PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 2, 1973

TAME Z^2-ACTIONS ON E^n

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Let $\mathscr{H}(E^n)$ denote the group of homeomorphisms of euclidean n-space, and G a subgroup isomorphic to $Z \oplus Z$. G is said to be a Z^2-action on E^n and two such actions are said to be equivalent if they are conjugate in $\mathscr{H}(E^n)$. In § 2, the notion of a tame Z^2-action is introduced and for $n \geqq 5$ tame Z^2-actions are shown to be classified by $\pi_1(SO_{n-2}) \cong Z_2$. In § 3, tameness is shown to be inherited by a subaction of a tame Z^2-action and an example of a nontame Z^2-action with tame subactions is given.

1. Introduction. Let U be an n-dimensional manifold and let $\mathscr{H}(U)$ denote the group of homeomorphisms of U onto itself with the compact open topology. If G is a subgroup of $\mathscr{H}(U)$, we say that G *acts* on U and refer to G as an *action*. If K is a topological group which is isomorphic to G, we refer to G as a *K-action*. Two actions are (*topologically*) *equivalent* if they are conjugate in $\mathscr{H}(U)$. We say that G satisfies *Sperner's condition* if for each compact set $X \subset U$ the set $\{g \in G \mid g(X) \cap X \neq \emptyset\}$ is finite. Unless otherwise stated, all actions on E^n will be assumed to be *orientation preserving*, i.e., we require that each member of an action be orientation preserving.

If Z^i denotes the free abelian group on i generators, we have, for $i \leqq n$, the *standard* Z^i-action on E^n, generated by the maps h_j, $j = 1, \cdots, i$, where $h_j(x_1, \cdots, x_n) = (x_1, \cdots, x_j + 1, \cdots, x_n)$. It is a classical result that a Z-action on E^2 is equivalent to the standard action if and only if it satisfies Sperner's condition, and Duvall and Husch [3] showed that for $n \neq 4$, a Z^n-action on E^n is equivalent to the standard action if and only if it satisfies Sperner's condition. In general, however, Sperner's condition is not sufficient to insure that a Z^k-action is equivalent to the standard action. Examples of nonstandard actions which satisfy Sperner's condition may be found in [9], [10], [6], and [3]. Husch [6], and Husch and Row [7] have shown that the standard Z-action for $n > 4$ and the standard Z and Z^2-actions for $n = 3$ are characterized by Sperner's condition together with an additional homotopy condition.

In this note, we define the notion of a tame action (inspired by [6]) and show that the tame Z^2-actions on E^n, $n \geqq 5$, are classified by $\pi_1(SO_{n-2}) \cong Z_2$. We also give examples of some nonstandard Z^2-actions. We will often use the fact that if G satisfies Sperner's condition and has no elements of finite order, then the orbit space U/G is a (T_2) manifold and the natural projection $U \to U/G$ is a regular covering map [8]. We use the symbol $\cong$ to denote homeomorphism or isomorphism, depending on the context. For equivalent formulations of Sperner's condition and any notation not specifically explained here, the reader is referred to [3].

2. Tame actions. Recall that a sequence $\{G_i, \alpha_i\}_{i=1}^{\infty}$

$$G_1 \xleftarrow{\alpha_1} G_2 \xleftarrow{\alpha_2} \cdots$$

of groups and homomorphisms is *stable* if for some subsequence

$$G_{i_1} \xleftarrow{\beta_1} G_{i_2} \xleftarrow{\beta_2} G_{i_3} \longleftarrow \cdots$$

we have

$$\beta_n \mid_{\text{image } \beta_{n+1}} : \text{image } \beta_{n+1} \longrightarrow \text{image } \beta_n$$

is an isomorphism for each n, where $\beta_n = \alpha_{i_n}\alpha_{i_n+1}\cdots\alpha_{i_{n+1}-1}$. We omit the proof of the following proposition. One implication is proved in [7]; the other may be verified by a routine diagram chase.

PROPOSITION 1. *In the commutative diagram*

$$\begin{array}{ccccccccccc}
 & & 1 & & 1 & & 1 & & & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \longrightarrow & A_{i+1} & \xrightarrow{\alpha_i} & A_i & \longrightarrow & A_{i-1} & \longrightarrow & \cdots & \longrightarrow & A_1 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \longrightarrow & B_{i+1} & \xrightarrow{\beta_i} & B_i & \longrightarrow & B_{i-1} & \longrightarrow & \cdots & \longrightarrow & B_1 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \longrightarrow & C_{i+1} & \xrightarrow{\gamma_i} & C_i & \xrightarrow{\gamma_{i-1}} & C_{i-1} & \longrightarrow & \cdots & \longrightarrow & C_1 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
 & & 1 & & 1 & & 1 & & & & 1
\end{array}$$

of groups and homomorphisms, suppose that the columns are exact and that the γ_i are isomorphisms. Then, $\{A_i, \alpha_i\}_{i=1}^{\infty}$ is stable if and only if $\{B_i, \beta_i\}_{i=1}^{\infty}$ is stable. If either sequence is stable, the induced sequence $1 \to \varprojlim A_i \to \varprojlim B_i \to \varprojlim C_i \to 1$ is exact.

Let G be a Z^2-action on E^n. For each $X \subset E^n$, let GX denote the set $\{g(X) \mid g \in G\}$. We say that G is *tame* provided that:

(1) G satisfies Sperner's condition and

(2) For each compact set $C \subset E^n$, there is a compact set $D \subset E^n$

containing C such that the inclusion induced map $l_*: \pi_j(E^n - GD) \to \pi_j(E^n - GC)$ is the zero map for $j = 0, 1$.

We assume now that G is a tame Z^2-action and that $n \geqq 5$. Let O_G be the orbit space E^n/G. Then O_G is a manifold and the projection $p: E^n \to O_G$ is a covering map. Since O_G is an Eilenberg-McLane $K(Z^2, 1)$ space, it follows [15] that O_G has the homotopy type of the torus $T^2 = S^1 \times S^1$. Since $H_n(O_G) \cong H_{n-1}(O_G) \cong 0$, it follows that O_G is noncompact and has one end [14].

Let $\{D_i\}_{i=1}^{\infty}$ be a collection of compact subsets of O_G such that $\bigcup D_i = O_G$, $D_{i+1} \supset D_i$ for each i, and $O_G - D_i$ is connected for each i. Using (2) above, we can find a nested sequence $\{C_i\}_{i=1}^{\infty}$ of compact subsets of E^n such that $GC_i \supset p^{-1}D_i$ and $l_*: \pi_j(E^n - GC_{i+1}) \to \pi_j(E^n - GC_i)$ is zero for each i and $j = 0, 1$. By choosing a subsequence of D_i's if necessary, we can assume without loss of generality that $GC_i = p^{-1}D_i$.

PROPOSITION 2. *$\pi_1(O_G, O_G - D_i) = 0$ for each i, and $E^n - GC_i$ is connected.*

Proof. Let x be a base point for $O_G - D_i$, and assume without loss of generality that $x \in O_G - D_{i+1}$. Let $\hat{x}$ be a point in $E^n - GC_{i+1}$ such that $p(\hat{x}) = x$. Let $\alpha: (I, \{0, 1\}, \{0\}) \to (O_G, O_G - D_i, x)$ be a map, and assume (without loss of generality) that $\alpha(1) \in O_G - D_{i+1}$. Let $\hat{\alpha}$ be the lift of α based at $\hat{x}$. We have $\hat{x}, \hat{\alpha}(1) \in E^n - GC_{i+1}$ so by (2) there is a path β in $E^n - GC_i$ joining $\hat{x}$ and $\hat{\alpha}(1)$. Since $\hat{\alpha}$ and β are homotopic with endpoints fixed in E^n, α and $p\beta$ are homotopic in O_G, so that $[\alpha] = 0$ in $\pi_1(O_G, O_G - D_i)$. The second conclusion follows from the first by a covering space argument.

PROPOSITION 3. *If ε is the end of O_G, π_1 is stable at ε and the natural projection $\pi_1(\varepsilon) \to \pi_1(O_G)$ is an isomorphism.*

Proof. For each i, let $x_i \in E^n - GC_i$ be a base point, $y_i = p(x_i)$, and let α_i, $p\alpha_i$ be connecting paths between x_i, x_{i+1} and y_i, y_{i+1}. We have the following diagram from the exact homotopy sequences of a fibration

$$
\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \uparrow & & \\
1 & \longrightarrow & \pi_1(E^n - GC_i, x_i) & \longrightarrow & \pi_1(O_G - D_i, y_i) & \longrightarrow & \pi_0(p^{-1}(y_i)) & \longrightarrow & 1 \\
 & & & & & \searrow \subseteq & \uparrow \cong & \nearrow & \\
 & & & & & & \pi_1(O_G, y_i) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 1 & &
\end{array}
$$

which gives rise to the commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \pi_1(E^n - GC_i, x_i) & \longrightarrow & \pi_1(O_G - D_i, y_i) & \longrightarrow & \pi_1(O_G, y_i) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \gamma_i} & & \\
1 & \longrightarrow & \pi_1(E^n - GC_{i+1}, x_{i+1}) & \longrightarrow & \pi_1(O_G - D_{i+1}, y_{i+1}) & \longrightarrow & \pi_1(O_G, y_{i+1}) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where the columns are change of base point maps and each γ_i is an isomorphism. We apply Proposition 1 to see that π_1 is stable at ε and that

$$\begin{aligned} 1 &\longrightarrow \varprojlim \{\pi_1(E^n - GC_i, x_i)\} \longrightarrow \varprojlim \{\pi_1(O_G - D_i, y_i)\} \\ &\longrightarrow \varprojlim \{\pi_1(O_G, y_i)\} \longrightarrow 1 \end{aligned}$$

is exact. This translates into $1 \to 1 \to \pi_1(\varepsilon) \to \pi_1(O_G) \to 1$, and the proof is complete.

Now by [11], we may assume that O_G has a (unique) PL structure. There is a map $f\colon T^2 \to O_G$ which is a homotopy equivalence. We may assume f to be an embedding by general position. Let $\tau_G = f(T^2)$. By a theorem of Hudson [5], τ_G is unique up to concordance, hence, up to ambient isotopy [4].

THEOREM 4. *If G is a tame Z^2-action on E^n, $n \geqq 5$, then O_G is PL-homeomorphic to the interior of a regular neighborhood of τ_G.*

Proof. Let N be a regular neighborhood of τ_G in O_G. From the exact sequence of the triad $(O_G, O_G - \tau_G, \operatorname{int} N)$ [1], we have

$$\begin{aligned} \cdots &\longrightarrow \pi_i(O_G - \tau_G, \operatorname{int} N - \tau_G) \longrightarrow \pi_i(O_G, \operatorname{int} N) \\ &\longrightarrow \pi_i(O_G; O_G - \tau_G, \operatorname{int} N) \longrightarrow \pi_{i-1}(O_G - \tau_G, \operatorname{int} N - \tau_G) \longrightarrow \cdots \\ &\longrightarrow \pi_2(O_G, O_G - \tau_G, \operatorname{int} N) \longrightarrow \pi_1(O_G - \tau_G, \operatorname{int} N - \tau_G) \\ &\longrightarrow \pi_1(O_G, \operatorname{int} N)\,. \end{aligned}$$

Since $\pi_i(O_G, \operatorname{int} N) \cong 0$ for all i, we can apply [4, Lemma 12.4] to get $\pi_i(O_G, O_G - \tau_G, \operatorname{int} N) \cong 0$ for all i, and thus $\pi_i(O_G - \tau_G, \operatorname{int} N - \tau_G) \cong 0$ for all i. It follows that the inclusion $\operatorname{int} N - \tau_G \to O_G - \tau_G$ is a homotopy equivalence and hence the inclusion bdy $N \to O_G - \operatorname{int} N$ is a homotopy equivalence. General position gives an inclusion induced isomorphism $\pi_1(O_G - \operatorname{int} N) \overset{\to}{\cong} \pi_1(O_G)$ so the projection $\pi_1(\varepsilon) \to \pi_1(O_G - \operatorname{int} N)$ is an isomorphism by Proposition 3. Applying Siebenmann's Open

Collar Theorem [14], we have that $O_G - \text{int}\, N$ is PL homeomorphic to bdy $N \times [0, 1)$. Thus, $O_G \cong \text{int}\, N$.

COROLLARY 5. *There are exactly two topological types of tame Z^2-actions on E^n, $n \geqq 5$.*

Proof. Two tame Z^2-actions are equivalent if and only if their orbit spaces are homeomorphic [3]. By a theorem of T. Price [13, p. 336] and uniqueness of PL structures [11], orientable regular neighborhoods of T^2 in codimension three or greater are SO_{n-2} bundles and are classified up to homeomorphism by $\pi_1(SO_{n-2}) \cong Z_2$, $n \geqq 5$.

3. Subactions.

PROPOSITION 6. *Suppose G is a Z^2-action on E^n, $n \geqq 2$ and $H \subset G$ is a subgroup of index two. Then G is a tame action if and only if H is a tame action.*

Proof. Suppose first that G is a tame action. Since G satisfies Sperner's condition, clearly H does. Now let $\{C_j\}$ be a sequence of compact sets in E^n such that $C_j \subset C_{j+1}$ for each j and the inclusion $E^n - GC_{j+1}$ into $E^n - GC_j$ is zero on π_0 and π_1. Since H is a subgroup of index two, one can find a homeomorphism k in G such that $GC_j = HD_j$ where $D_j = C_j \cup k(C_j)$. Hence, H is tame.

Now suppose H is tame and suppose G fails to satisfy Sperner's condition. Then for some compact set $C \subset E^n$, the set

$$M = \{g \in G \mid g(C) \cap C \neq \varnothing\}$$

is infinite. We can find a basis $\{h, k\}$ for G such that $\{h, k^2\}$ is a basis for H. Since H satisfies Sperner's condition, M must contain an infinite number of elements of the form $h^i k^{2l+1}$. Let $D = k(C) \cup C$. Then if $h^i k^{2l+1} \in M$, $h^i k^{2l}(D) = h^i k^{2l+1}(C) \cup h^i k^{2l}(C)$ so that $h^i k^{2l}(D) \cap D \neq \varnothing$. But then H does not satisfy Sperner's condition, a contradiction. Thus, G satisfies Sperner's condition.

Now let $\{C_j\}$ be a sequence of compact sets in E^n such that $C_j \subset C_{j+1}$ for each j and $\bigcup C_j = E^n$. As above there is a homeomorphism k in G such that $GC_j = HD_j$ where $D_j = C_j \cup k(C_j)$ (the homeomorphism k in the above basis will suffice). Taking a subsequence of the C_j's we find that inclusion $E^n - HD_{j+1} = E^n - GC_{j+1}$ into $E^n - HD_j = E^n - GC_j$ is zero on π_0 and π_1. Thus, G is a tame action.

PROPOSITION 7. *Suppose that X is the total space of an orientable $O(q)$ bundle over T^2, $q \geqq 3$ and $p: \tilde{X} \to X$ is a double cover of X. Then $\tilde{X}$ is the total space of the trivial $O(q)$ bundle over T^2.*

Proof. Orientable $O(q)$ bundles over T^2 are classified by $\pi_1(SO_q)$ which is Z_2 for $q \geqq 3$. Applying the classification described by Price in [13] yields the proposition.

THEOREM 8. *Suppose that H is a tame Z^2-action on E^n, $n \geqq 5$. Then H is topologically equivalent to the standard Z^2-action if and only if there exists a Z^2-action G on E^n such that H is a subgroup of G and $G/H \cong Z_2$.*

Proof. If H is equivalent to the standard action, the required G clearly exists.

Now suppose H is tame and G is given such that $G/H \cong Z^2$. By Proposition 6, one has that G is tame. Thus, O_G is homeomorphic to the interior of a orientable regular neighborhood of T^2 and, therefore, is the total space of an orientable $O(q)$ bundle over T^2. But the natural covering projection of O_H onto O_G is a double cover of O_G. The theorem follows from Proposition 7.

COROLLARY 9. *Suppose G is a tame Z^2-action on E^n, $n \geqq 5$. Then every homeomorphism in G except the identity is topologically equivalent to a translation of E^n.*

Proof. Suppose f is a member of G. Then one can find a basis $\{h, k\}$ for G such that there is a positive integer n for which $h^n = f$. Clearly if h is topologically equivalent to a translation then so is f. Let H be the subgroup of G generated by $\{h, k^2\}$. Then H is a subgroup of index two and by Proposition 6 is tame. Applying Theorem 8, one gets H topologically equivalent to the standard Z^2-action and, hence, h is topologically equivalent to a translation.

REMARK. Proposition 6 is true whenever H is a subgroup of finite index. This together with Corollary 9 says that every subaction of a tame Z^2-action is tame.

EXAMPLE. For $n \geqq 5$, let W^{n-2} be a contractible open manifold and let $Q^n = S^1 \times S^1 \times W^{n-2}$. Then the universal cover of Q^n is $E^2 \times W^{n-2} \cong E^n$ [12], so that if G is the corresponding group of covering transformations, G is a Z^2-action satisfying Sperner's condition. Let ε_Q and ε_W denote the ends of Q^n and W^{n-2} respectively. It follows by applying Proposition 1 that if $\pi_1(\varepsilon_Q)$ is stable so is $\pi_1(\varepsilon_W)$ and there is a short exact sequence $1 \to \pi_1(\varepsilon_W) \to \pi_1(\varepsilon_Q) \to Z^2 \to 1$. If W^{n-2} is the Whitehead example [17] for $n = 5$ or the interior of a contractible manifold with nonsimply connected boundary for $n > 5$ [2], the above shows that Q^n is not the interior of a regular neighborhood of T^2, so that G is not a tame action. However, if h_1, h_2 are the generators of G corresponding to the standard cover of $S^1 \times S^1$, the orbit space of the subgroup generated by h_i is homeomorphic to $S^1 \times E^{n-1}$, so that h_1 and h_2 are both topologically equivalent to translations.

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Received August 3, 1972. Research partially supported by the College of Arts and Sciences Office of Research and Graduate Studies, Oklahoma State University.

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