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ON THE ADDITIVITY THEOREM FOR n-DIMENSIONAL ASYMPTOTIC DENSITY

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"Additivity theorems" are approximations to countable additivity for set functions. A special case of the additivity theorem for natural density in n-dimensions is proved. To accomplish this a slightly different n-dimensional asymptotic density is defined for which the additivity theorem holds in general.

The additivity theorem (AT) for the case of sets of nonnegative integers reads as follows: If $A_1, A_2, \cdots$ is a disjoint sequence of sets each of which possesses natural density (i.e., for each i,

$$\nu(A_i) = \lim_{n\to\infty} \frac{A_i(n)}{n}$$

exists), then there exists a sequence $B_1, B_2, \cdots$ such that, for each i, $A_i \sim B_i$ (i.e., the symmetric difference of A_i and B_i is finite), the natural density of $V = \bigcup_{i=1}^{\infty} B_i$ exists and $\nu(V) = \sum_{i=1}^{\infty} \nu(A_i)$. This is as close as we can get to countable additivity for natural density.

In [3] the author has generalized asymptotic density and natural density to sets of n-dimensional lattice points. The reader is referred to that paper with respect to all concepts and notations which remain undefined below. The question naturally arises as to the validity of the AT for this generalized natural density. Firstly, it seems natural to define $A \sim B$ to mean that the symmetric difference of A and B is contained in $J(N)$ for some N. Then we may conjecture: If A_1, $A_2, \cdots$ is a disjoint sequence of subsets of S (S = all n-tuples of nonnegative integers) where $\nu(A_i)$ exists for each i, then there exists $B_1, B_2, \cdots$ with $B_i \sim A_i$ for each i such that $\nu(\bigcup_{i=1}^{\infty} B_i)$ exists and equals $\sum_{i=1}^{\infty} \nu(A_i)$.

At present we are able to prove only a special case of this conjecture, namely, when $\nu(A_i) = 0$ for each i. (Using [3, Theorem 5.5] we could trivially extend this to the case where $\nu(A_i) \neq 0$ for at most finitely many i.) To accomplish the proof of the special case we shall, in § 2, introduce a slightly different asymptotic density, upper asymptotic density and natural density in n-dimensions. For this last density we prove, in § 3, the AT and apply it, in § 4, to the special case involving our original density.

2. The $\mathscr{C}$-densities. We consider first the Schnirelmann type $\mathscr{C}$-density of a set $A \subseteq S$ (see [2]). It is

$$d_{\mathscr{C}}(A) = glb\left\{\frac{A(L(\boldsymbol{x}))}{S(L(\boldsymbol{x}))} \mid \boldsymbol{x} \in S\backslash\mathbf{0}\right\}.$$

We then define the $\mathscr{C}$-asymptotic density of A:

$$\delta_{\mathscr{C}}(A) = \lim_{N\to\infty} d_{\mathscr{C}}(A \cup J(N)).$$

Continuing, in analogy to [3, § 5], we define the upper $\mathscr{C}$-density of A to be

$$\bar{d}_{\mathscr{C}}(A) = lub\left\{\frac{A(L(\boldsymbol{x}))}{S(L(\boldsymbol{x}))} \mid \boldsymbol{x} \in S\backslash\mathbf{0}\right\},$$

the upper $\mathscr{C}$-asymptotic density of A to be

$$\bar{\delta}_{\mathscr{C}}(A) = \lim_{N\to\infty} \bar{d}_{\mathscr{C}}(A\backslash J(N)),$$

and finally, the $\mathscr{C}$-natural density of A to be $\nu_{\mathscr{C}}(A) = \delta_{\mathscr{C}}(A) = \bar{\delta}_{\mathscr{C}}(A)$ when the second equality holds.

The densities d, δ, $\bar{d}$, $\bar{\delta}$, and ν of [3] will be referred to as K-densities since they depend on "d" which is commonly called "K-density". In § 4 we shall need to compare the K- and $\mathscr{C}$-densities.

3. **The additivity theorem for $\mathscr{C}$-natural density.** We shall say that a sequence $(\boldsymbol{x}_j)$ of point of S *converges to infinity* ($(\boldsymbol{x}_j) \to \infty$) if, for each $i = 1, 2, \cdots, n$, the ith coordinate of $\boldsymbol{x}_j$ converges to infinity as $j \to \infty$. Two useful corollaries will follow from the

LEMMA. *Let $A \subseteqq S$. Then $\nu_{\mathscr{C}}(A)$ exists if and only if, for each sequence $(\boldsymbol{x}_j) \to \infty$, we have*

$$\lim_{j\to\infty} \frac{A(L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))} \text{ exists}.$$

In this case all the limits are the same and

$$\nu_{\mathscr{C}}(A) = \lim_{j\to\infty} \frac{A(L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))}$$

for each sequence $(\boldsymbol{x}_j) \to \infty$.

Proof. Our methods follow closely to those of [3, Theorems 2.6, 2.7, 5.2, 5.3, 5.4]. Therefore, much will be left for the reader. By [3, Lemma 2.2] we have, when $(\boldsymbol{x}_j) \to \infty$, that

$$\frac{S(L(\boldsymbol{x}_j) \cap J(N))}{S(L(\boldsymbol{x}_j))} \longrightarrow 0 \qquad (j \longrightarrow \infty) \tag{1}$$

for each N. Thus, for each N,

$$d_{\mathscr{C}}(A \cup J(N)) \leqq \underset{j \geqq 1}{glb} \frac{[A \cup J(N)](L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))}$$
$$\leqq \varliminf_{j \to \infty} \frac{[A \cup J(N)](L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))} = \varliminf_{j \to \infty} \frac{A(L(\boldsymbol{x}_j))}{(L(\boldsymbol{x}_j))} ,$$

the last equality following from (1) above. Hence, letting $N \to \infty$, we obtain

$$\delta_{\mathscr{C}}(A) \leqq \varliminf_{j \to \infty} \frac{A(L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))} . \tag{2}$$

Similarly we may obtain

$$\bar{\delta}_{\mathscr{C}}(A) \geqq \varlimsup_{j \to \infty} \frac{A(L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))} . \tag{3}$$

Now, using [3, Lemma 2.2], for each j, choose $M_j > 0$ and $\boldsymbol{x}_j \in S$ such that

$$\frac{S(J(j) \cap L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))} < \frac{1}{2^j} \text{ and } \frac{[A \cup J(M_j)](L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))} < d_{\mathscr{C}}(A \cup J(M_j)) + \frac{1}{2^j} .$$

Clearly $(\boldsymbol{x}_j) \to \infty$ and

$$d_{\mathscr{C}}(A \cup J(j)) \leqq \frac{[A \cup J(j)](L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))}$$
$$\leqq \frac{[A \cup J(M_j)](L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))} \leqq d_{\mathscr{C}}(A \cup J(M_j)) + \frac{1}{2^j}$$

so that $[A \cup J(j)](L(\boldsymbol{x}_j))/S(L(\boldsymbol{x}_j)) \to \delta_{\mathscr{C}}(A)$ as $j \to \infty$. But

$$\frac{[A \cup J(j)](L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))} - \frac{S(J(j) \cap L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))} \leqq \frac{A(L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))} \leqq \frac{[A \cup J(j)](L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))}$$

whence we have proved that there exists a sequence $(\boldsymbol{x}_j) \to \infty$ such that

$$\delta_{\mathscr{C}}(A) = \lim_{j \to \infty} \frac{A(L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))} . \tag{4}$$

Similarly we may find $(\boldsymbol{y}_j) \to \infty$ such that

$$\bar{\delta}_{\mathscr{C}}(A) = \lim_{j \to \infty} \frac{A(L(\boldsymbol{y}_j))}{S(L(\boldsymbol{y}_j))} . \tag{5}$$

Finally, using (2), (3), (4), and (5) we may, as in [3, Theorem 5.4], complete the proof of the Lemma.

COROLLARY 1. *If $\nu_{\mathscr{C}}(A)$ exists, then, for each $\varepsilon > 0$, there exists an $N = N(\varepsilon)$, such that, if $\boldsymbol{x} \in S\backslash J(N)$, then*

$$\left| \frac{A(L(\boldsymbol{x}))}{S(L(\boldsymbol{x}))} - \nu_{\mathscr{C}}(A) \right| < \varepsilon .$$

Proof. If there is an $\varepsilon > 0$ such that for each N there is an $\boldsymbol{x}_N \in S\backslash J(N)$ with

$$\left| \frac{A(L(\boldsymbol{x}_N))}{S(L(\boldsymbol{x}_N))} - \nu_{\mathscr{C}}(A) \right| \geqq \varepsilon ,$$

then $(\boldsymbol{x}_N) \to \infty$ but $A(L(\boldsymbol{x}_N))/S(L(\boldsymbol{x}_N))$ does not converge to $\nu_{\mathscr{C}}(A)$ contrary to the Lemma.

COROLLARY 2. *$\nu_{\mathscr{C}}(A)$ is a finitely additive set function.*

Proof. For $A_1, \cdots, A_k$ disjoint and each possessing $\mathscr{C}$-natural density, take any $(\boldsymbol{x}_j) \to \infty$. Then, by the Lemma,

$$\frac{\left(\bigcup_{i=1}^{k} A_i\right)(L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))} = \sum_{i=1}^{k} \frac{A_i(L(\boldsymbol{x}_j))}{S(L(\boldsymbol{x}_j))} \to \sum_{i=1}^{k} \nu_{\mathscr{C}}(A_i)$$

as $j \to \infty$.

THEOREM. (*Additivity theorem for $\mathscr{C}$-natural density.*) *Let (A_i) be a sequence of subsets of S, pairwise disjoint and such that $\nu_{\mathscr{C}}(A_i)$ exists for each i. Then there exists a sequence (B_i) of sets such that $B_i \sim A_i$ for each i, $\nu_{\mathscr{C}}(\bigcup_{i=1}^{\infty} B_i)$ exists and equals $\sum_{i=1}^{\infty} \nu_{\mathscr{C}}(A_i)$.*

Proof. The B_i's shall be of the form $B_i = A_i\backslash J(N_i)$ for suitable integers N_i. Firstly, for arbitrary N_i, let $V = \bigcup_{i=1}^{\infty} B_i$. Then, for each k, since $V \supset B_1 \cup B_2 \cup \cdots \cup B_k$ and applying Corollary 2 we have

$$\begin{aligned} \underline{\delta}_{\mathscr{C}}(V) &\geqq \underline{\delta}_{\mathscr{C}}(B_1 \cup \cdots \cup B_k) = \nu_{\mathscr{C}}(B_1 \cup \cdots \cup B_k) \\ &= \sum_{i=1}^{k} \nu_{\mathscr{C}}(B_i) = \sum_{i=1}^{k} \nu_{\mathscr{C}}(A_i) . \end{aligned}$$

Thus $\sigma = \sum_{i=1}^{\infty} \nu_{\mathscr{C}}(A_i)$ exists and $\underline{\delta}_{\mathscr{C}}(V) \geqq \sigma$.

We now choose suitable N_i's to assure that $\bar{\delta}_{\mathscr{C}}(V) \leqq \sigma$. We apply Corollary 1 noting that, by Corollary 2, the sets $A_1, A_1 \cup A_2, A_1 \cup A_2 \cup A_3, \cdots$ possess $\mathscr{C}$-natural density. Take $N_1 = -1$ (i.e., $B_1 = A_1$) and, for $k \geqq 1$, choose $N_{k+1} > N_k$ such that $\boldsymbol{x} \in S\backslash J(N_{k+1})$ implies

$$\frac{A_1(L(\boldsymbol{x})) + \cdots + A_{k+1}(L(\boldsymbol{x}))}{S(L(\boldsymbol{x}))} \leqq \nu_{\mathscr{C}}(A_1) + \cdots + \nu_{\mathscr{C}}(A_{k+1}) + \frac{1}{k} .$$

Now, in accordance with (5) above, find a sequence $(x_j) \to \infty$ such that

$$\bar{\delta}_{\mathscr{C}}(V) = \lim_{j\to\infty} \frac{V(L(x_j))}{S(L(x_j))} .$$

Let $m(j)$ be such that $x_j \in J(N_{m(j)+1}) \backslash J(N_{m(j)})$. Clearly $m(j) \to \infty$ as $j \to \infty$. We have

$$\begin{aligned}\bar{\delta}_{\mathscr{C}}(V) &= \lim_{j\to\infty} \frac{V(L(x_j))}{S(L(x_j))} = \lim_{j\to\infty} \frac{B_1(L(x_j)) + \cdots + B_{m(j)}(L(x_j))}{S(L(x_j))} \\ &\leqq \overline{\lim_{j\to\infty}} \frac{A_1(L(x_j)) + \cdots + A_{m(j)}(L(x_j))}{S(L(x_j))} \\ &\leqq \lim_{j\to\infty} \left(\nu_{\mathscr{C}}(A_1) + \cdots + \nu_{\mathscr{C}}(A_{m(j)}) + \frac{1}{m(j)-1}\right) = \sigma .\end{aligned}$$

This completes the proof of the Theorem.

We mention here that the author's student, Mr. F. Wong, has obtained, in the course of his M.Sc. research, the following related but slightly different result (see [1]): Let $A_1 \subseteqq A_2 \subseteqq \cdots \subseteqq S$ be an increasing sequence of sets. Then there is a sequence (B_i) such that $B_i \sim A_i$ for each i,

$$\delta_{\mathscr{C}}\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{i\to\infty} \delta_{\mathscr{C}}(A_i)$$

and

$$\bar{\delta}_{\mathscr{C}}\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{i\to\infty} \bar{\delta}_{\mathscr{C}}(A_i) .$$

4. Comparison of $\mathscr{C}$- and K-asymptotic densities and the special case of the AT for K-natural density. It is evident from the definitions and (2) and (3) above that, for each $A \subseteqq S$, $\delta(A) \leqq \delta_{\mathscr{C}}(A) \leqq \bar{\delta}_{\mathscr{C}}(A) \leqq \bar{\delta}(A)$. Thus, if the K-natural density of A exists, then so does the $\mathscr{C}$-natural density and the two are equal. We venture to conjecture the converse, namely, if the $\mathscr{C}$-natural density of A exists then so does the K-natural density. If this conjecture is ture, then the AT for K-natural density is clearly proved by applying the above theorem. However, we can prove only a little part of this conjecture, namely, that $\nu_{\mathscr{C}}(A) = 0$ if and only if $\nu(A) = 0$. From this, with the AT for $\mathscr{C}$-natural density, follows immediately the special case of the AT for K-natural density promised above.

For $\nu_{\mathscr{C}}(A) = 0 \Leftrightarrow \nu(A) = 0$ it suffices to show that

$$(7) \qquad \bar{\delta}_{\mathscr{C}}(A) = 0 \Longrightarrow \bar{\delta}(A) = 0 .$$

We use the remarkable result of B. Müller [4, Satz 8] which implies, for our purposes, that

$$\frac{A(F)+1}{S(F)+1} \geqq (1-(1-d_{\mathscr{C}}(A))^{1/n})^n , \tag{8}$$

where $F \in \mathscr{K}$ and, of course, n is the dimension of S (see [2, Theorem 17]).

Now (7) will follow immediately from

$$\bar{\delta}(A) \leqq 1-(1-(\bar{\delta}_{\mathscr{C}}(A))^{1/n})^n$$

which in turn follows, upon taking the limit as $N \to \infty$, from

$$\bar{d}(A \backslash J(N)) \leqq 1-\left(\frac{(N+1)^n-1}{(N+1)^n}\right)(1-(\bar{d}_{\mathscr{C}}(A \backslash J(N)))^{1/n})^n . \tag{9}$$

To prove (9) we take an arbitrary $F \in \mathscr{K}$ and show that $[A \backslash J(N)](F)/S(F)$ does not exceed the right hand side of (9). If the point $(N, N, \cdots, N) \notin F$, then $[A \backslash J(N)](F)=0$ and we are done. Hence, assuming that $(N, \cdots, N) \in F$, we have $[\bar{A} \cup J(N)](F) \geqq (N+1)^n-1$ (here $\bar{A}=S \backslash A$). We have, applying (8),

$$\begin{aligned}\frac{[\bar{A} \cup J(N)](F)}{S(F)} &> \frac{[\bar{A} \cup J(N)](F)}{[\bar{A} \cup J(N)](F)+1} \cdot \frac{[\bar{A} \cup J(N)](F)+1}{S(F)+1} \\ &\geqq \left(\frac{(N+1)^n-1}{(N+1)^n}\right)[1-(1-d_{\mathscr{C}}(\bar{A} \cup J(N)))^{1/n}]^n ,\end{aligned}$$

so that

$$\begin{aligned}\frac{[A \backslash J(N)](F)}{S(F)} &= 1-\frac{[\bar{A} \cup J(N)](F)}{S(F)} \\ &\leqq 1-\left(\frac{(N+1)^n-1}{(N+1)^n}\right)[1-(1-d_{\mathscr{C}}(\bar{A} \cup J(N)))^{1/n}]^n.\end{aligned}$$

We obtain (9) if we can show that $d_{\mathscr{C}}(\bar{A} \cup J(N)) \geqq 1-\bar{d}_{\mathscr{C}}(A \backslash J(N))$. But this is easy since, for each $\boldsymbol{x} \in S \backslash \mathbf{0}$, we have

$$\frac{[\bar{A} \cup J(N)](L(\boldsymbol{x}))}{S(L(\boldsymbol{x}))} = 1-\frac{[A \backslash J(N)](L(\boldsymbol{x}))}{S(L(\boldsymbol{x}))} \geqq 1-\bar{d}_{\mathscr{C}}(A \backslash J(N)) .$$

We conclude with an example which shows that, while ν and $\nu_{\mathscr{C}}$ may be the same, δ and $\delta_{\mathscr{C}}$ are not. We leave to the reader the task of proving that $\delta(A) \neq \delta_{\mathscr{C}}(A)$ where A is defined presently.

EXAMPLE. Let S be of dimension two. For integers a and b let

$$\begin{aligned}D(a, b) = {} & [U((a+b, a)) \cap L((a+2b, a+b))] \\ & \cup [U((a, a+b)) \cap L((a+b, a+2b))] .\end{aligned}$$

Take $a_1 = b_1 = 1$, $a_{i+1} = a_i + 2b_i$, $b_{i+1} = (a_{i+1})!$ and define $A = S \setminus \bigcup_{i=1}^{\infty} D(a_i, b_i)$.

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