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SELF ADJOINT STRICTLY CYCLIC OPERATOR ALGEBRAS

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A *strictly cyclic* operator algebra $\mathscr{A}$ on a Hilbert space X is a uniformly closed subalgebra of $\mathscr{L}(X)$ such that $\mathscr{A}x_0 = X$ for some x_0 in X. In this paper it is shown that if $\mathscr{A}$ is a strictly cyclic self-adjoint algebra, then (i) there exists a finite orthogonal decomposition of $X, X = \sum_{j=1}^{n} \oplus M_j$, such that each M_j reduces $\mathscr{A}$ and the restriction of $\mathscr{A}$ to M_j is strongly dense in $\mathscr{L}(M_j)$ and (ii) the commutant of $\mathscr{A}$ is finite dimensional.

1. Notation and terminology. Throughout the paper X is a complex Hilbert space and $\mathscr{L}(X)$ is the algebra of continuous linear operators on X. $\mathscr{A}$ will denote a uniformly closed subalgebra of $\mathscr{L}(X)$ which is *strictly cyclic* and x_0 will be a *strictly cyclic vector* for $\mathscr{A}$: That is, $\mathscr{A}x_0 = X$. We do not insist that the identity element I of $\mathscr{L}(X)$ be an element of $\mathscr{A}$. We say that $\mathscr{A}$ is *self-adjoint* if $A^* \in \mathscr{A}$ whenever $A \in \mathscr{A}$.

If $\mathscr{B} \subset \mathscr{L}(X)$, then the *commutant* of $\mathscr{B}$ is $\mathscr{B}' = \{E: E \in \mathscr{L}(X)$ and $EB = BE$ for all B in $\mathscr{B}\}$. A closed linear subspace M of X *reduces* $\mathscr{B}$ if the projection of X onto M is in $\mathscr{B}'$. In this case M is a *minimal reducing subspace* of $\mathscr{B}$ if $M \neq \{\theta\}$ and $\{\theta\}$ is the only reducing subspace of $\mathscr{B}$ properly contained in M.

We say that a collection $\{M_j\}_{j=1}^{n}$ of closed linear subspaces of X is an *orthogonal decomposition* of X if and only if the M_j are pairwise orthogonal and span X. A collection $\{P_j\}_{j=1}^{n}$ of projections is a *resolution of identity* if and only if the collection $\{P_j(X)\}_{j=1}^{n}$ of ranges of the P_j is an orthogonal decomposition of X.

2. Introduction. Strictly cyclic operator algebras have been studied by R. Bolstein, A. Lambert, the author of this paper and others. (See for example [1], [2], and [4].) In Lemma 1 of [1] Bolstein shows that if N is a normal operator on X and $\{N\}'$ is strictly cyclic, then $\{N\}''$ is finite dimensional. This raised questions about the nature of arbitrary self-adjoint, strictly cyclic operator algebras. In this paper we show that if $\mathscr{A}$ is such an operator algebra, then there exists a finite orthogonal decomposition $\{M_j\}$ of X such that each M_j reduces $\mathscr{A}$ and $\mathscr{A}/M_j$ is strongly dense in $\mathscr{L}(M_j)$. From this it follows that $\mathscr{A}'$ is finite dimensional; indeed we show that $\mathscr{A}' = \sum_{j,k=1}^{n} P_j\mathscr{A}'P_k$ (where P_j is the projection of X onto M_j) and that for each j and k, $P_j\mathscr{A}'P_k$ is of dimension zero or one. If $\mathscr{A}'$

is abelian, we are able to show more; namely that $\mathscr{A}' = \{\sum_{j=1}^{n} \lambda_j P_j : \lambda_j$ complex$\}$, giving us a complete generalization of Bolstein's result.

Each of the results mentioned above is a consequence of two basic facts concerning a self-adjoint strictly cyclic operator algebra $\mathscr{A}$: (1) (Lemma 1) each collection of pairwise orthogonal projections in $\mathscr{A}'$ is finite and (2) (Theorems 1 and 2 of [3]) $\mathscr{A}$ has minimal reducing subspaces.

3. Decomposition theorem. The first lemma in this section demonstrates a very special characteristic of strictly cyclic operator algebras on a Hilbert space.

LEMMA 1. *Let $\mathscr{A}$ be a strictly cyclic operator algebra on X. Each collection of mutually orthogonal projections in $\mathscr{A}'$ is finite.*

Proof. Let $\{P_j\}$ be a collection of mutually orthogonal projections in $\mathscr{A}'$. Without loss of generality we may assume that $\{P_j\}$ is countable. Let $Q_n = \sum_{j=1}^{n} P_j$ and note that Q_n converges strongly to $Q = \sum_{j \geq 1} P_j$. Thus by Lemma 2.1 in [2] Q_n converges uniformly to $Q = \sum_{j \geq 1} P_j$. However, $Q - Q_n$ is a projection and hence has norm zero or one. Thus for n sufficiently large $Q_n = Q$ and thus $\{P_j\}$ is finite.

This lemma and its proof were suggested by Robert Kallman, University of Florida.

COROLLARY 2. *Let $\mathscr{A}$ be a strictly cyclic operator algebra on X. Each normal element of $\mathscr{A}'$ has finite spectrum.*

Proof. By Lemma 3.6 in [2] if $E \in \mathscr{A}'$, then E has no continuous spectrum. Thus if E is a normal element of $\mathscr{A}'$, the spectrum of E consists entirely of point spectrum and by Lemma 1 E has only a finite number of distinct eigenspaces. Thus the spectrum of E is finite.

Corollary 2 was proven by R. Bolstein in [1] in the special case in which $\mathscr{A}$ is the commutant of a normal operator N.

Before considering further the nature of the commutant of a self-adjoint, strictly cyclic operator algebra $\mathscr{A}$, we shall study the algebra $\mathscr{A}$ itself.

THEOREM 3. *If $\mathscr{A}$ is a self-adjoint strictly cyclic operator algebra on X, then there exists a finite orthogonal decomposition $\{M_k\}_{k=1}^{n}$ of X such that each M_k reduces $\mathscr{A}$ and $\mathscr{A}/M_k$ is strongly dense in $\mathscr{L}(M_k)$.*

Proof. By Theorem 1 of [3] if X and $\{\theta\}$ are the only reducing subspaces of $\mathscr{A}$, then $\mathscr{A}$ is strongly dense in $\mathscr{L}(X)$ and the trivial decomposition $\{X\}$ of X satisfies the requirements of the theorem.

Assume that $\{M_k\}_{k=1}^p$ is a collection of mutually orthogonal subspaces of X such that each M_k reduces $\mathscr{A}$ and $\mathscr{A}/M_k$ is strongly dense in $\mathscr{L}(M_k)$. If the M_k span X, the conclusion of the theorem is satisfied. Otherwise consider $\mathscr{A}_1 = \mathscr{A}/\{M_1, \cdots, M_p\}^\perp$. If P is the orthogonal projection of X onto $\{M_1, \cdots, M_p\}^\perp$, then $P \in \mathscr{A}'$, and if x_0 is a strictly cyclic vector for $\mathscr{A}$, then $\mathscr{A}_1 Px_0 = \mathscr{A}Px_0 = P\mathscr{A}x_0 = P(X) = \{M_1, \cdots, M_p\}^\perp$. Thus $\mathscr{A}_1$ is strictly cyclic. Again by Theorem 1 of [3], if $\mathscr{A}_1$ has only trivial reducing subspaces, $\mathscr{A}_1$ is strongly dense in $\mathscr{L}(\{M_1, \cdots, M_p\})^\perp$ and the construction is complete. Otherwise $\mathscr{A}_1$ has a nontrivial reducing subspace. Then by Theorem 2 of [3] $\mathscr{A}_1$ has a minimal reducing subspace M_{p+1} and by Theorem 3 of [3] $\mathscr{A}_1/M_{p+1}$ is strongly dense in $\mathscr{L}(M_{p+1})$. Thus $M_1, \cdots, M_{p+1}$ are pairwise orthogonal reducing subspaces for $\mathscr{A}$ and $\mathscr{A}/M_k$ is strongly dense in $\mathscr{L}(M_k)$ for $k = 1, \cdots, p+1$. By Lemma 1 the construction will terminate with a finite number of pairwise orthogonal reducing subspaces.

In view of Theorem 3 it is tempting to write $\mathscr{A} = \bigoplus \sum_{k=1}^n \mathscr{L}(M_k)$. However, this is misleading since $\mathscr{A}$ may not be the full direct sum of the $\mathscr{L}(M_k)$. The following simple finite dimensional example demonstrates this:

$$\mathscr{A} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \text{ a } 2 \times 2 \text{ complex matrix} \right\}.$$

Here $\mathscr{A}$ is a strictly cyclic self-adjoint operator algebra on $\mathscr{C}^4$.

We shall use the decomposition of $\mathscr{A}$ developed in Theorem 3 to study the commutant of $\mathscr{A}$. It is worthwhile noting at this point that the decomposition in Theorem 3 may not be unique. We shall investigate this further in Corollary 7.

THEOREM 4. *Let $\mathscr{A}$ be a self-adjoint strictly cyclic operator algebra and $\{M_k\}_{k=1}^n$ a decomposition of X as required in Theorem 3. Let P_k be the orthogonal projection of X onto M_k. Then $\mathscr{A}' = \sum_{j,k=1}^n P_j\mathscr{A}'P_k$ and for each value of j and of k, $P_j\mathscr{A}'P_k$ is of dimension one or zero. In particular $\mathscr{A}'$ is finite dimensional.*

Proof. We note that $\sum_{k=1}^n P_k = I$ and that since M_k is a minimal reducing subspace of $\mathscr{A}$, then P_k is a minimal projection in $\mathscr{A}'$. Further $\mathscr{A}' = (\sum_{j=1}^n P_j)\mathscr{A}'(\sum_{k=1}^n P_k) = \sum_{j,k=1}^n P_j\mathscr{A}'P_k$.

We first show that $P_j\mathscr{A}'P_j = \{\lambda P_j\}$. Assume that $C = P_jEP_j$ is a projection. Note that $C \in \mathscr{A}'$ and $C = P_jCP_j \ll P_j$. Thus since P_j is minimal, either $C = 0$ or $C = P_j$ and the only projections in $P_j\mathscr{A}'P_j$

are 0 and P_j. Therefore $P_j\mathscr{A}'P_j = \{\lambda P_j\}$.

Secondly we show that either $P_j\mathscr{A}'P_k = 0$ or $P_j\mathscr{A}'P_k = \{\lambda\, U_{jk}\}$ where U_{jk} is the partial isometry with initial set $P_k(X)$ and final set $P_j(X)$. Let $F = P_jEP_k$, $E \in \mathscr{A}'$. Then $FF^* \in P_j\mathscr{A}'P_j$ and hence by the preceding paragraph $FF^* = \lambda P_j$ for some complex λ. Therefore, $FF^*F = \lambda F$. If $P_j\mathscr{A}'P_k \neq 0$, then some $F \neq 0$. Since $FF^*F = \lambda F = \lambda P_jEP_k$, F is a scalar multiple of the partial isometry with initial set $P_k(X)$ and final set $P_j(X)$.

The proof of Theorem 4 was provided by T. Hoover.

COROLLARY 5. *If $\mathscr{A}$ is a self-adjoint strictly cyclic operator algebra with an abelian commutant, then $\mathscr{A}' = \{\sum_{j=1}^n \lambda_j P_j: \lambda_j$ complex$\}$ where $\{P_j\}$ is a resolution of identity as required in Theorem 4. In particular $\mathscr{A}'$ consists of normal operators with finite spectra.*

Proof. By Theorem 4 $\mathscr{A}' = \sum_{j,k=1}^n P_j\mathscr{A}'P_k$. Thus if $\mathscr{A}'$ is abelian, $\mathscr{A}' = \sum_{j=1}^n P_j\mathscr{A}'P_j$. Moreover, by Theorem 4, $P_j\mathscr{A}'P_j = \{\lambda_jP_j: \lambda_j$ complex$\}$.

The following corollary due to Bolstein, inspired the ideas which have been developed in this paper. The techniques used by Bolstein in [1] to arrive at this result differ radically from those used in this paper.

COROLLARY 6. (Bolstein) *Let N be a normal operator with a strictly cyclic commutant $\{N\}'$. Then there exist orthogonal projections $P_1, \cdots, P_n$ such that*

$$\{N\}'' = \left\{\sum_{j=1}^n \lambda_j P_j: \lambda_j \text{ complex}\right\} .$$

Proof. By the Fuglede theorem $\{N\}'$ is self-adjoint. Thus since $\{N\}''$ is abelian, we can apply Corollary 5.

We return now to the question of the uniqueness of the decomposition $\{M_k\}_{k=1}^n$ in Theorem 3 or equivalently the uniqueness of a resolution of identity $\{P_k\}_{k=1}^n$ in $\mathscr{A}'$, consisting of minimal projections.

COROLLARY 7. *The decomposition $\{M_k\}_{k=1}^n$ in Theorem 3 is unique if and only if $\mathscr{A}'$ is abelian.*

Proof. Assume first that $\mathscr{A}'$ is abelian. By Corollary 5 $\mathscr{A}' = \{\sum_{j=1}^n \lambda_jP_j: \lambda_j$ complex$\}$. If Q is any projection in $\mathscr{A}'$, $QP_j = P_jQ$ for each j. Hence QP_j is a projection and since P_j is minimal, either

$QP_j = 0$ or $QP_j = P_j$. Therefore, if Q is a minimal projection in $\mathscr{A}'$, or equivalently $Q(X)$ is a minimal reducing subspace of X, then $Q = P_j$ for some j. Thus the decomposition $\{M_k\}_{k=1}^n$ is unique.

Now assume that the decomposition $\{M_k\}_{k=1}^n$ of Theorem 3 is unique. Let P be any nonzero projection in $\mathscr{A}'$ and P_0 a minimal projection dominated by P. Since the decomposition is unique, necessarily $P_0(X) = M_k$ for some k. Consequently $P = \sum_{j=1}^n \lambda_j P_j$ where λ_j is zero or one. Thus all projections (and hence all elements) in $\mathscr{A}'$ commute.

In conclusion we note that if $\mathscr{A}$ is an arbitrary strictly cyclic operator algebra on X, then $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2$ where $\mathscr{A}_1$ is self-adjoint strictly cyclic and $\mathscr{A}_2$ is strictly cyclic but has no reducing subspaces on which it is self-adjoint. To see this we argue as follows: Let $\mathscr{K}$ be the class of all reducing subspaces of $\mathscr{A}$ on which $\mathscr{A}$ is self-adjoint. Order $\mathscr{K}$ by inclusion and note that Lemma 1 implies that any linearly ordered subset of $\mathscr{K}$ is finite. Thus the Maximal Principle can be applied and there exists a maximal reducing subspace M such that $\mathscr{A}/M$ is self-adjoint. Finally if x_0 is a strictly cyclic vector for $\mathscr{A}$ and P the projection of X onto M, then Px_0 is a strictly cyclic vector for $\mathscr{A}/M$.

ADDENDUM. The referee kindly pointed out that Rickart (Section 3, pp. 622–623, of "The uniqueness of norm problems in Banach spaces", Annals of Mathematics, 51 (1950), 615–628) showed that the commutant of a strictly cyclic transitive algebra consists only of scalars and that the algebra is n-transitive for every n. Thus $\mathscr{A}$ is strongly dense in $\mathscr{L}(X)$. These facts make it unnecessary to quote Theorem 1 of [3] in the proof of Theorem 3 of this paper.

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