdivision-refinement type, and functions are from the subset $\{(x, y): x < y\}$ of $R \times R$ to $R$, where $R$ represents the set of real numbers. If $D = \{x_q\}$ is a subdivision of $[a, b]$ and $G$ is a function, then $D(I) = \{[x_{q-1}, x_q]\}$ and $G_q = G(x_{q-1}, x_q)$. The statements that $G$ is bounded, $G \in O^\circ$, $G \in O^\circ Q^\circ$ and $G \in O^\circ B^\circ$ on $[a, b]$ mean there exist a subdivision $D$ of $[a, b]$ and a positive number $B$ such that if $J = \{x_q\}$ is a refinement of $D$, then

(1) $|G(u)| < B$ for $u \in J(I)$,
(2) $|\Pi_r(1 + G_q)| < B$ for $1 \leq r \leq s \leq n$,
(3) $|\Pi_r(1 + G_q)| > B$ for $1 \leq r \leq s \leq n$, and
(4) $\sum_{J(I)} |G| < B$,

respectively. The notation $\{x_{qr}\}$ represents a subdivision of an interval $[x_{q-1}, x_q]$ defined by a subdivision $\{x_q\}$. If $G$ is a function, then $G \in S_1$ on $[a, b]$ only if $\lim_{x \to y} G(x, y)$ and $\lim_{x \to y} - G(x, y)$ exist and are zero for $p \in [a, b]$, and $G \in S_2$ on $[a, b]$ only if $\lim_{x \to y} G(p, x)$ and $\lim_{x \to y} - G(x, p)$ exist for $p \in [a, b]$. Further, $G \in O^\circ A^\circ$ on $[a, b]$ only if $G(x) \in O^\circ \{1 + G\}$ exists and $|\Pi^\circ(1 + G)| = 0$, and $G \in O\{1 + G\}$ on $[a, b]$ only if $G \in O^\circ B^\circ$ exists for $a \leq x < y \leq b$ and $\sum_{a}^{b} |1 + G - \Pi(1 + G)| = 0$. Also, $G \in O^\circ Q^\circ$ and $G \in O^\circ B^\circ$ on $[a, b]$ if there exists a subdivision $D = \{x_q\}$ of $[a, b]$ such that

(1) if $1 \leq q \leq n$ then $x_{q-1} < x < y < x_q$, then $G \in O^\circ Q^\circ$ on $[x, y]$, and
(2) if $1 \leq q \leq n$, then either $G \in O^\circ B^\circ$ on $[x_{q-1}, x_q]$ or $G - 1 \in O^\circ B^\circ$ on $[x_{q-1}, x_q]$, respectively. The statement that $G$ is almost bounded above by $\beta$ (or, almost bounded below by $\beta$) on $[a, b]$ means there exists a positive integer $N$ such that if $D$ is a subdivision of $[a, b]$ and $u \in H$ only if $u \in D(I)$ and $G(u) > \beta$ (or, $G(u) < \beta$) then $H$ has less than $N$ elements. Consult B. W. Helton [2] and J. S. MacNerney [4] for
additional details.

**Theorem 1.** If $G$ is a function, then the following are equivalent:

1. $G \in OB^o$ on $[a, b]$, and
2. if $F \in OP^o$ on $[a, b]$, then $F + G \in OP^o$ on $[a, b]$.

**Proof** ($2 \rightarrow 1$). Let $F$ be the function such that $F(x, y) = 0$ if $G(x, y) \geq 0$ and $F(x, y) = -2$ if $G(x, y) < 0$. Hence, if $J$ is a subdivision of $[a, b]$, then

$$|\prod_{J(I)}(1 + F + G)| = |\prod_{J(I)}(1 + |G|)|,$$

which can be bounded only if $G \in OB^o$.

**Proof** ($1 \rightarrow 2$). Suppose $F \in OP^o$. There exist positive numbers $B$ and $C$ with $B > 1$, a positive integer $i$ and a subdivision $D$ of $[a, b]$ such that if $J = \{x_q\}_q$ is a refinement of $D$, then

1. $|\prod_{r}(1 + F_q)| < B$ for $1 \leq r \leq s \leq w$,
2. $\exp[4B \sum_k |G|] < C$,
3. if $T$ is a collection of nonintersecting subsets of $J(I)$, then the number of $t \in T$ such that $\exp[4B \sum_k |G|] > 2$ is less than $i$, and
4. the number of $u \in J(I)$ such that $|G(u)| > 1/4B$ is less than $i$.

Let $J = \{x_q\}_q$ be a refinement of $D$ and suppose $1 \leq r \leq s \leq w$. Let $L = \{[x_{q-1}, x_q]\}_q$, and let $H$ be the subset of $L$ such that $u \in H$ only if $|1 + F(u)| \leq 1/4B$. Further, let $K$ be the collection of subsets of $L$ such that $k \in K$ only if there exist $u, v \in H$ such that $u$ precedes $v$ on $[a, b]$ and either

1. $k = \{t | t$ precedes $v$ and follows $u\}$ and $k \cap H = \emptyset$,
2. $u$ is the first element in $H$ and $k = \{t | t$ precedes $u\}$, or
3. $v$ is the last element in $H$ and $k = \{t | t$ follows $v\}$.

Let $u \in M$ only if $u \in H$ and $|G(u)| > 1/4B$, and let $k \in N$ only if $k \in K$ and $\exp[4B \sum_k |G|] > 2$. Hence, $M$ and $N$ each has less than $i$ elements. Also, $K$ has at most one more element than $H$. Hence, $K - N$ can have at most $i$ more elements than $H - M$. Let $j, m$ and $n$ denote the number of elements in $M, H - M$ and $K - N$, respectively, and suppose $U = \bigcup_{k \in K} k$. Hence,

$$|\prod_{U}(1 + F + G)|$$

$$\leq |\prod_{H}[1 + F + |G|]| \cdot |\prod_{U}(1 + F + G)|$$

$$\leq |\prod_{K}[1/4B + |G|]| \cdot |\prod_{U}(1 + F + G)|$$

$$\leq |\prod_{K}(1/4B + |G|)| \cdot |\prod_{U}(1 + F + G)|$$

$$\leq C(1/2B)^{\ast} \cdot \{\prod_{k \in K}[I_k(1 + F)]|I_k(1 + 4B |G|)]\right\}$$

$$= C(1/2B)^{\ast} \cdot \{\prod_{k \in K}[I_k(1 + F)]|I_k(1 + 4B |G|)]\right\}.$$
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\[ \{ \Pi_{k \in \mathbb{K} \cap \mathbb{N}} [ \Pi_k(1 + F) | \Pi_k(1 + 4B \cdot G) ] \} \]
\[ \leq C(1/2B)^m \cdot \{ BC \}^i \cdot \{ 2B \}^n \]
\[ = B^i C^{i+1} (2B)^{n-m} \leq B^i C^{i+1} (2B)^i . \]

**Lemma 1.1.** If \( \int_a^b F \) exists, then \( F \in OA^o \) on \([a, b] \).

This result is due to A. Kolmogoroff [3, p. 669]. Further, related results have also been obtained by W. D. L. Appling [1, Th. 2, p. 155] and B. W. Helton [2, Th. 4.1, p. 304].

**Corollary 1.1.** If \( \int_a^b F \) exists, then the following are equivalent:

1. \( F \in OP^o \) on \([a, b]\), and
2. \( \int_a^b F \in OP^o \) on \([a, b]\).

**Indication of proof.** Since \( \int_a^b F \) exists, \( F \in OA^o \) [Lemma 1.1]. The result now follows by using Theorem 1.

**Corollary 1.2.** If \( F \in OP^o \) on \([a, b]\), \( \Pi^i(1 + F) \) exists and \( \int_a^b |G| = 0 \), then \( \Pi^i(1 + F + G) \) exists and is \( \Pi^i(1 + F) \).

**Indication of proof.** A related result is proved by B. W. Helton [2, Th. 5.6, p. 315]. This result follows by an argument similar to the one used in that theorem since Theorem 1 implies that \( F + G \in OP^o \).

**Corollary 1.3.** If \( G \) is a function, then the following are equivalent:

1. \( G \in OP^o \) on \([a, b]\), and
2. if \( F \in OB^o \) on \([a, b]\), then \( F + G \in OP^o \) on \([a, b]\).

**Proof.** Theorem 1 establishes that (1) implies (2). Further, (2) implies (1) since \( F \equiv 0 \) belongs to \( OB^o \).

B. W. Helton has shown that \( G \) is a function from \( S \times S \) to \( N \) such that \( G \in OA^o \) and \( G \in OB^o \), then \( G \in OM^o \), where \( S \) represents a linearly ordered set and \( N \) represents a ring which has a multiplicative identity element denoted by 1 and has a norm \( | \cdot | \) with respect to which \( N \) is complete and \( |1| = 1 \) [2, Th. 3.4 (1 \( \to \) 2), p. 301]. We now use Theorem 1 to establish a related result. In particular, we show that if \( F \) and \( G \) are functions from \( R \times R \) to \( R \) such that \( F \in OM^o \), \( F \in OP^o \), \( F \in S \cap S \), and \( G \in OB^o \) on \([a, b]\) and \( \int_a^b G \) exists, then \( F + G \in OM^o \) on \([a, b]\).

**Lemma 2.1.** If \( F \) and \( G \) are functions such that \( F \in OM^o \), \( F \in
If $F \in O^\circ$, $F \in S_1$ and $G \in O^B$, then there exists a sub-
division $\{y_i\}$ of $[a, b]$ such that if $y_{q-1} < x < y < y_q$ and $H$ is a sub-
division of $[x, y]$, then

$$|1 - \Pi_{H,I}(1 + F + G)| < \varepsilon.$$ 

Further, if $F \in S_2$ and $G \in S_2$ on $[a, b]$, then there exists a subdivision
$\{z_i\}$ of $[a, b]$ such that if $z_{q-1} \leq x < y \leq z_q$ and $H$ is a subdivision of
$[x, y]$, then

$$|1 + F(x, y) + G(x, y) - \Pi_{H,I}(1 + F + G)| < \varepsilon.$$

**Proof.** Suppose $F$ and $G$ are functions such that $F \in O^M$, $F \in O^\circ$, $F \in S_1$ and $G \in O^B$, and $\varepsilon > 0$. It follows from Theorem 1 that $F + G \in O^\circ$. There exist a subdivision $D = \{y_i\}$ of $[a, b]$ and a number $B > 1$ such that if $J = \{x_i\}$ is a subdivision of $D$, then

1. $|\Pi_j(1 + F_j)| < B$ and $|\Pi_j(1 + F_j + G_j)| < B$ for $1 \leq i \leq j \leq n$,
2. $|F(x, y)| < \varepsilon/9B$ and $\Sigma_{y_i=1}^n|G| < \varepsilon/9B$ if $1 \leq q \leq n$, $x_{q-1} < x < y < x_q$ and $H$ is a subdivision of $[x, y]$, and
3. $\Sigma_{y_i=1}^n(1 + F_j) - \Pi_{H,J}(1 + F) < \varepsilon/9B$, where $H_i$ is a subdivision of $[x_{q-1}, x_q]$ for $q = 1, 2, \ldots, n$.

Suppose $1 \leq q \leq n$ and $y_{q-1} < x < y < y_q$. If $H = \{h_i\}$ is a subdivision of $[x, y]$, then

$$|1 - \Pi_{H,I}(1 + F + G)| = |1 + F(x, y) - F(x, y) - \Pi_{x=1}^q(1 + F_j)| + |\Pi_{x=q+1}^n(1 + F) - F(x, y)| + \Sigma_{x_i=1}^n|G| |\Pi_{x=q+1}^n(1 + F) + G_k|$$

$$< \varepsilon/9B + \varepsilon/9B + B\varepsilon/9B^3 < \varepsilon.$$

We now make the additional suppositions that $F \in S_2$ and $G \in S_2$ on $[a, b]$. There exists a subdivision $E = \{w_i\}$ of $[a, b]$ such that

1. $y_q \in (w_{2q}, w_{2q+1})$ for $1 \leq q < u$,
2. $|F(y_q, w_{2q+1}) + G(y_q, w_{2q+1}) - F(y_q, x) - G(y_q, x)| < \varepsilon/2$ for $0 \leq q < u$ and $x \in (y_q, w_{2q+1})$,
3. $|F(w_{2q}, y_q) + G(w_{2q}, y_q) - F(x, y_q) - G(x, y_q)| < \varepsilon/2$ for $0 < q \leq u$ and $x \in (w_{2q}, y_q)$.

Let $D_1 = \{z_i\}$ be the subdivision $D \cup E$ of $[a, b]$. Suppose $1 \leq q \leq 3u$, $z_{q-1} \leq x < y \leq z_q$ and $H$ is a subdivision of $[x, y]$. If either $z_{q-1} < x < y < z_q$ or neither $z_{q-1} < x < y < z_q$ is in $D$, then

$$|1 + F(x, y) + G(x, y) - \Pi_{H,I}(1 + F + G)|$$

$$\leq |F(x, y)| + |G(x, y)| + |1 - \Pi_{H,I}(1 + F + G)|$$

$$< \varepsilon/9B + \varepsilon/9B^3 + \varepsilon/3B < \varepsilon.$$
If $z_{q-1} \in D_i$, $x = z_{q-1}$ and $H = \{h_q\}$, then
\[
|1 + F(x, y) + G(x, y) - \Pi_{H(t)}(1 + F + G)|
\leq |F(x, y) + G(x, y) - F(x, h_i) - G(x, h_i)|
+ |1 + F(x, h_i) + G(x, h_i)||1 - \Pi_{\Delta}^*[1 + F(h_{q-1}, h_q)]
+ G(h_{q-1}, h_q)||
< \varepsilon/2 + B\varepsilon/3B < \varepsilon.
\]

If $z_q \in D_i$ and $y = z_q$, the necessary inequality follows in a similar manner. Therefore, $D_i$ is the desired subdivision.

**Theorem 2.** If $F$ and $G$ are functions such that $F \in OM^\circ$, $F \in OP^\circ$, $F \in S_1 \cap S_2$ and $G \in OB^\circ$ on $[a, b]$ and $\int_a^b G$ exists, then $F + G \in OM^\circ$ on $[a, b]$.

**Proof.** We initially show that if $\varepsilon > 0$ then there exists a subdivision $D$ of $[a, b]$ such that if $H = \{h_q\}$ is a refinement of $D$ and $H_q$ is a subdivision of $[x_{q-1}, x_q]$ for $q = 1, 2, \cdots, n$, then
\[
\Sigma^* \left|1 + F_q + G_q - \Pi_{H(t)}(1 + F + G)\right| < \varepsilon.
\]
Let $\varepsilon > 0$. It follows from Lemma 1.1 that $G \in OA^\circ$ and from Theorem 1 that $F + G \in OP^\circ$. Thus, by employing the hypothesis and Lemma 2.1, there exist a subdivision $D_1 = \{y_q\}$ of $[a, b]$ and a number $B > 1$ such that if $J = \{y_q\}$ is a refinement of $D_1$, then

1. $\Sigma^* |F_q| < B$,
2. $|\Pi_{\Delta}^*(1 + F_q)| < B$ for $1 \leq i \leq j \leq n$,
3. $\Sigma^* |G_q - \Sigma^* L_q(I) G| < \varepsilon/5$ and $\Sigma^* \left|1 + F_q - \Pi_{\Delta}^*(1 + F)\right| < \varepsilon/5$, where $L_q$ is a subdivision of $[x_{q-1}, x_q]$ for $1 \leq q \leq n$,
4. $|1 - \Pi_{H(t)}(1 + F)| < \varepsilon/5B$ and $|1 - \Pi_{H(t)}(1 + F + G)| < \varepsilon/5B^2$ for $1 \leq q \leq n, x_{q-1} < x < y < x_q$ and $H$ a subdivision of $[x, y]$.

Further, it also follows from Lemma 2.1 that there exists a subdivision $D_2 = \{z_q\}$ of $[a, b]$ such that if $1 \leq q \leq n, z_q-1 < x < y < z_q$ and $H$ is a subdivision of $[x, y]$, then
\[
|1 + F(x, y) + G(x, y) - \Pi_{H(t)}(1 + F + G)| < \varepsilon/10B.
\]
Let $D = D_1 \cup D_2$, and suppose $H = \{z_q\}$ is a refinement of $D$ and $H_q = \{z_q\}$ is a subdivision of $[x_{q-1}, x_q]$ for $1 \leq q \leq n$. Let $P$ be the set such that $q \in P$ only if $[x_{q-1}, x_q]$ has an end point in $D_i$, and let $Q = \{i\} - P$. Further, to simplify notation, let $F_{q,r} = F(x_{q,r-1}, x_{q,r})$, $G_{q,r} = G(x_{q,r-1}, x_{q,r})$, $A_{q,r} = \Pi_{\Delta}^*[1 + F_{q,r}]$ and $B_{q,r} = \Pi_{\Delta}^*[1 + F_{q,r} + G_{q,r}]$. Thus,
\[
\Sigma^* |1 + F_q + G_q - \Pi_{H(t)}(1 + F + G)|
\leq \Sigma_{q \in P} |1 + F_q + G_q - B_q|.$
+ \sum_{q \in Q} |1 + F_q + G_q - B_{q_0}|
< 2\mu\varepsilon/10u + \sum_{q \in Q} |1 + F_q + G_q - [A_{q,n(q)} + 1
+ \sum_{r=1}^{n(q)} A_{qr}G_{qr}B_{qr}]|
\leq \varepsilon/5 + \sum_{q \in Q} |1 + F_q - A_{q,n(q)}|
+ \sum_{q \in Q} |G_q - \sum_{r=1}^{n(q)} A_{qr}G_{qr}B_{qr}|
< 2\varepsilon/5 + \sum_{q \in Q} |G_q - \sum_{r=1}^{n(q)} A_{qr}G_{qr}B_{qr}|
\leq 3\varepsilon/5 + \sum_{q \in Q} \sum_{r=1}^{n(q)} |1 - A_{qr}||G_q|
+ \sum_{q \in Q} \sum_{r=1}^{n(q)} |A_{qr}||1 - B_{qr}|
< 3\varepsilon/5 + (\varepsilon/5B)B + (\varepsilon/5B^2)B^2 = \varepsilon.

Hence, if \(a \leq x < y \leq b\) and \(\varepsilon > 0\), then there exist a subdivision \(D\) of \([a, b]\) and a number \(B\) such that if \(H = \{x_i\}_{i=1}^n\) is a refinement of \(D\) and \(H_q\) is a subdivision of \([x_{q-1}, x_q]\), then

1. \(|H_i(1 + F_q + G_q)| < B\) for \(1 \leq i \leq j \leq n\), and
2. \(|\sum_{q \in Q} (1 + F_q + G_q - H_q(1 + F + G))| < \varepsilon/B^2\).

Thus, if \(H\) and \(H_q\) are defined as above, then

\[
|\prod_i^n (1 + F_q + G_q) - \prod_{i=1}^j (1 + F_q + G_q)|
\leq B^i \sum_{q \in Q} |1 + F_q + G_q - H_q(1 + F + G)|
< B^i (\varepsilon/B^2) = \varepsilon.
\]

Therefore, \(\prod_i^n (1 + F + G)\) exists.

It now follows that \(|1 + F + G - H(1 + F + G)| = 0\). Hence, \(F + G \in OM^o\) on \([a, b]\).

**Theorem 3.** If \(F \in OQ^o\), \(G \in OB^o\) and \(1 + F + G\) is bounded away from zero on \([a, b]\), then \(F + G \in OQ^o\) on \([a, b]\).

**Proof.** There exist a subdivision \(D\) of \([a, b]\), a positive number \(c < 1\) and a positive integer \(m\) such that if \(J = \{x_i\}_{i=1}^n\) is a refinement of \(D\), then

1. \(|1 + F_q + G_q| > c\) for \(1 \leq q \leq n\),
2. \(|\prod_i^n (1 + F_q)| > c\) for \(1 \leq i \leq j \leq n\), and
3. if \(K\) is any collection of nonintersecting subsets of \(J(I)\), then the number of \(k \in K\) such that \(\sum_k |G_k|/c > 1/2\) is less than \(m\).

Suppose \(J = \{x_i\}_{i=1}^n\) is a refinement of \(D\) and \(1 \leq r \leq s \leq n\). Let \(K = \{k_i\}\) be the collection of nonintersecting subsets of \(\{x_{q-1}, x_q\}\) such that

1. \(k_i = \{x_{q-1}, x_q\}\), where \(m(1)\) is the first integer such that \(m(1) \geq r\) and \(|G_{m(1)}|/c \leq 1/2\) and \(n(1)\) is the largest integer such that \(n(1) < s\), \(\sum_{1}^{n(1)} |G_q|/c \leq 1/2\) and \(\sum_{m(1)+1}^{n(1)} |G_q|/c > 1/2\) if such an integer...
exists and $s$ otherwise, and

\[(2) \quad k_i = \left[\{x_{q-1}, x_q]\right]_{m+1}^n, \text{ where } m(j) \text{ is the first integer such that } m(j) > n(j - 1) \text{ and } |G_{m(j)}|/c < 1/2 \text{ and } n(j) \text{ is the largest integer such that } n(j) \leq s, \Sigma_{m(j)+1}^{n(j)} |G_q|/c \leq 1/2 \text{ and } \Sigma_{m(j)+1}^{n(j)+1} |G_q|/c > 1/2 \text{ if such an integer exists and } s \text{ otherwise.} \]

Let $U = \bigcup_{k \in K} k$ and $V = \{[x_{q-1}, x_q] \cap U$. Note that $K$ and $V$ each has a maximum of $m$ elements. Thus,

$$|\Pi_r(1 + F_q + G_q)|$$

$$= |\Pi_V|1 + F + G||\Pi_V|1 + F + G|$$

$$\geq c^m|\Pi_V||1 + F| - |G||$$

$$= c^m\Pi_{k \in K}\Pi_k|1 + F||\Pi_k[1 - |G|(|1 + F|)^{-1}||$$

$$\geq c^m\Pi_{k \in K}\Pi_k(1 - |G|/c)$$

$$\geq c^m\Pi_{k \in K}[1 - \Sigma_k|G|/c] \geq c^m/2^m.$$

**Corollary 3.1.** If \(\int_a^b F\) exists, then the following are equivalent:

1. \(F \in OQ^\circ\) on \([a, b]\), and
2. \(\int_a^b F \in OQ^\circ\) on \([a, b]\).

*Indication of proof.* Since \(\int_a^b F\) exists, \(F \in OA^\circ\) [Lemma 1.1]. The result now follows by using Theorem 3.

**Corollary 3.2.** If \(G\) is a function, then the following are equivalent:

1. \(G \in OQ^1\) on \([a, b]\), and
2. if \(F \in OB^\circ\) on \([a, b]\), then \(F + G \in OQ^1\) on \([a, b]\).

*Indication of proof.* Since \(F \equiv 0\) is in \(OB^\circ\), (2) implies (1). Further, it follows from Theorem 3 that (1) implies (2).

**Lemma 3.1.** If \(0 \leq G \leq 1\) and \(G \in OB^\circ\) on \([a, b]\), then \(-G \in OQ^\circ\) on \([a, b]\).

*Indication of proof.* If \(H\) is a subdivision of \([a, b]\), then

\[\Pi_{H(I)}(1 - G) = \exp[\Sigma_{H(I)} \ln(1 - G)]\]

\[= \exp[-\Sigma_{H(I)} \Sigma_1 G^i/i].\]

Thus, \(\Pi_{H(I)}(1 - G) \to 0\) as \(\Sigma_{H(I)} G \to \infty\).

**Corollary 3.3.** If \(G\) is a function, then the following are equivalent:

1. \(G \in OB^\circ\) on \([a, b]\), and
2. if \(F \in OQ^1\) on \([a, b]\), then \(F + G \in OQ^1\) on \([a, b]\).
Proof. Since it follows from Theorem 3 that (1) implies (2), we need only show that (2) implies (1). The function $|G|$ is almost bounded above on $[a, b]$ by $1/2$. If this is not so, then a contradiction follows by considering the function $F$ such that

1. $F(x, y) = 0$ if $-1/2 \leq G(x, y) \leq 0$,
2. $F(x, y) = -G(x, y) -1/2$ if $G(x, y) < -1/2$,
3. $F(x, y) = -2$ if $0 < G(x, y) \leq 1/2$, and
4. $F(x, y) = -G(x, y) -3/2$ if $G(x, y) > 1/2$.

Thus, although $F \in OQ^1$, $F + G \in OQ^1$ since $|1 + F + G| \leq 1$ and the number of intervals for which $|1 + F + G| = 1/2$ is unbounded. Now, if $G \not\in OB^\circ$, a contradiction follows from Lemma 3.1 by using the function $F$ such that

1. $F(x, y) = -2$ if $G(x, y) \geq 0$, and
2. $F(x, y) = 0$ if $G(x, y) < 0$.

Theorem 4. If $G$ is a function, then the following are equivalent:

1. if $\int_a^b |F| = 0$, then $FG \in OB^\circ$,
2. if $\int_a^b |F| = 0$, then $FG \in OP^\circ$,
3. if $\int_a^b |F| = 0$, then $FG \in OQ^\circ$, and
4. $G$ is bounded on $[a, b]$.

Proof. It follows readily that (4) implies (1). Further, it follows that (4) implies (2) and (3) by using Theorems 1 and 3, respectively. If $G(x, y)$ as $x, y \to p^-$, $G(x, y)$ as $x, y \to p^+$, $G(x, p)$ as $x \to p^-$ and $G(p, x)$ as $x \to p^+$ are bounded for each $p \in [a, b]$, then it follows from the covering theorem that $G$ is bounded on $[a, b]$. If one or more of these bounds fail to exist for some $p \in [a, b]$, then there exists a sequence $\{(y_\ell, z_\ell)\}$ of distinct subintervals of $[a, b]$ such that $|G(y_\ell, z_\ell)| > q^\ell$ for $q = 1, 2, \ldots$, and if $\{x_i\}$ is a subdivision of $[a, b]$ and $r$ is a positive integer then there exist positive integers $i$ and $j$ such that $j > r$ and $x_{i-1} \leq y_j < z_j \leq x_i$. Contradictions to (1) and (2) now follow by considering the function $F$ such that

$$F(x, y) = \frac{|G(x, y)|}{[q^\ell |G(x, y)|]}$$

if there exists a positive integer $q$ such that $x = y_\ell$ and $y = z_\ell$ and $F(x, y) = 0$ otherwise. Here $\int_a^b |F| = 0$, but $FG$ is in neither $OB^\circ$ nor $OP^\circ$. Further, a contradiction to (3) follows by considering the function $F$ such that $F(x, y) = [-G(x, y)]^{-1}$ if there exists a positive integer $q$ such that $x = y_\ell$ and $y = z_\ell$ and $F(x, y) = 0$ otherwise.
LEMMA 5.1. If $G$ is a function such that

(1) $G$ is almost bounded above by $1/3$ on $[a, b]$, and

(2) if $F \in OP^\circ$ on $[a, b]$, then $FG \in OP^\circ$ on $[a, b]$,

then $G \in OB^\circ$ on $[a, b]$.

Proof. Suppose $G \in OB^\circ$ on $[a, b]$. It follows from Theorem 4 that $G$ is bounded on $[a, b]$. There exists a set \( \{C(i)\}_{i=1}^n \) such that

(1) $C(i)$ is a finite set of nonoverlapping subintervals of $[a, b]$ which can be grouped into a collection $D(i)$ of nonintersecting pairs of adjacent intervals,

(2) no interval in $C(i + 1)$ has an end point which is also the end point of an interval in $C(q)$, $q = 1, 2, \ldots, i$,

(3) if $(x, y) \in C(i)$, then $G(x, y) < 1/3$, and

(4) $\sum_{c(\in C)\setminus|G|} > i$.

Let $C = \bigcup_{i=1}^\infty D(i)$, and let $F$ be the function on $[a, b]$ such that if $\{(u, v), (r, s)\} \in C$ and $G(u, v) \geq G(r, s)$, then

(a) $F(u, v) = -2$ if $G(u, v) < 0$,
(b) $F(u, v) = 2$ if $G(u, v) \geq 0$,
(c) $F(r, x) = -1$ if $r = v$ and $r < x$, and
(d) $F(x, s) = -1$ if $s = u$ and $x < s$,

and $F(x, y) = 0$ otherwise. Thus, $F \in OP^\circ$ on $[a, b]$. However,

$$[1 + F(u, v)G(u, v)][1 + F(r, s)G(r, s)] \geq 1 + |G(u, v)|/3.$$  

Hence, since $G$ is bounded and $\sum_{c(\in C)\setminus|G|}^\infty$ is unbounded, $FG \in OP^\circ$. This is a contradiction, and therefore, $G \in OB^\circ$ on $[a, b]$.

LEMMA 5.2. If $G$ is a function such that

(1) $G$ is almost bounded below by $1/10$ on $[a, b]$, and

(2) if $F \in OP^\circ$ on $[a, b]$, then $FG \in OP^\circ$ on $[a, b]$,

then $G - 1 \in OB^\circ$ on $[a, b]$.

Proof. Suppose $G - 1 \in OB^\circ$ on $[a, b]$. It follows from Theorem 4 that $G$ is bounded on $[a, b]$. There exists a set \( \{C(i)\}_{i=1}^n \) satisfying conditions (1) and (2) in Lemma 5.1 plus the additional conditions

(3) if $(x, y) \in C(i)$, then $G(x, y) > 1/10$,

(4) $\sum_{c(\in C)\setminus|G-1|} > i$.

Let $C = \bigcup_{i=1}^\infty D(i)$, where $D(i)$ is defined as in Lemma 5.1. Note that if $\{(u, v), (r, s)\} \in C$ and $G(u, v) \geq G(r, s)$, then either

(5) $G(u, v) \geq 1$ and $|1 - G(u, v)| \geq |1 - G(r, s)|$, or

(6) $G(r, s) < 1$ and either $G(u, v) = G(r, s)$ or

$$|1 - G(u, v)| < |1 - G(r, s)|.$$  

Let $F'$ be the function on $[a, b]$ such that if $\{(u, v), (r, s)\} \in C$ and $G(u, v) \geq G(r, s)$, then
(a) \( F(u, v) = -2 \) and \( F(r, s) = 0 \) if (5) is true,
(b) \( F(u, v) = 1 \) and \( F(r, x) = -1/2 \) if (6) is true, \( r = v \) and \( r < x \), and
(c) \( F(u, v) = 1 \) and \( F(x, s) = -1/2 \) if (6) is true, \( s = u \) and \( x < s \),
and \( F(x, y) = 0 \) otherwise. Thus, \( F \in OP^\circ \) on \([a, b]\). Observe that if (5) is true, then

\[
[1 + F(u, v)G(u, v)][1 + F(r, s)G(r, s)] = -(1 + 2[G(u, v) - 1]),
\]

and if (6) is true, then

\[
[1 + F(u, v)G(u, v)][1 + F(r, s)G(r, s)] \
\geq [1 + G(r, s)][1 - G(r, s)/2] \
> 1 + [1/20][1 - G(r, s)].
\]

Hence, since \( G \) is bounded and \( \{\Sigma_{C(i)} | G - 1\} \) is unbounded, \( FG \in OP^\circ \).

This is a contradiction, and therefore, \( G - 1 \in OB^\circ \) on \([a, b]\).

**Theorem 5.** If \( G \) is a function, then the following are equivalent:

1. \( G \in OB^* \) on \([a, b]\), and
2. if \( F \in OP^\circ \) on \([a, b]\), then \( FG \in OP^\circ \) on \([a, b]\).

**Proof** \((2 \rightarrow 1)\). If \( a \leq \alpha < b \), then there exists a number \( \beta \) such that \( \alpha < \beta \leq b \) and either \( G \in OB^\circ \) on \([\alpha, \beta]\) or \( G - 1 \in OB^\circ \) on \([\alpha, \beta]\).

If this is false and \( a \leq \alpha < \beta < b \), then it follows from Lemmas 5.1 and 5.2 that \( G \) is neither almost bounded above by \( 1/3 \) nor almost bounded below by \( 1/10 \) on \([\alpha, \beta]\); hence, there exist sequences \( \{s_p\}_i \) and \( \{r_p\}_i \) such that

1. \( s_p \) and \( r_p \) are subintervals of \([a, b]\) with a common end point,
2. \( s_p \) precedes \( r_p \) and \( r_{p+1} \) precedes \( s_p \), and
3. \( G(s_p) < 1/10 \) and \( G(r_p) \geq 1/10 \).

Let \( H = \{s_p\}_i \cup \{r_p\}_i \), and let \( F \) be the function on \([a, b]\) such that

1. \( F(x, y) = -1 \) if there exists an interval \((z, y) \in H\) such that \( x < y \) and \( G(z, y) < 1/10 \),
2. \( F(x, y) = 2 \) if \((x, y) \in H\) and \( G(x, y) \geq 1/10 \), and
3. \( F(x, y) = 0 \) otherwise.

Thus, \( F \in OP^\circ \) on \([a, b]\). However, it follows that \( FG \in OP^\circ \) on \([a, b]\) since

\[
[1 + F(s_p)G(s_p)][1 + F(r_p)G(r_p)] > (.9)(1.2) = 1.08.
\]

Similarly, if \( a < \beta \leq b \), then there exists a number \( \alpha \) such that \( a \leq \alpha < \beta \) and either \( G \in OB^\circ \) on \([\alpha, \beta]\) or \( G - 1 \in OB^\circ \) on \([\alpha, \beta]\).

It now follows that \( G \in OB^* \) on \([a, b]\) by using the covering theorem.
Proof (1 → 2). Since $OB^° \subseteq OP^°$, if $G \in OB^°$ and $F \in OP^°$ on $[x, y]$, then $FG \in OP^°$ on $[x, y]$. Note that

$$1 + FG = 1 + F + F(G - 1).$$

Thus, it follows from Theorem 1 that if $G - 1 \in OB^°$ and $F \in OP^°$ on $[x, y]$, then $FG \in OP^°$ on $[x, y]$. Therefore, (1) must imply (2).

Corollary 5.1. If $G$ is a function, then the following are equivalent:

1. $G \in OP^°$ on $[a, b]$, and
2. if $F \in OB^*$ on $[a, b]$, then $FG \in OP^°$ on $[a, b]$.

Indication of proof. It follows that (1) implies (2) by using Theorem 5 and that (2) implies (1) by considering the function $F = 1$.

Lemma 6.1. If $G$ is a bounded function such that

1. $G$ is almost bounded above by $1/3$ on $[a, b]$, and
2. if $F \in OQ^°$ and is bounded on $[a, b]$ and $1 + FG$ is bounded away from zero, then $FG \in OQ^°$ on $[a, b]$,

then $G \in OB^°$ on $[a, b]$.

Proof. Suppose $G \in OB^°$ on $[a, b]$. There exist a subdivision $D$ of $[a, b]$ and a positive integer $m$ such that if $J$ is a refinement of $D$ and $u \in J(I)$ then $|G(u)|/m < 1/2$. Let $H$ be the set such that $u \in H$ only if there exists a refinement $J$ of $D$ such that $u \in J(I)$, and let $F$ be the function such that

1. $F(u) = -2$ if $u \in H$ and $0 \leq G(u) \leq 1/3$,
2. $F(u) = 1/m$ if $u \in H$ and $G(u) < 0$, and
3. $F(x, y) = 0$ otherwise.

Since $F \in OQ^°$ and $1 + FG$ is bounded away from zero, $FG \in OQ^°$. However, it follows from Lemma 3.1 that $FG \in OQ^°$. This is a contradiction, and therefore, $G \in OB^°$.

Lemma 6.2. If $G$ is a bounded function such that

1. $G$ is almost bounded below by $1/10$ on $[a, b]$, and
2. if $F \in OQ^°$ and is bounded on $[a, b]$ and $1 + FG$ is bounded away from zero, then $FG \in OQ^°$ on $[a, b]$,

then $G - 1 \in OB^°$ on $[a, b]$.

Proof. There exist a subdivision $D$ of $[a, b]$ and a number $B$ such that if $J$ is a refinement of $D$ and $u \in J(I)$ then $|G(u)| < B$. Let $H$ be the set such that $u \in H$ only if there exists a refinement $J$ of $D$ such that $u \in J(I)$. Let $H_1$ and $H_2$ be the subsets of $H$ such that $u \in H_1$ only if $G(u) \leq 1$ and $u \in H_2$ only if $G(u) > 1$. For $i = 1, 2$, let $G_i(x, y) = G(x, y)$ if $(x, y) \in H_i$ and $G_i(x, y) = 0$ if $(x, y) \notin H_i$. 
Suppose \( G_1 - 1 \in OB^\circ \) on \([a, b]\). Let \( F \) be the function such that

1. \( F(u) = -2 \) if \( u \in H_1 \) and \( G(u) < 5/12 \) or \( 7/12 < G(u) \leq 1 \),
2. \( F(u) = -3 \) if \( u \in H_1 \) and \( 5/12 \leq G(u) \leq 7/12 \), and
3. \( F(x, y) = 0 \) otherwise.

Since \( F \in OQ^\circ \) and \( 1 + FG \) is bounded away from zero, \( FG \in OQ^\circ \). However, it follows from Lemma 3.1 that \( FG \notin OQ^\circ \). This is a contradiction, and therefore, \( G_1 - 1 \in OB^\circ \).

Suppose \( G_2 - 1 \in OB^\circ \) on \([a, b]\). There exist a set \( \{C(i)\}_i \) and an integer \( m > 1 \) such that

1. \( C(i) \) is a finite set of nonoverlapping subintervals of \([a, b]\) which can be grouped into a collection \( D(i) \) of nonintersecting pairs \( \{(u, v), (r, s)\} \) of adjacent intervals such that either \( G(u, v) > 1 \) or \( G(r, s) > 1 \),
2. no interval in \( C(i + 1) \) has an end point which is also the end point of an interval in \( C(q) \), \( q = 1, 2, \ldots, i \),
3. if \( (x, y) \in C(i) \) then \( G(x, y) > 1/10 \) and \( G(x, y)/m < 1/2 \), and
4. \( \sum_{C(i) \mid |G| - 1 | > i.} \)

Let \( C = \bigcup_i D(i) \), and let \( F \) be the function such that if \( \{(u, v), (r, s)\} \in C \) and \( G(u, v) \geq G(r, s) \) then \( F(u, v) = -1/m \), \( F(r, s) = 1/(m - 1) \) if \( r = v \) and \( F(x, s) = 1/(m - 1) \) if \( s = u \), and \( F(x, y) = 0 \) otherwise. Since \( F \in OQ^\circ \) and \( 1 + FG \) is bounded away from zero, \( FG \in OQ^\circ \). However, if \( \{(u, v), (r, s)\} \in C \) and \( G(u, v) \geq G(r, s) \), then

\[
0 < [1 + F(u, v)G(u, v)][1 + F(r, s)G(r, s)] \\
\leq [1 - G(u, v)/m][1 + G(u, v)/(m - 1)] \\
< 1 + [1 - G(u, v)/m]/m(m - 1).
\]

It follows from Lemma 3.1 that \( FG \notin OQ^\circ \). This is a contradiction, and therefore, \( G_2 - 1 \in OB^\circ \).

Thus, since \( G_i - 1 \in OB^\circ \) on \([a, b]\) for \( i = 1, 2 \), it follows that \( G - 1 \in OB^\circ \) on \([a, b]\).

**Theorem 6.** If \( G \) is a bounded function, then the following are equivalent:

1. \( G \in OB^\ast \) on \([a, b]\), and
2. if \( F \in OQ^\circ \) and is bounded on \([a, b]\) and \( 1 + FG \) is bounded away from zero, then \( FG \in OQ^\circ \) on \([a, b]\).

**Proof** (2 \( \rightarrow \) 1). If \( a \leq \alpha < \beta \leq b \), then there exists a number \( \beta \) such that \( \alpha < \beta \leq b \) and either \( G \in OB^\circ \) on \([\alpha, \beta]\) or \( G - 1 \in OB^\circ \) on \([\alpha, \beta]\). If this is false, then it follows from Lemmas 6.1 and 6.2 that there exist sequences \( \{s_i\}_i \) and \( \{r_i\}_i \) and a set \( H \) defined as in Theorem 5. Let \( F \) be a function on \([a, b]\) such that if \( (u, v) \) and \( (v, s) \) are intervals in \( H \) such that \( G(u, v) \leq 1/10 \) and \( G(v, s) \geq 1/10 \), then
(1) \(1 + F(u, v)G(u, v) = 1/2\) and \(F(v, s) = 0\) if \(G(u, v) < -1/10\),
(2) \(F(x, v) = 1, -1/2 \leq F(v, s) < 0\) and \(1/2 \leq 1 + F(v, s)G(v, s) \leq .95\) if \(-1/10 \leq G(u, v) \leq 0\), and
(3) \(F(x, v) = -3, -1/2 \leq F(v, s) < 0\) and \(1/2 \leq 1 + F(v, s)G(v, s) \leq .95\) if \(0 < G(u, v) < 1/10\),
and \(F(x, y) = 0\) otherwise. Since \(F\) is a bounded function in \(OQ^\circ\) such that \(1 + FG\) is bounded away from zero, \(FG \in OQ^\circ\). However,
\[
|[1 + F(s_p)G(s_p)][1 + F(r_p)G(r_p)]| \leq .95.
\]
Hence, \(FG \in OQ^\circ\). Similarly, if \(a < \beta \leq b\), then there exists a number \(\alpha\) such that \(a \leq \alpha < \beta\) and either \(G \in OB^\circ\) on \([\alpha, \beta]\) or \(G - 1 \in OB^\circ\) on \([\alpha, \beta]\). It now follows that \(G \in OB^*\) on \([a, b]\) by using the covering theorem.

**Proof** \((1 \rightarrow 2)\). This follows from Theorem 3 by a procedure similar to that used in Theorem 5.

**References**


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