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REGULAR COMPLETIONS OF CAUCHY SPACES

D. C. KENT AND G. D. RICHARDSON

A uniform convergence space is a generalization of a uniform space. The set of all Cauchy filters of some uniform convergence space is called a Cauchy structure. We give necessary and sufficient conditions for the Cauchy structure of some totally bounded uniform convergence space to be precompact; i.e., have a regular completion. Also, it is shown that there is an isomorphism between the set of ordered equivalence classes of strict regular compactifications of a completely regular convergence space and the set of ordered precompact Cauchy structures inducing the given convergence structure.

Preliminaries. Kowalsky [5] has studied completions using only Cauchy filters, described axiomatically, and not necessarily those of a uniform convergence space. This has led others to the notion of a Cauchy space, which is described axiomatically in [2]. The reader is referred to [6], [7], and [8] for a discussion of completions of Cauchy spaces.

For basic definitions of convergence spaces and uniform convergence space, see [3] and [1]. A Hausdorff convergence space (S, q) is compatible with a uniform convergence space iff it satisfies the "Limitierungsaxiom": $\mathfrak{F} \cap \mathfrak{G}$ q-converges to x whenever $\mathfrak{F}$ and $\mathfrak{G}$ both q-converge to x. *We will make the assumption that all convergence spaces in this paper satisfy this axiom.* The closure operator in a convergence space (S, q) will be denoted by Γ_q. A Hausdorff convergence space (S, q) is called *regular* if it has the property that $\Gamma_q\mathfrak{F}$ (the filter generated by $\{\Gamma_q F \mid F \in \mathfrak{F}\}$) q-converges to x whenever $\mathfrak{F}$ q-converges to x. The filter $\dot{x}$ denotes the set of all subsets of S containing the set $\{x\}$. If filters $\mathfrak{F}$ and $\mathfrak{G}$ contain disjoint sets, we write "$\mathfrak{F} \vee \mathfrak{G} = 0$". The term "ultrafilter" will be abbreviated "u.f."; uniform convergence space will be abbreviated "u.c.s.".

A *Cauchy structure* $\mathscr{C}$ on a set S is characterized axiomatically in [2] as follows: (1) $\dot{x} \in \mathscr{C}$ for each $x \in S$; (2) $\mathfrak{F} \in \mathscr{C}$ and $\mathfrak{G}$ finer than $\mathfrak{F}$ implies $\mathfrak{G} \in \mathscr{C}$; (3) $\mathfrak{F}, \mathfrak{G} \in \mathscr{C}$ and $\mathfrak{F} \vee \mathfrak{G} \neq 0$ implies $\mathfrak{F} \cap \mathfrak{G} \in \mathscr{C}$. The pair $(S, \mathscr{C})$ is called a *Cauchy space.* It should be pointed out that the Cauchy space axioms of [2] are stricter than those of [5] and [7].

A Cauchy space $(S, \mathscr{C})$ induces a convergence structure q in the following way: $\mathfrak{F}$ q-converges to x iff $\mathfrak{F} \cap \dot{x} \in \mathscr{C}$. Conversely, if (S, q) is a Hausdorff convergence space, then define the *associated Cauchy structure* $\mathscr{C}$ on S: $\mathfrak{F} \in \mathscr{C}$ iff $\mathfrak{F}$ q-converges. Note that $(S, \mathscr{C})$ induces

q on S. A Cauchy space $(S, \mathscr{C})$ is called *Hausdorff* if the induced convergence space is Hausdorff, and *complete* if each Cauchy filter converges. *We will assume that all spaces are Hausdorff unless otherwise indicated.* The above describes a one-to-one correspondence between the convergence spaces and the complete Cauchy spaces. If $\mathscr{C}$ is a Cauchy structure on S, then we often write C_q for C if q is the induced convergence structure. $(S, \mathscr{C}_q)$ is called *regular* if $\Gamma_q\mathfrak{F} \in \mathscr{C}$ whenever $\mathfrak{F} \in \mathscr{C}$. This definition was suggested by the referee, and corresponds to the definition of regularity for u.s.c.'s given in [10] and [12].

Let $(S, \mathscr{C})$ be a Cauchy space and define (as in [1]): $\mathfrak{F} \sim \mathfrak{G}$ iff $\mathfrak{F} \cap \mathfrak{G} \in \mathscr{C}$. This equivalence relation partitions $\mathscr{C}$ into equivalence classes of the form $[\mathfrak{F}] = \{\mathfrak{G} \in \mathscr{C} \mid \mathfrak{G} \sim \mathfrak{F}\}$. Let T be the set of equivalence classes, and let $j: S \rightarrow T$ denote the canonical mapping, i.e., $j(x) = [\dot{x}]$.

DEFINITION 1.1. $(P, \mathscr{D}_r, f)$ is called a *completion* of the Cauchy space $(S, \mathscr{C})$, if $(P, \mathscr{D})$ is complete, and f is a dense embedding from $(S, \mathscr{C})$ into $(P, \mathscr{D})$. If in addition, whenever a filter $\mathfrak{F}$ r-converges to y in P, there is a filter $\mathfrak{G}$ on fS which r-converges to y and such that $\Gamma_r\mathfrak{G} \leqq \mathfrak{F}$, then $(P, \mathscr{D}_r, f)$ is called a *strict completion.*

The notion of a completion of a u.c.s. is defined similarly. Wyler [11] has shown that each u.c.s. has a completion, with the universal property, and so each Cauchy space has a completion. If P denotes any convergence space property, then we say that a (strict) completion is a (*strict*) *P completion* if it possesses property P.

The next two definitions follow the terminology of [8]. Analogous definitions apply in the u.c.s. setting.

DEFINITION 1.2. A completion $(P, \mathscr{D}, f)$ of the Cauchy space $(S, \mathscr{C})$ is said to be in *standard form* if $P = T, f = j$, and $f\mathfrak{F}$ converges to $[\mathfrak{F}]$ for each $\mathfrak{F} \in \mathscr{C}$.

DEFINITION 1.3. The completions $(P_i, \mathscr{D}_i, f_i)$, $i = 1, 2$, of the Cauchy space $(S, \mathscr{C})$ are said to be *equivalent* if there is an isomorphism g from $(P_1, \mathscr{D}_1)$ onto $(P_2, \mathscr{D}_2)$ such that $gf_1 = f_2$.

PROPOSITION 1.4 ([8]). *Each Cauchy space (u.c.s.) completion is equivalent to exactly one in standard form.*

Let $(S, \mathscr{C})$ be a Cauchy space and T, j defined as above. Let A be a subset of S; then define ΣA to be $\{[\mathfrak{F}] \in T \mid A \in \mathfrak{G}$ for some $\mathfrak{G} \in [\mathfrak{F}]\}$. If $\mathfrak{G}$ is a filter on S, then $\Sigma\mathfrak{G}$ denotes the filter on T

generated by $\{\Sigma G \mid G \in \mathfrak{G}\}$. Further, the convergence structure p on T is defined as follows: $\mathscr{H}$ p-converges to $[\mathfrak{F}]$ iff $\mathfrak{H} \geqq \Sigma\mathfrak{G}$ for some $\mathfrak{G} \in [\mathfrak{F}]$. In general p is not Hausdorff, and so we use the notation $\mathscr{C}_p$ only whenever p is Hausdorff. The following is straightforward to verify: if p is Hausdorff, then $(T, \mathscr{C}_p, j)$ is a completion of $(S, \mathscr{C})$ iff $(S, \mathscr{C})$ is regular. Our next result follows immediately from the definitions.

PROPOSITION 1.5. *Let $(T, \mathscr{D}_s, j)$ be any completion of $(S, \mathscr{C}_q)$ in standard form. The following are true.*

(1) *If $A \subset S$, then $\Gamma_s jA = \Sigma A$ and $\Gamma_q A = j^{-1}\Sigma A$.*

(2) *The completion is strict iff $s \geqq p$.*

COROLLARY 1.6. *$(T, \mathscr{C}_p, j)$ is the only possible candidate for a strict regular completion of $(S, \mathscr{C})$ in standard form. Moreover, $(T, \mathscr{C}_p, j)$ has the universal property for regular Cauchy spaces.*

Proof. The first part follows from (2) of Proposition 1.5 since the convergence structure induced by any regular completion on T must be coarser than p. The second part following from Theorem 4.11 of [7], since $(T, \mathscr{C}_p)$ is the quotient space of the quasi-completion mentioned there.

Finally, we remark that if $(T, \mathscr{C}_p, j)$ is a completion of $(S, \mathscr{C})$, and if $\mathscr{I}$ is a u.c.s. with Cauchy filters $\mathscr{C}$, then by Theorem 15 of [8], $(S, \mathscr{I})$ has a u.c.s. completion with Cauchy structure $\mathscr{C}_p$.

Almost topological completions. In [9] it is shown that a regular compactification (R, r, f) of a convergence space (S, q) is *almost topological*, which means that r and its topological modification, λr, coincide relative to the convergence of u.f.'s. The next theorem characterizes those Cauchy spaces which have almost topological completions. The proof of this theorem uses the following lemma proved in [4].

LEMMA 2.1. *Let (S, q) be a convergence space, $A \subset S$, $\mathfrak{F}$ an u.f. on S. $\Gamma_q A \in \mathfrak{F}$, then there is an u.f. $\mathfrak{G}$ containing A such that $\mathfrak{F} \geqq \Gamma_q \mathfrak{G}$.*

THEOREM 2.2. *The following conditions are equivalent for a regular Cauchy space $(S, \mathscr{C})$.*

(1) *$(S, \mathscr{C})$ has an almost topological regular completion.*

(2) *If $\Sigma\mathfrak{F} \vee \Sigma\mathfrak{G} \neq 0$ for $\mathfrak{F} \in \mathscr{C}$, $\mathfrak{G}$ an u.f. on S, then $\mathfrak{F} \cap \mathfrak{G} \in \mathscr{C}$.*

(3) *$(T, \mathscr{C}_p, j)$ is an almost topological regular completion.*

Proof. (1) *implies* (2). Let $(T, \mathscr{D}_r, j)$ be such a completion in standard form, and let $\mathfrak{F}, \mathfrak{G}$ be as in the hypothesis of (2). Then $\Gamma_r j\mathfrak{F} \vee \Gamma_r j\mathfrak{G} \neq 0$, and so $[\mathfrak{F}]$ is an adherent point of the u.f. $j\mathfrak{G}$ in $(T, \lambda r)$. Thus $j\mathfrak{G}$ λr-converges to $[\mathfrak{F}]$, and so by hypothesis r-converges to $[F]$. Hence $\mathfrak{F} \cap \mathfrak{G} \in \mathscr{C}$, and (2) is satisfied.

(2) *implies* (3). First we show that if $\Sigma\mathfrak{F} \vee \Sigma\mathfrak{G} \neq 0$, for $\mathfrak{F}, \mathfrak{G} \in \mathscr{C}$, then $[\mathfrak{F}] = [\mathfrak{G}]$. Let $\mathfrak{U}$ be an u.f. on T such that $\mathfrak{U} \geqq \Sigma\mathfrak{F} \vee \Sigma\mathfrak{G}$. From Lemma 2.1 there is an u.f. $\mathfrak{H}$ on S such that $\Sigma\mathfrak{H} \leqq \mathfrak{U}$. Thus $\Sigma\mathfrak{H} \vee \Sigma\mathfrak{F} \neq 0$, and by condition (2) $[\mathfrak{H}] = [\mathfrak{F}]$. Similarly, $[\mathfrak{H}] = [\mathfrak{G}]$, and so $[\mathfrak{F}] = [\mathfrak{G}]$. Thus (T, p) is Hausdorff, and so $(T, \mathscr{C}_p, j)$ is a completion of $(S, \mathscr{C})$.

Two steps are needed to prove Γ_p is idempotent. First let $A \subset S$ and $[\mathfrak{F}] \in \Gamma_p^2 jA = \Gamma_p \Sigma A$. Then there is an u.f. $\mathfrak{U}$ p-converging to $[\mathfrak{F}]$ such that $\Sigma A \in \mathfrak{U}$. Thus $\mathfrak{U} \geqq \Sigma\mathfrak{G}$ for some $\mathfrak{G} \in [\mathfrak{F}]$. By Lemma 2.1, there is an u.f. $\mathfrak{H}$ on A such that $\mathfrak{U} \geqq \Sigma\mathfrak{H}$, and we have $\Sigma\mathfrak{G} \vee \Sigma\mathfrak{H} \neq 0$. By condition (2), $[\mathfrak{F}] = [\mathfrak{H}]$, and since $A \in \mathfrak{H}$, then $[\mathfrak{F}] \in \Sigma A$. Thus $\Gamma_p^2 jA = \Gamma_p jA$ whenever $A \subset S$.

Next let $B \subset T$ and $[\mathfrak{F}] \in \Gamma_p^2 B$. Then there is an u.f. $\mathfrak{U}$ p-converging to $[\mathfrak{F}]$ such that $\Gamma_p B \in \mathfrak{U}$. Thus $\mathfrak{U} \geqq \Sigma\mathfrak{G}$ for some $\mathfrak{G} \in [\mathfrak{F}]$. Using Lemma 2.1 again, there is an u.f. $\mathfrak{K}$ on B such that $\mathfrak{U} \geqq \Gamma_p\mathfrak{K}$, and also an u.f. $\mathfrak{H}$ on S such that $\mathfrak{K} \geqq \Sigma\mathfrak{H}$. Thus $\mathfrak{U} \geqq \Gamma_p\mathfrak{K} \geqq \Gamma_p\Sigma\mathfrak{H} = \Sigma\mathfrak{H}$, and so $\Sigma\mathfrak{G} \vee \Sigma\mathfrak{H} \neq 0$, which implies that $[\mathfrak{F}] = [\mathfrak{H}]$. Since $\mathfrak{K}$ p-converges to $[\mathfrak{H}]$ and $B \in \mathfrak{K}$, then $[\mathfrak{F}] \in \Gamma_p B$. Thus $\Gamma_p^2 B = \Gamma_p B$, for $B \subset T$. If $\mathfrak{F} \in \mathscr{C}$, then $\Gamma_p \Sigma\mathfrak{F} = \Gamma_p^2 j\mathfrak{F} = \Gamma_p j\mathfrak{F} = \Sigma\mathfrak{F}$, and so $(T, \mathscr{C}_p, j)$ is a regular completion of $(S, \mathscr{C})$.

Finally, we show that p and λp coincide on u.f.'s. Let $[\mathfrak{F}] \in T$ and $\mathfrak{U}$ an u.f. such that $\mathfrak{U} \geqq \bigcap \{\mathfrak{H} \mid \mathfrak{H}\ p\text{-converges to } [\mathfrak{F}]\}$. The latter intersection is the p-neighborhood filter at $[\mathfrak{F}]$, and since Γ_p is idempotent, the λp-neighborhood filter at $[\mathfrak{F}]$. Note that $[\dot{\mathfrak{F}}] \geqq \Gamma_p\mathfrak{U} \vee \Sigma\mathfrak{F}$. From Lemma 2.1, there is an u.f. $\mathfrak{G}$ on S such that $\mathfrak{U} \geqq \Sigma\mathfrak{G}$, and so $\Gamma_p\mathfrak{U} \geqq \Gamma_p\Sigma\mathfrak{G} = \Sigma\mathfrak{G}$. Thus $\Sigma\mathfrak{G} \vee \Sigma\mathfrak{F} \neq 0$, which implies $[\mathfrak{G}] = [\mathfrak{F}]$, and so $\mathfrak{U}$ p-converges to $[\mathfrak{F}]$, which completes the proof.

PROPOSITION 2.3. *If $(T, \mathscr{C}_s, j)$ is any almost topological regular completion of $(S, \mathscr{C})$ in standard form, then $s \leqq p$ and $s = p$ on u.f.'s.*

Proof. Clearly $s \leqq p$. Let $\mathfrak{U}$ be an u.f. which s-converges to $[\mathfrak{F}]$ in T. Then from Lemma 2.1, there is an u.f. $\mathfrak{G}$ on S such that $\Gamma_s j\mathfrak{G} \leqq \mathfrak{U}$. Thus the u.f. $j\mathfrak{G}$ λs-converges to $[\mathfrak{F}]$, which implies by hypothesis that $j\mathfrak{G}$ s-converges to $[\mathfrak{F}]$. Hence $\mathfrak{G} \in [\mathfrak{F}]$, and from Proposition 1.5 $\Gamma_s j\mathfrak{G} = \Sigma\mathfrak{G}$, which implies that $\mathfrak{U}$ p-converges to $[\mathfrak{F}]$, and so $s = p$ on u.f.'s.

Precompact Cauchy spaces. A Cauchy space (u.c.s.) is said to be *totally bounded* if every u.f. is Cauchy. A totally bounded Cauchy space with a regular completion will be termed *precompact*. From Theorem 2.2 we conclude the following.

PROPOSITION 3.1. *A precompact Cauchy space is almost topological and has an almost topological regular completion.*

Another characterization of precompact Cauchy spaces is given by Theorem 3.4. First, we need two preliminary results, the first of which is proved in [9].

LEMMA 3.2. *A convergence space (S, q) is compact and regular iff (S, q) is almost topological and λq is a compact Hausdorff topology.*

PROPOSITION 3.3. *Let $(S, \mathscr{I})$ be a u.c.s. with a compact regular induced convergence structure q, and let $\mathscr{U}$ be the filter of λq-neighborhoods of the diagonal Δ_S in $S \times S$. Then each element of $\mathscr{I}$ is finer than $\mathscr{U}$.*

Proof. It follows from Lemma 3.2 that $\mathscr{U}$ is a Hausdorff uniformity. Let $\Phi \in \mathscr{I}$, and assume $\mathscr{U} \not\leqq \Phi$. Then there is an u.f. $\mathfrak{F}$ on $S \times S$ such that $\Phi \leqq \mathfrak{F}$ and $\mathscr{U} \not\leqq \mathfrak{F}$. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be the first and second projections, respectively, of $\mathfrak{F}$ onto S; then by the assumption of compactness, there are points x and y in S such that $\mathfrak{F}_1$ q-converges to x and $\mathfrak{F}_2$ q-converges to y. Since $\mathscr{U} \not\leqq \mathfrak{F}$, x and y must be distinct. But $(\mathfrak{F}_1 \times \mathfrak{F}_2) \vee \Phi \neq 0$, and so $\dot{x} \times \dot{y} = (\dot{x} \times \mathfrak{F}_1) \circ \Phi \circ (\mathfrak{F}_2 \times \dot{y}) \in \mathscr{I}$. This contradicts the fact that (S, q) is Hausdorff.

THEOREM 3.4. *The Cauchy structure $(S, \mathscr{C})$ of a totally bounded u.c.s. $(S, \mathscr{I})$ is precompact iff the following conditions are satisfied.*

(1) $\mathscr{U} = \bigcap \{\Phi \mid \Phi \in \mathscr{I}\}$ *is a Hausdorff uniformity on S.*

(2) $(S, \mathscr{C})$ *is regular.*

(3) *If $\mathfrak{F}$ and $\mathfrak{G}$ are u.f.'s on S such that $\mathfrak{F} \times \mathfrak{G} \geqq \mathscr{U}$, then $\mathfrak{F} \times \mathfrak{G} \in \mathscr{I}$.*

Proof. Assume the three conditions. Let $(S', \mathscr{W})$ denote the Hausdorff uniform completion of $(S, \mathscr{U})$. For each $\mathscr{U}$-Cauchy filter $\mathfrak{F}$ on S, let $[\mathfrak{F}]_{\mathscr{U}} = \{\mathfrak{G} \mid \mathfrak{G}$ is $\mathscr{U}$-Cauchy and $\mathfrak{F} \times \mathfrak{G} \geqq \mathscr{U}\}$. From condition (3) and the fact that $(S, \mathfrak{J})$ is totally bounded, it follows that an u.f. $\mathfrak{G}$ is in $[\mathfrak{F}]$ (the $\mathscr{C}$-equivalence class) iff $\mathfrak{G} \in [\mathfrak{F}]_{\mathscr{U}}$. Since S' can be identified with $\{[\mathfrak{F}]_{\mathscr{U}} \mid \mathfrak{F}$ is an u.f. on $S\}$, we can identify S' with $T = \{[\mathfrak{F}] \mid \mathfrak{F} \in \mathscr{C}\}$. If r is the topology on T associated with $\mathscr{W}$, then $\Sigma A \subset \Gamma_r jA$, $A \subset S$. If $\mathfrak{F} \in \mathscr{C}$ and $\mathfrak{G}$ is an u.f. on S such that

$\Sigma\mathfrak{F} \vee \Sigma\mathfrak{G} \neq 0$, then $\Gamma_r j\mathfrak{F} \vee \Gamma_r j\mathfrak{G} \neq 0$, and so $[\mathfrak{F}]_{\mathscr{U}} = [\mathfrak{G}]_{\mathscr{U}}$. Hence $[\mathfrak{F}] = [\mathfrak{G}]$, or $\mathfrak{F} \cap \mathfrak{G} \in \mathscr{C}$, and by Theorem 2.2, $(S, \mathscr{C})$ is precompact.

Conversely, assume $(S, \mathscr{C})$ is precompact. From our previous results, $(T, \mathscr{C}_p, j)$ is a regular completion of $(S, \mathscr{C})$. From Theorem 15 of [8], there is a u.c.s. $\mathscr{J}$ on T which has Cauchy filters $\mathscr{C}_p$ and such that $(T, \mathscr{J}, j)$ is a completion of $(S, \mathscr{I})$. Note that $\mathscr{J}$ induces the convergence structure p on T. Let $\mathscr{W}$ be the uniformity on T of λp-neighborhoods of the diagonal Δ_T. By Proposition 3.3, each $\psi \in \mathscr{J}$ is finer than $\mathscr{W}$. Let $\mathscr{U} = (j \times j)^{-1}(\mathscr{W})$; then $\mathscr{U}$ is a uniformity and each $\Phi \in \mathscr{I}$ is finer than $\mathscr{U}$. Thus $\mathscr{U} \leqq \bigcap \{\Phi \mid \Phi \in \mathscr{I}\}$. If $\mathscr{U}$ is strictly coarser than $\bigcap \Phi$, then there is an u.f. $\mathfrak{F}$ on $S \times S$ such that $\mathfrak{F} \geqq \mathscr{U}$, but $\mathfrak{F} \ngeq \bigcap \Phi$. Let $\mathfrak{F}_1, \mathfrak{F}_2$ denote the projections of $\mathfrak{F}$; then since $(j \times j)\mathfrak{F} \geqq \mathscr{W}$, $j\mathfrak{F}_1$ and $j\mathfrak{F}_2$ converge to the same point in (T, p). Thus $\mathfrak{F}_1 \times \mathfrak{F}_2 \in \mathscr{I}$, and so $\mathfrak{F} \in \mathscr{I}$, which contradicts $\mathfrak{F} \ngeq \bigcap \Phi$. Hence $\mathscr{U} = \bigcap \Phi$ is a Hausdorff uniformity on S, and (1) follows.

Of course (2) is clear. If $\mathfrak{F}, \mathfrak{G}$ are u.f.'s on S such that $\mathfrak{F} \times \mathfrak{G} \geqq \mathscr{U}$, then $j\mathfrak{F} \times j\mathfrak{G} \geqq \mathscr{W}$, and so they λp-converge to the same point. Since (T, p) is almost topological, then $j\mathfrak{F}$ and $j\mathfrak{G}$ p-converge to the same point, and so $\mathfrak{F} \times \mathfrak{G} \in \mathscr{I}$, which implies (3).

Strict regular compactifications. One of the more significant results in uniform space theory is the existence of an isomorphism from the ordered set of equivalence classes of the Hausdorff compactifications of a completely regular topological space and the ordered set of compatible precompact uniformities.

(R, r, f) is said to be a *strict compactification* of the convergence space (S, q), if f is a dense embedding, (R, r) is a compact convergence space, and if $\mathfrak{F}$ r-converges to $y \in R$, then there is a filter $\mathfrak{G}$ on fS which r-converges to y and $\Gamma_r\mathfrak{G} \leqq \mathfrak{F}$. We define equivalence classes of compactifications of (S, q), and an ordering among the classes, analogous to the topological setting. Also if $(S, \mathscr{C}_1)$, $(S, \mathscr{C}_2)$ are two Cauchy spaces, then $\mathscr{C}_1 \geqq \mathscr{C}_2$ is defined to be $\mathscr{C}_1 \subset \mathscr{C}_2$.

A convergence space will be called *completely regular* if it has a strict regular compactification.

PROPOSITION 4.1. *A convergence space (S, q) is completely regular iff it is almost topological and λq is a completely regular topology.*

This result is essentially proved in [9], but the following two points need to be added. The compactification in [9] is in fact a strict regular compactification. The "Limitierungsaxiom" was not assumed in [9], but causes no difficulty if imposed.

THEOREM 4.2. *If (S, q) is a completely regular convergence space,* *then the ordered set of equivalence classes of strict regular compactifications of (S, q) is isomorphic to the ordered set of precompact Cauchy structures on S which induce q.*

Proof. Let $(S, \mathscr{C})$ be a precompact Cauchy space which induces q on S; then $(T, \mathscr{C}_p, j)$ is a strict regular completion of $(S, \mathscr{C})$. Thus (T, p, j) is a strict regular compactification of (S, q). Define $\gamma(S, \mathscr{C}) = (T, p, j)$. We show that γ is an isomorphism.

Suppose $\mathscr{C}_1$ and $\mathscr{C}_2$ are distinct Cauchy structures on S; with no loss of generality assume $\mathfrak{F} \in \mathscr{C}_1 - \mathscr{C}_2$. We claim that $\gamma(S, \mathscr{C}_1) = (T_1, p_1, j_1)$ is not equivalent to $\gamma(S, \mathscr{C}_2) = (T_2, p_2, j_2)$. Assume, on the contrary, that there is a homeomorphism $f: (T_1, p_1) \rightarrow (T_2, p_2)$ such that $fj_1 = j_2$. Then $j_1\mathfrak{F}$ p_1-converges to $[\mathfrak{F}]_1 \in T_1$, and so $fj_1\mathfrak{F} = j_2\mathfrak{F}$ p_2-converges to an element in T_2. This can occur only if $\mathfrak{F} \in \mathscr{C}_2$, which contradicts the choice of $\mathfrak{F}$, and it follows that γ is injective.

Next to show γ is onto. Let (R, r, f) be any strict regular compactification of (S, q). Let $\mathscr{C}$ be the set of all filters $\mathfrak{F}$ on S such that $f\mathfrak{F}$ r-converges in R. By a straightforward argument, it can be shown that $\mathscr{C}$ satisfies the Cauchy space axioms, and is also totally bounded and induces q. Since $(R, \mathscr{C}_r, f)$ is strict regular completion of $(S, \mathscr{C})$, then by Theorem 2.2 $(T, \mathscr{C}_p, j)$ is also a strict regular completion of $(S, \mathscr{C})$. By Corollary 1.6, (P, p, j) and (R, r, f) are equivalent. Hence γ is subjective.

Finally to show that γ and γ^{-1} are order preserving. Suppose $\mathscr{C}_1 \geqq \mathscr{C}_2$, i.e., $\mathscr{C}_1 \subset \mathscr{C}_2$. Let $\gamma(S, \mathscr{C}_i) = (T_i, p_i, j_i)$, $i = 1, 2$. It is straightforward to check that if $f: T_1 \rightarrow T_2$ such that $f([\mathfrak{F}]_1) = [\mathfrak{F}]_2$, where $[\mathfrak{F}]_i$ is ths equivalence class in T_i of $\mathfrak{F} \in \mathscr{C}_i$, then $f(\Sigma_1 A) \subset \Sigma_2 A$. Thus $f(\Sigma_1\mathfrak{F}) \geqq \Sigma_2\mathfrak{F}$, and so f is continuous. It follows that $(T_1, p_1, j_1) \geqq (T_2, p_2, j_2)$. The proof that γ^{-1} is order preserving is straightforward, and the theorem follows.

We conclude with the following remarks concerning Theorem 4.2. In either of the ordered sets of Theorem 4.2, each nonempty subset has a supremum; the finest precompact Cauchy structure on S which induces q corresponds to the Stone-Čech compactification (S, q).

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WASHINGTON STATE UNIVERSITY
AND
EAST CAROLINA UNIVERSITY