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PROJECTIVE PSEUDO COMPLEMENTED SEMILATTICES

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This paper is concerned with the properties of free, and projective pseudo complemented semilattices (PCSL).

It is proved that a projective PCSL is complemented and all its chains and disjointed subsets are countable, and that a Boolean algebra is projective in the category of PCSL if and only if it is projective in the category of Boolean algebras. Further, necessary and sufficient conditions are established for a finite PCSL to be projective.

1. Preliminaries. A *semilattice* A is a partially ordered set closed under meets. If A has a least element we will denote it by 0. We say that a^* is the *pseudo complement* of $a \in A$, A a semilattice with 0, if we have (i) $a \cdot a^* = 0$, (ii) If $ab = 0$ then $b \leqq a^*$, for $b \in A$. Clearly pseudo complements are unique when they exist. A semilattice with 0 called a *pseudocomplemented semilattice* (PCSL) if each element has a pseudo-complement. A PCSL has a greatest element, 0^*, which we denote by 1. A function $f: A \rightarrow B$, A, B PCSL's, is called a homomorphism if $f(ab) = f(a) \cdot f(b)$, $f(a^*) = f(a)^*$ for $a, b \in A$. We observe that $f(0) = 0$, and $f(1) = 1$. For $S \subseteqq A$ let $S^* = \{x^* : x \in S\}$.

It is easily shown that the following identities are true in any PCSL.

(1)	$xy = yx$	(13)	$(xy)^* = (x^{**}y^{**})^*$
(2)	$x(yz) = (xy)z$	(14)	$x^*y^{**} = 0 \leftrightarrow x^*y^* = x^*$
(3)	$xx = x$	(15)	$xy = 0 \leftrightarrow x \leqq y^*$
(4)	$0 \cdot x = 0$	(16)	$x(xy)^* = xy^*$
(5)	$x(xy)^* = xy^*$	(17)	$x(x^*y)^* = x$
(6)	$x0^* = x$	(18)	$x^*(xy)^* = x^*$
(7)	$0^{**} = 0$	(19)	$x^*(x^*y)^* = x^*y^*$
(8)	$x \leqq x^{**}$	(20)	$x^{**}(x^*y)^* = x^{**}$
(9)	$x \leqq y \rightarrow y^* \leqq x^*$	(21)	$x^{**}(xy)^* = x^{**}y^*$
(10)	$x \leqq y \rightarrow x^{**} \leqq y^{**}$	(22)	$(xy)^*(xy^*)^* = x^*$
(11)	$x^{***} = x^*$	(23)	$(xy)^{**} = x^{**}y^{**}$
(12)	$x^*y^* = (x^*y^*)^{**}$		

The definitions of the concepts discussed in this paper may be found in References 3, 4, 5, and 7.

2. Free PCSL.

LEMMA 2.1. *Let X freely generate the* PCSL *F. Then*

(1) $0 \notin X$, $1 \notin X$.
(2) *If* $S \subseteqq X$, *S finite then* $\Pi(S) \neq 0$.
(3) *If* $S \subseteqq X^*$, *S finite then* $\Pi(S) \neq 0$.
(4) *If* $x \in X$ *then* $x \neq x^{**}$.
(5) *If* $x_1, x_2 \in X$, $x_1 \neq x_2$, *then* $x_1^* \neq x_2^*$, $x_1^{**} \neq x_2^{**}$.
(6) $S \subseteqq X$, *then* $|S| = |S^*| = |S^{**}|$.
(7) $x_1, x_2 \in X$ *and* $x_1 \leqq x_2$ *then* $x_1 = x_2$.
(8) *If* $x \geqq \Pi(T)$, $T \subseteqq X$ *then* $x \in T$, where $x \in X$.

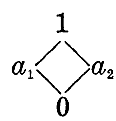
Proof. (1) If $0 \in X$, then let f be a homomorphism $f: F \rightarrow 2$ such that $f(0) = 1$. But $f(0) = f(a \cdot a^*) = f(a) \cdot f(a)^* = 0$. Thus $0 = 1$ in 2, a contradiction. Also, suppose $1 \in X$. Let g be a homomorphism $g: F \rightarrow 2$ such that $g(1) = 0$. But $1 = 0^* = g(1)^* = g(0) = 0$, again, a contradiction.

(2) Suppose $\Pi(S) = 0$. S finite, $S \subseteqq X$. Then there is a homomorphism $f: F \rightarrow 2$ so that $f(x) = 1$ all $x \in S$. Thus $1 = f(\Pi(S)) = f(0) = 0$ a contradiction.

(3) Suppose $\Pi(S) = 0$, S finite $S \subseteqq X^*$. Then there is a homomorphism $f: F \rightarrow 2$ so that $f(x) = 0$ for all $x^* \in S$. Thus $1 = f(\Pi(S)) = f(0) = 0$ a contradiction.

(4) Suppose $x = x^{**}$. Then there is a homomorphism $f: F \rightarrow 3$, the 3 element chain $3 = (0, a, 1)$ such that $f(x) = a$. But $a^{**} = f(x^{**}) = f(x) = a$ is false since $a^{**} = 0$ in 3.

(5) Let $x_1 \neq x_2$ and suppose $x_1^* = x_2^*$. Since F is free let f be the homomorphism from F onto the boolean algebra, such that $x_1 \rightarrow a_1$,



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and $x_2 \rightarrow a_2$. Since $x_1^* = x_2^*$ then $a_1^* = a_2^*$. That is $a_2 = a_1$, a contradiction. Thus $x_1^* \neq x_2^*$.

If $x_1^{**} = x_2^{**}$ then we have $x_1^{***} = x_2^{***}$, i.e., $x_1^* = x_2^*$, a contradiction. Thus $x_1^{**} \neq x_2^{**}$.

(6) Let $S \subseteqq X$. Then $S^* = \{x^* : x \in S\}$. Let $f: S \rightarrow S^*$ be defined by $f(x) = x^*$. Clearly f is onto. Suppose $f(x_1) = f(x_2)$, i.e., $x_1^* = x_2^* \therefore x_1 = x_2$, i.e., f is $1-1$. Thus $|S| = |S^*|$. Also let $g: S \rightarrow S^{**}$ be defined by $g(x) = x^{**}$. If $g(x_1) = g(x_2)$ hence $x_1^{**} = x_2^{**}$ and $x_1 = x_2$, i.e., g is $1-1$. Thus $|S| = |S^{**}|$.

(7) Suppose $x_1 \neq x_2$. Let $f: F \rightarrow 2$ be a homomorphism such that $f(x_1) = 1$ and $f(x_2) = 0$. But since $x_1 < x_2$ thus $1 \leqq 0$ — a contradiction.

(8) Let $x \geqq \Pi(T)$ and suppose $x \notin T$. Let $f: F \rightarrow 2$ be a homomorphism such that $f(x) = 0$ and $f(x_i) = 1$, $x_i \in T$. Then we have $1 \leqq 0$ — a contradiction.

LEMMA 2.2. *If A is a* PCSL *then A^* is a retract of A.*

Proof. $\varphi: A \to A^*$ defined by $\varphi(x) = x^{**}$ is a homomorphism onto A^*. If $x \in A^*$ then $\varphi(x) = x$, since $x^{***} = x^*$. Hence, A^* is a retract of A.

THEOREM 2.1. *If F is a* PCSL *freely generated by X, then F^* is freely Boolean generated by X^{**};* i.e., *F^* is free in the class of boolean algebras.*

Proof. Let $\varphi: F \to F^*$ the homomorphism $\varphi(x) = x^{**}$, and let $\psi: F^* \to F$ be the inclusion map. Then $\varphi\psi = I_{F^*}$. Let $X = \{x_i: i \in I\}$ and let B be any Boolean algebra and suppose $b_i \in B$, for $i \in I$ then there exists a homomorphism $f: F \to B$ such that $f(x_i) = b_i$. Let $h = f\psi: F^* \to B$. Then $h(x_i^{**}) = f(x_i^{**}) = b_i^{**} = b_i$. Also we note that h is a Boolean homomorphism.

THEOREM 2.2. *Let A be any free* PCSL *and let X freely generate A. Then every element of A is of the form $\Pi(T)\cdot(\Pi(P_1))^* \cdots (\Pi(P_n))^*$, where $T \subseteqq X$, $P_i = R_i \cup S_i^*$, $R_i \cup S_i \subseteqq X$, $R_i \cap S_i = \varnothing$, P_i finite for $i = 1, 2, \cdots, n$, $n \geqq 0$, using the convention that $\Pi(\varnothing) = 1$.*

Proof. Let $B = \{\Pi(T)\cdot r: T \subseteqq X, r \in A^*, T \text{ finite}\}$. Then B is a subalgebra of A, since $0 \in B$, and B is closed undermeets. Also if $b \in B$, then $b^* \in A^*$, and thus $b^* \in B$. Further, we note that $X \subseteqq B$, hence $B = A$. Since the homomorphism $\varphi: A \to A^*$ given by $\varphi(x) = x^{**}$ is onto, then A^* is freely Boolean generated by X^{**}. Hence any element $r \neq 1$ of A^* is a product of elements of the form $\alpha = \sum_{A^*}(U \cup V^*)$ where U and V are finite disjoint subsets of X^{**}. But $U = S^{**}$ and $V = R^{**}$ for some R, S subsets of X. Clearly $R \cap S = \varnothing$ and $V^* = R^*$. Hence

$$\alpha = \sum_{A^*}(S^{**} \cup R^*) = (\Pi(S^* \cup R^{**}))^* = (\Pi(R \cup S^*))^* ,$$

by [2, Theorem. 2] and (13) of § 1. Since $x(xy)^* = xy^*$, $x(x^*y)^* = x$, ((16), (17) of § 1) we may assume that $T \cap R_i = T \cap S_i = \varnothing$ for all $i \leqq n$.

THEOREM 2.3. *Let X, Y freely generate a* PCSL *F. Then $X = Y$.*

Proof. Let $x \in X$. Then $x = \Pi(T)\cdot r$ where $\varnothing \neq T \subseteqq Y$ and $r \in F^*$. Then $x \leqq y_i$ for all $y_i \in T = \{y_1, y_2, \cdots, y_n\}$. Also, $y_i = \Pi(T_i)\cdot r_i$ for $\varnothing \neq T_i \subseteqq X$ and $r_i \in F^*$. Hence $x = \Pi(T)\cdot r = \Pi(\bigcup T_i)(\Pi r_i)\cdot r$ from which we see that $x \leqq \Pi(\bigcup T_i)$, and conclude that $\bigcup T_i = \{x\}$, using Lemma 2.1(7). Hence $y_i = x \cdot r_i$ and thus $y_i \leqq x$ and hence $x = y_i$, i.e., $x \in Y$. Thus $X \subseteqq Y$, and by a similar argument $Y \subseteqq X$.

LEMMA 2.3. *Suppose X freely generates a* PCSL F, *and let $x \in X$, $R \cup S \cup T \subseteqq X$, $R \cup S \cup T$ finite. If $0 \neq \Pi(T \cup R^{**} \cup S^*) \leqq x$, then $x \in T$.*

Proof. Since $0 \neq \Pi(T \cup R^{**} \cup S^*)$, then $T \cap S = R \cap S = \varnothing$. Clearly $x \notin S$. Suppose $x \notin T$. Let $f\colon F \to F$ be a homomorphism such that

$$f(y) = \begin{cases} 1 \text{ if } y \in R \cup T - \{x\} \\ x \text{ if } y = x \\ 0 \text{ if } y \in S\,. \end{cases}$$

This is possible since X freely generates F. Then

$$f(\Pi(T \cup R^{**} \cup S^*)) = \begin{cases} 1 & \text{if } x \notin R \\ x^{**} & \text{if } x \in R\,. \end{cases}$$

Hence $1 \leqq x$ or $x^{**} \leqq x$, so $x = 1$, or $x = x^{**}$. But this is impossible by Lemma 2.1, and the result follows.

LEMMA 2.4. *Let X freely generate F, and $T \subseteqq X$, and $r \in F^*$, $x \in X$. If $0 < \Pi(T) \cdot r \leqq x$. Then $x \in T$.*

Proof. r is a sum in F^* of elements of the form $\Pi(R^{**} \cup S^*)$, where $R \cup S \subseteqq X$, $R \cap S = \varnothing$. Hence for some R and S we have $0 < \Pi(T \cup R^{**} \cup S^*) \leqq x$. Then by Lemma 2, $x \in T$.

THEOREM 2.4. *Let X freely generate a* PCSL F. *Then the elements of X are super-meet irreducible. That is, let $a_1, a_2, \cdots, a_n \in F$, $x \in X$, and $0 < a_1 a_2 \cdots a_n \leqq x$, then $a_i \leqq x$ for some i.*

Proof. For each i, $a_i = \Pi(P_i) \cdot r_i$, $P_i \subseteqq X$, $r_i \in F^*$. Hence $0 < \Pi(P_1 \cup \cdots \cup P_n) \cdot r_1 \cdots r_n \leqq x$, then by Lemma 2.5 $x \in P_1 \cup \cdots \cup P_n$ and thus $x \in P_i$ for some i. Therefore $a_i \leqq x$.

LEMMA 2.5. *Let X freely generate F, and $a \in F$, $r \in F^*$. If $0 < r < a$, then $a \in F^*$.*

Proof. Suppose $a \notin F^*$ then $a \leqq x$, for some $x \in X$. Hence $0 < r \leqq x$. But r is a sum (in F^*) of elements of the form $\Pi(R^{**} \cup S^*)$, where $R \cup S \subseteqq X$, $R \cap S = \varnothing$. Hence for some such R, S, we have, $0 < \Pi(R^{**} \cup S^*) = \Pi(\varnothing \cup R^{**} \cup S^*) \leqq x$, and then by Lemma 2.4 we have $x \in \varnothing$, *a contradiction.*

LEMMA 2.6. *Let X freely generate F, and $a \in F$. If $a^* = 0$, then $a = 1$.*

Proof. We have $a = \Pi(T) \cdot r$ where $T \subseteqq X$, $r \in F^*$. Since $a^* = 0$, then $1 = a^{**} = \Pi(T)^{**} \cdot r \leqq r$, thus $r = 1$. Hence $a = \Pi(T)$. If $T \neq \varnothing$ then $a \leqq x$ for some $x \in X$. Thus $1 \leqq x^{**}$. But this is impossible by Lemma 2.1(3).

THEOREM 2.5. *If F is a free* PCSL, *then F is complemented,* i.e., *if $a \in F$, the $a + a^*$ exists and equals* 1.

Proof. Suppose $a \leqq b$, $a^* \leqq b$, then $b^* \leqq a^* a^{**} = 0$. Hence $b = 1$ by Lemma 2.6.

THEOREM 2.6. *Let F be a free* PCSL.

(1) *Let $S \subseteqq F^*$, S finite, and $a = \sum_{F^*}(S)$. Then $\sum_F(S)$ exists and equals a.*

(2) *$a^* + b^* = (ab)^*$ for $a, b \in F$.*

Proof. (1) Clearly true if $S = \{0\}$.

We may assume $S \neq \{0\}$. Now $a \geqq s$ for all $s \in S$. If $b \in F$ and $b \geqq s$ all $s \in S$, then $b \in F^*$ by Lemma 2.6 and thus $b \geqq a$. Thus $\sum_F(S)$ exists and equals a.

$$\begin{aligned}(2)\quad a^* +_F b^* &= a^* +_{F^*} b^* \\ &= (a^{**}b^{**})^* \quad \text{since } F^* \text{ is a Boolean algebra} \\ &= (ab^*) \qquad \text{by (13) of § 1.}\end{aligned}$$

LEMMA 2.7. *Let F be a free* PCSL *and $r \in F^*$. Then $\{a \in F\colon a^{**} = r\}$ is finite.*

Proof. By Lemma 2.6, $a^* = 0$ iff $a = 1$, and in any PCSL $a^* = 1$ iff $a = 0$. Hence we may assume $0 < r < 1$. Let X freely generate F. By Theorem 2.2 there exists a finite subset X_1 of X such that $r \in F_1$, the algebra generated by X_1. Now F_1 is finite. We need only show that if $a^{**} = r$, then $a \in F_1$. If $a \in F^*$, then $a = a^{**} = r \in F_1$. Now suppose $a \notin F^*$. Then $a = \Pi(T) \cdot s$, where $\varnothing \neq T \subseteqq X$, and $s \in F^*$. Further, from Theorem 2.2 we may assume that s is in the subalgebra generated by a subset of X which is disjoint from T. If $T \not\subseteqq X_1$, then there exists an element $x \in T - X_1$. Let $f\colon F \to F$ be a homomorphism such that $f(x) = 0$, and $f(y) = y$, for all $y \in X - \{x\}$. Then $f(a) = 0$ and hence, $0 = f(a^{**}) = f(r)$. But $f(r) = r$ since

$x \notin X_1$. Then $r = 0$, a contradiction. This proves that $T \subseteqq X_1$, and so $\Pi(T) \in F_1$. Let $g: F \to F$ be a homomorphism such that $g(y) = 1$ for all $y \in T$, and $g(y) = y$ for $y \in X - T$. Then $g(s) = s$. Hence $s = g(s) = g(s \cdot \Pi(T)^{**}) = g(a^{**}) = g(r)$. But by definition of F_1, and g, $g(r) \in F_1$. Thus $s \in F_1$ and hence $a = \Pi(T) \cdot s \in F_1$.

COROLLARY 2.1. *Let F be a free* PCSL *and let $r \in F^*$, then $\{a \in F: a^* = r\}$ is finite.*

Proof. $\{a \in F: a^* = r\} = \{a \in F: a^{**} = r^*\}$ which is finite.

COROLLARY 2.2. *Let F be an infinite free* PCSL *and let $S \subseteqq F$, S infinite. Then,* $|S^*| = |S|$. Proof is clear.

THEOREM 2.7. *If B is a free Boolean algebra, then there exists a free* PCSL *F such that $F^* = B$.*

Proof. Let $X \subseteqq B$, freely Boolean generate B. Let F be the free PCSL on a set S of $|X|$ free generators. Then F^* is a free Boolean algebra freely generated by S^{**}. Since $|X| = |S| = |S^{**}|$, by Lemma 2.1(6), then $F^* = B$.

LEMMA 2.8. *Every free Boolean algebra is a retract (in the category of* PCSL) *of a free* PCSL.

Proof. Let B be a free Boolean algebra. By Theorem 2.7, there exists a free PCSL F such that $F^* = B$. But F^* is a retract of F, hence B is a retract of F.

THEOREM 2.8. *In a free* PCSL, *all chains are countable.*

Proof. Let F be a free PCSL, and let $C = \{a_i \in F: i \in I\}$ be an infinite chain. Then C^* is an infinite chain in F^* a free Boolean algebra. But chains in F^* are countable [6], and since $|C| = |C^*|$, hence C is a countable chain.

THEOREM 2.9. *All disjointed subsets of a free* PCSL *are countable.*

Proof. Let S be an infinite disjointed subset of F, a free PCSL. Now $|S| = |S^{**}|$. Also $a^{**}b^{**} = (ab)^{**} = 0^{**} = 0$, for $a, b \in S$. Thus S^{**} is a disjointed subset of F^*. But in a free Boolean algebra all disjointed sets are countable [7, p. 51], hence S is countable.

3. Projective PCSL.

THEOREM 3.1. *B, a Boolean algebra is projective in the category of boolean algebras, iff it is projective in the category of* PCSL.

Proof. B is a retract of a free Boolean algebra $\bar{B}$. By Theorem 2.7 there exists a free PCSL F such that $\bar{B} = F^*$, and thus $\bar{B}$ is a retract of F in the category of PCSL. Hence B is a retract of F in the category of PCSL and thus B is projective. Conversely, let B be a Boolean algebra which is projective in the category of PCSL. Thus there is a free PCSL F such that B is a retract of F. Then by Lemma 2, it follows that B is a retract of F^* in the category of Boolean algebras, and the result follows.

REMARK. The definition of projectivity makes it clear that the results of the preceding section following Theorem 2.4, hold for projective PCSL.

4. Finite projective PCSL.

DEFINITION 4.1. If P is a partially ordered set and $M \subseteqq P$, $p \in P$, let $M_p = \{m \in M: m \geqq p\}$.

DEFINITION 4.2. Let P be a finite partially ordered set and let M be the set of maximal elements of P. Then a semi-lattice A with least element, 0, is said to be *freely generated by P with the defining relation $\Pi(M) = 0$* if there is an order preserving function $\theta: P \rightarrow A$ such that $\Pi(\theta(M)) = 0$, $\theta(P)$ generates A, and such that if B is any semi-lattice with 0, and $h: P \rightarrow B$ is any order preserving function such that $\Pi(h(M)) = 0$, then there exists a semi-lattice homomorphism $g: A \rightarrow B$ such that $g(0) = 0$, and $g\theta = h$. The existence of A is guaranteed by a known theorem of universal algebra. A is unique up to isomorphism. See [4, p. 182, 183].

LEMMA 4.1. *Let P be a finite partially ordered set and M be the set of maximal elements of P. Suppose for each $p \in P - M$ we have $M_p \neq M$. Let A be a semi-lattice with 0 freely generated by P with defining relation $\Pi(M) = 0$ and let $\theta: P \rightarrow A$ be an in Definition 4.2. Then,*

(a) *θ is an order isomorphism. (So we may consider P contained in A, and θ as the inclusion function.)*

(b) *If $p_1, \cdots, p_n \in P$ then $p_1 p_2 \cdot \cdots \cdot p_n = 0$ iff $\bigcup \{M_{p_i}: i \leqq n\} = M$.*

(c) *If $p, p_1, \cdots, p_n \in P$ and $0 < p_1 p_2 \cdot \cdots \cdot p_n \leqq p$ then $p_i \leqq p$, some i.*

(d) *P is the set of meet irreducible elements of A.*

Proof. (a) If $S \subseteqq P$ let $M(S) = \bigcup \{M_p: p \in S\}$, $m(S)$ be the set of minimal elements of S. Define

$$B = \{M\} \cup \{S: \varnothing \neq S \subseteqq P,\ M(S) \neq M \text{ and for } x, y \in S,\ x \neq y \rightarrow x \,||\, y\}$$

where $a \,||\, b$ means $a \not\leqq b$ and $b \not\leqq a$. For $S_1, S_2 \in B$, define

$$S_1 \cdot S_2 = \begin{cases} M & \text{if } M(S_1 \cup S_2) = M \\ m(S_1 \cup S_2) & \text{if } M(S_1 \cup S_2) \neq M\,. \end{cases}$$

Then $S_1 \cdot S_2 = S_2 \cdot S_1$, $S_1 \cdot S_1 = S_1$, $S_1 \cdot M = M$ for any $S_1, S_2 \in B$. It is easy to verify that

$$S_1 \cdot (S_2 \cdot S_3) = \begin{cases} M & \textit{if } M(S_1 \cup S_2 \cup S_3) = M \\ m(S_1 \cup S_2 \cup S_3) & \text{if } (S_1 \cup S_2 \cup S_3) \neq M\,. \end{cases}$$

Therefore, $(S_1 \cdot S_2) \cdot S_3 = S_3 \cdot (S_1 \cdot S_2) = S_1 \cdot (S_2 \cdot S_3)$, and thus B is a semi-lattice with smallest element M if we define $S_1 \leqq S_2$ whenever $S_1 \cdot S_2 = S_1$. Note that $S_1 \leqq S_2$ iff either $S_1 = M$, or for any $x \in S_2$ there exists $y \in S_1$ such that $x \geqq y$. Define $h: P \rightarrow B$ by $g(p) = \{p\}$ for $p \in P$. If $p_1 \leqq p_2$ then $\{p_1\} \leqq \{p_2\}$. Also, $\Pi(h(M)) = \Pi\{\{m\}: m \in M\} = M$. Thus there exists a homomorphism $g: A \rightarrow B$ such that $g\theta = h$. If $\theta(p_1) \leqq \theta(p_2)$, then $h(p_1) \leqq h(p_2)$. But $\{p_1\} \leqq \{p_2\}$ implies $p_1 \leqq p_2$ or $M_{p_1} = M$. If $M_{p_1} = M$ then $p_1 \in M$ and hence $M = P = \{p_1\}$, so $p_1 = p_2$. Therefore, θ is an order isomorphism. Henceforth we may assume $P \subseteqq A$ and $\theta(p) = p$ for all $p \in P$.

(b) If $p_1 p_2 \cdot \cdots \cdot p_n = 0$, then $\{p_1\} \cdot \cdots \cdot \{p_n\} = M$. Therefore, $M = M(\{p_1, \cdots, p_n\}) = \bigcup \{M_{p_i}: i \leqq n\}$. If $\bigcup \{M_{p_i}: i \leqq n\} = M$ then $p_1 p_2 \cdots p_n \leqq \Pi(M) = 0$.

(c) Suppose $0 < p_1 \cdots p_n \leqq p$. Then $\{p_1\} \cdot \cdots \cdot \{p_n\} \leqq \{p\}$ and $\{p_1\} \cdot \cdots \cdot \{p_n\} \neq M$. Therefore, $p \geqq p_i$ for some i.

(d) Since P generates A, every element of A is a product of elements of P. Therefore, any meet irreducible element of A is in P. Conversely if $p \in P$, and $p \neq 0$ then p is meet irreducible by (c) and the fact that P generates A. If $0 \in P$ then $0 \in M$ because $M_p \neq M$ for $p \in P - M$, thus $P = \{0\}$ and $A = \{0\}$ and thus (d) is proved.

LEMMA 4.2. *Let A be a finite semi-lattice, P be the set of meet irreducible elements of A, and M the set of maximal elements of P. If*

(a) *If $p_1, \cdots, p_n \in P$ then $p_1 \cdots p_n = 0$ iff $\bigcup \{M_{p_i}: i \leqq n\} = M$.*

(b) *If $p, p_1, \cdots, p_n \in P$ and $0 < p_1 \cdots p_n \leqq p$ then $p_i \leqq p$ some i.*

Then for each $p \in P - M$, $M_p \neq M$ *and* A *is freely generated by* P *with defining relation* $\Pi(M) = 0$.

Proof. If $p \in P - M$ and $M_p = M$, then by (a) $p = 0$. But $\Pi(M) = 0$ by (a) hence $p = \Pi(M)$ which contradicts the fact that P is meet irreducible. If $x \in A$, $x \neq 0$, then $x = \Pi(S)$ for some $S \subseteqq P$, and we may assume the elements of S to be pairwise incomparable. If $x \in \Pi(S')$, $S' \subseteqq P$ and the elements of S' incomparable, then by (b) every member of S' is greater than or equal to a member of S, and vice-versa. Therefore $S = S'$. Thus for $x \in A$ there exists a unique set S_x of incomparable elements of P such that $x = \Pi(S_x)$.

Suppose B is any semi-lattice with 0, and $h: P \rightarrow B$ is an order preserving function such that $\Pi(h(M)) = 0$.

Define $g: A \rightarrow B$ by $g(x) = \Pi(h(S_x))$ for $x \neq 0$ and $g(0) = 0$. To show g is a semi-lattice homomorphism, first note $g(xy) = g(x) \cdot g(y) = 0$ if $x = 0$, or $y = 0$. Suppose $x \neq 0$ and $y \neq 0$. If $xy \neq 0$ then $S_{xy} = m(S_x \cup S_y)$, the set of minimal elements of $S_x \cup S_y$. Since g is order preserving we have $g(x) \cdot g(y) = \Pi(h(S_x)) \cdot \Pi(h(S_y)) = \Pi(h(S_x \cup S_y)) = \Pi h(S_{xy}) = g(xy)$.

If $xy = 0$ then by (a) $\bigcup \{M_p: p \in S_x\} \cup \bigcup \{M_p: p \in S_y\} = M$. Therefore $g(x) \cdot g(y) = \Pi(h(S_x \cup S_y)) \leqq \Pi(h(M)) = 0 = g(xy)$. Clearly $g \mid P = h$, and the proof is complete.

LEMMA 4.3. *Let* A *be a finite semi-lattice with* 1 *and suppose* $A - \{1\}$ *satisfies the hypothesis of Lemma* 4.2. *Then*

(a) *A is pseudo complemented and for each* $x \in A - \{1\}$, $x^* = \Pi(M - M_x)$ *and* $x^{**} = \Pi(M_x)$ *where* M_x, M *as in Lemma* 4.2.

(b) $A^* - \{1\} = \{\Pi(S): S \subseteqq M\}$.

(c) *M is the set of dual atoms of A which is also the set of dual atoms of* A^*.

(d) *If* $S \subseteqq A^*$, *then* $\sum_A (S)$ *exists and equals* $\sum_{A^*} (S)$.

Proof. Firstly we show that if $S \subseteqq M$, $m \in M$ and $\Pi(S) \leqq m$, then $m \in S$. We prove this as follows: If $\Pi(S) = 0$ then $S = M$ by hypothesis (a), and thus $m \in S$. If $\Pi(S) \neq 0$ then by hypothesis (b) $m' \leqq m$ for some $m' \in S$, but then $m = m' \in S$, so $m \in S$.

(a) Let $x \in A - \{1\}$ and let $y = \Pi(M - M_x)$. Then

$$xy \leqq \Pi(M_x) \cdot \Pi(M - M_x) = \Pi(M) = 0 .$$

Now suppose $xz = 0$ for some $x \in A$. Using the notation of the proof of Lemma 4.2 we have $\Pi(S_x \cup S_z) = 0$. Therefore by hypothesis (b), $M = \bigcup \{M_p; p \in S_x \cup S_z\}$. If $m \in M - M_x$, it follows that $m \geqq p$ for some $p \in S_x \cup S_z$. If $p \in S_x$, then $m \geqq x$ contradicting $m \in M - M_x$.

Therefore, $p \in S_z$ and so $m \geqq p \geqq z$. Therefore, $y = \Pi(M - M_x) \geqq z$ and this proves that $y = x^*$.

Now

$$m \in M_y \leftrightarrow y = \Pi(M - M_x) \leqq m$$
$$\leftrightarrow m \in M - M_x \text{ by hypothesis (a)}.$$

Therefore, $M_y = M - M_x$ and $x^{**} = y^* = \Pi(M - M_y) = \Pi(M_x)$. This proves (a).

(b) By (a), every element of A^* is of the form $\Pi(S)$ for some $S \subseteqq M$. If $m \in M$, then $m^{**} = \Pi(M_m) = m$, hence $M \subseteqq A^*$. This proves (b).

(c) If $m \in M$ and $m < x < 1, x \in A$, then $m < p < 1$ for some $p \in P$. This is a contradiction and so m is a dual atom of A. If x is a dual atom of A, then x is meet irreducible and hence $x \in M$. By (b), the dual atoms of A^* are in M. Therefore, M is the set of dual atoms of A^*.

(d) By hypothesis (b) of Lemma 4.2, and (b) above, it is easy to see that if $a \in A^*$, $x \in A$, and $0 < a \leqq x$ then $x \in A^*$. This implies (d) just as it did for a free PCSL, in the proof of Theorem 2.6.

REMARK. By Lemmas 4.2 and 4.3, the free finite PCSL F with n generators may be described as follows. Let P be the set

$$\{x_i: i \leqq n\} \cup \{z_s: S \subseteqq \{1, 2, \cdots, n\}\},$$

and suppose $x_i \leqq z_s$ iff $i \in S$. Then F is the semi-lattice with 0 which is freely generated by P with defining relation $\Pi\{z_s: S \subseteqq \{1, \cdots, n\}\} = 0$.

THEOREM 4.1. *Let A be a finite projective* PCSL, *let P be the set of meet irreducible elements of $A - \{1\}$, and M be the set of maximal elements of P. Then*

(a) *If $S \subseteqq P, p \in P$, and $0 < \Pi(S) \leqq p$, then $s \leqq p$ for some $s \in S$.*

(b) *If $S \subseteqq P$, then $\Pi(S) = 0$ iff $\bigcup\{M_s: s \in S\} = M$.*

(c) $\bigcap\{M_p: p \in P - M\} \neq \varnothing$.

Proof. As in the proof of Lemma 4.3, it is easy to see that M is the set of dual atoms of A. M is also the set of dual atoms of A^*. It follows that $(P - M) \cap A^* = \varnothing$.

(a) Let F be a PCSL freely generated by a set X such that $|X| = |P|$. Let $h: X \rightarrow P$ be $1 - 1$ and onto. Then there exists a homomorphism $f: F \rightarrow A$ such that $f \mid X = h$. Since P generates A, f is onto. Since A is projective, there exists a homomorphism $g: A \rightarrow F$ such that $fg = I_A$. Let $p \in P - M$ and $x = h^{-1}(p)$. Now $g(p) = \Pi(T) \cdot r$

for some $T \subseteqq X$, $r \in F^*$. Hence $p = fg(p) = \Pi(f(T)) \cdot f(r)$. Since p is meet irreducible and $p \notin A^*$, it follows that $p = f(y)$ for some $y \in T$. But $p = f(x)$ and $f \mid X$ is $1-1$, therefore $x = y \in T$ and so $g(p) \leqq \Pi(T) \leqq x$. We have therefore shown that for any $p \in P - M$, $g(p) \leqq h^{-1}(p)$.

Now suppose $S \subseteqq P$, $p \in P - M$ and $0 < \Pi(S) \leqq p$. Since g is $1-1$, $0 < \Pi(g(S)) \leqq g(p) \leqq h^{-1}(p)$, so by Theorem 2.4, $g(s) \leqq h^{-1}(p)$ for some $s \in S$. Hence $s = fg(s) \leqq fh^{-1}(p) = p$. This proves (a) for the case when $p \notin M$. If $p \in M$ and $\Pi(S) \leqq p$ for some $S \subseteqq P$, then $\Pi(S^{**}) \leqq p^{**} = p$. Since p is super-meet irreducible in A^*, it follows that for some $s \in S$, $s \leqq s^{**} \leqq p$, and so (a) holds.

(b) If $S \subseteqq P$ and $\Pi(S) = 0$, then for any $m \in M$, $\Pi(S) \leqq m$ and so $m \in M_s$ for some $s \in S$, by the preceding paragraph. This proves (b).

(c) We have shown that A satisfies the hypothesis of Lemmas 4.2 and 4.3. Therefore, for each $x \in A - \{1\}$, $x^* = \Pi(M - M_x)$. Suppose $\bigcap \{M_p : p \in P - M\} = \varnothing$. Then

$$\begin{aligned}\Pi\{p^* : p \in P - M\} &= \Pi\{\Pi(M - M_p) : p \in P - M\}\\ &= \Pi(\bigcup \{M - M_p : p \in P - M\})\\ &= \Pi(M - \bigcap \{M_p : p \in P - M\}) = 0 .\end{aligned}$$

Therefore,

$$\begin{aligned}0 = g(0) &= g(\Pi\{p^* : p \in P - M\})\\ &= \Pi\{g(p)^* : p \in P - M\}\\ &\geqq \Pi\{h^{-1}(p)^* : p \in P - M\}\end{aligned}$$

since $g(p) \leqq h^{-1}(p)$ for all $p \in P - M$.
But this is impossible, because if T is any finite subset of X, then $\Pi(T^*) \neq 0$ by Lemma 2.1.

LEMMA 4.4. *Suppose a* PCSL A *satisfied the hypotheses of Lemma* 4.3. *Let* B *be* PCSL *and* $g: A \to B$ *is a semi-lattice homomorphism such that* $g(0) = 0$, $g(p^{**}) = g(p)^{**}$ *for all* $p \in P$, P *the set of meet irreducible elements of* A, *and* $g(u^*) = g(u)^*$ *for all* $u \in A^*$. *Then* g *is a* PCSL *homomorphism.*

Proof. Let x by any element of A. We first prove that $g(x^{**}) = g(x)^{**}$. We have $x = \Pi\{p_i : i \leqq n\}$ for some $\{p_1, \cdots, p_n\} \subseteqq P$. Hence

$$g(x^{**}) = g(\Pi\{p_i : i \leqq n\}^{**}) = g(\Pi\{p_i^{**} : i \leqq n\})$$

by (23) of § 1

$$\begin{aligned}&= \Pi\{g(p_i^{**}) : i \leqq n\} = \Pi\{g(p_i)^{**} : i \leqq n\}\\ &= (\Pi\{g(p_i) : i \leqq n\})^{**} = g(\Pi\{p_i : i \leqq n\})^{**} = g(x)^{**} .\end{aligned}$$

Let $x \in A$ and let $u = x^{**}$, we have

$$g(x^*) = g(x^{***}) = g(u^*) = g(u)^* = g(x^{**})^* = g(x)^{***} = g(x)^* .$$

Hence g is a * homomorphism.

THEOREM 4.2. *Let A be a finite semi-lattice with* 1, *and let P be the meet irreducible element of $A - \{1\}$ and M the maximal elements of P. If*

(a) *If $p_1, \cdots, p_n \in P$ then $p_1 \cdots p_n = 0$ iff $\bigcup \{M_{p_i}: i \leqq n\} = M$.*

(b) *If $p, p_1, \cdots, p_n \in P$ and $0 < p_1 \cdots p_n \leqq p$, then $p_i < p$ for some i.*

(c) $\bigcap \{M_p: p \in P - M\} \neq \varnothing$.

Then A is a projective PCSL.

Proof. By Lemma 4.3 A is a PCSL. Let $M = \{a_1, \cdots, a_n\}$ and $P - M = \{b_1, \cdots, b_m\}$. Let F be a PCSL freely generated by $\{x_1, \cdots, x_n\} \cup \{y_1, \cdots, y_m\}$ and let $f: F \rightarrow A$ be a homomorphism such that $f(x_i) = a_i$, and $f(y_j) = b_j$ for all i, j. If $1 \leqq i \leqq n$, let

$$c_i = x_i^{**} + \sum \{x_k^*: k \neq i\} + \sum \{y_j^{**}: b_j < a_i\} + \sum \{y_j^*: b_j \not< a_i\} .$$

We observe that c_i is a dual atom of F^*. Let D be the set of all dual atoms of F^* which are not in $\{c_1, c_2, \cdots, c_n\}$. Since $\bigcap \{M_p: p \in P - M\} \neq \varnothing$ we may assume that $a_1 \geqq b_j$ for all $j = 1, 2, \cdots, m$. Let $h: P \rightarrow F$ be defined by

$$\begin{aligned} h(a_1) &= c_1 \Pi(D) & & \\ h(a_i) &= c_i & &\text{for } 1 < i \leqq n \\ h(b_j) &= \Pi\{y_k: b_j \leqq b_k\} \cdot \Pi(D) , & &\text{for } 1 \leqq j \leqq m . \end{aligned}$$

To show h is order preserving we observe the following: If $b_j < a_i$ then

$$h(b_j) \leqq y_j \cdot \Pi(D) \leqq y_j^{**} \cdot \Pi(D) \leqq \Sigma\{y_k^{**}: b_k < a_i\} \cdot \Pi(D) \leqq c_i \cdot \Pi(D) \leqq h(a_i) .$$

If $b_j \leqq b_r$ then $h(b_j) \leqq h(b_r)$ since

$$\{b_k: b_j \leqq b_k\} \supseteqq \{b_k: b_r \leqq b_k\} .$$

Also, $h(a_1) \cdots h(a_n)$ is the product of all the dual atoms of F^*, which is 0. By Lemma 4.2, there exists a semi-lattice homomorphism $g: A - \{1\} \rightarrow F$ such that $g \mid P = h$. Extend g to A by defining $g(1) = 1$. By Lemma 4.3, $z^* = \Pi(M - M_z)$ for all $z \in A$. Now $f(c_i) = a_i^{**} + \Sigma\{a_k^*: k \neq i\} + \Sigma\{b_j^{**}: b_j < a_i\} + \Sigma\{b_j^*: b_j \not< a_i\} = a_i$ for all i, since

$a_i^{**} = a_i$, $a_k^* \leqq a_i$ for $k \neq i$, $b_j^{**} \leqq a_i$ if $b_j < a_i$, and $b_j^* \leqq a_i$ if $b_j \not< a_i$. If $d \in D$, then either $d \geqq x_k^{**} + x_l^{**}$ for some $k \neq l$, in which case $f(d) \geqq a_k + a_l = 1$, or $d \geqq \Sigma\{x_k^*; k \leqq n\}$ in which case $f(d) \geqq \Sigma\{a_k^*: k \leqq n\} = (\Pi\{a_k: k \leqq n\})^* = 0^* = 1$, or $d \geqq x_i^{**} + y_j^*$ for some i and some j such that $b_j < a_i$, in which case $f(d) \geqq a_i + \Pi\{a_l: b_j \not< a_l\} = \Pi\{a_i + a_l: b_j \not< a_l\} = 1$, or $d \geqq x_i^{**} + y_j^{**}$ for some i and some j such that $b_j \not< a_i$ in which case $f(d) \geqq a_i + \Pi\{a_l: b_j < a_l\} = \Pi\{a_i + a_l: b_j < a_l\} = 1$. Thus $f(d) = 1$ for all $d \in D$. Now

$$fg(a_i) = f(c_i) = a_i\ , \qquad \text{for } i > 1$$
$$fg(a_1) = f(c_1) \cdot \Pi(f(D)) = a_1\ , \qquad \text{and}$$
$$fg(b_j) = \Pi\{b_k: b_j \leqq b_k\} \cdot \Pi f(D) = b_j\ , \qquad \text{for all } j\ .$$

Since P generates A we have $fg = I_A$. It remains to show that g is a * homomorphism.

For any k, y_k^{**} is the product of all dual atoms of F^* which are $\geqq y_k^{**}$. Since F^* is a free Boolean algebra, the only such dual atoms are the ones of the form $\Sigma(S^* \cup T^{**})$ where $y_k \in T$ and $S \cup T = \{x_1, \cdots, x_n, y_1, \cdots, y_m\}$. Thus $y_k^{**} = \Pi\{c_i: b_k < a_i\} \cdot \Pi(D_k)$ where $D_k \subseteqq D$. Therefore for any j,

$$\begin{aligned} g(b_j)^{**} &= \Pi\{y_k^{**}: b_j \leqq b_k\} \cdot \Pi(D) \\ &= \Pi\{\Pi\{c_i: b_k \leqq a_i\}: b_j \leqq b_k\} \cdot \Pi(D) = \Pi\{c_i: b_j < a_i\} \cdot \Pi(D) \\ &= \Pi\{g(a_i): b_j < a_i\} = g(\Pi\{a_i: b_j < a_i\}) = g(b_j^{**})\ . \end{aligned}$$

Also $g(a_i)^{**} = g(a_i^{**})$ since $a_i \in A^*$ and $g(a_i) \in F$. We observe that if R is the set of dual atoms of a finite Boolean algebra, then for any $T \subseteqq R$, $\Pi(T)^* = \Pi(R - T)$. Hence if $u \in A^*$, then $u = \Pi(S)$, for some $S \subseteqq M$, and $u^* = \Pi(M - S)$. If $a_1 \in S$,

$$\begin{aligned} g(u)^* &= (\Pi\{c_i: a_i \in S\} \cdot \Pi(D))^* = \Pi\{c_i: a_i \notin S\} \\ &= g(\Pi(M - S)) = g(u^*)\ . \end{aligned}$$

While if $a_1 \notin S$, $g(u)^* = (\Pi\{c_i: a_i \in S\})^* = \Pi\{c_i: a_i \notin S\} \cdot \Pi(D) = g(\Pi(M - S)) = g(u^*)$. We have now satisfied the hypothesis of Lemma 4.4, so g is a * homomorphism. Since A has been shown to be a retract of a free PCSL, then A is projective.

THEOREM 4.3. *Let A be a finite semi-lattice with* 1. *Let P be the set of meet irreducible elements of $A - \{1\}$, and M the set of maximal elements of P. Then A is projective if and only if the following hold.*

(a) *If $Q \subseteqq P$, then $\Pi(Q) = 0$ iff for each $m \in M$, there is a $q \in Q$ such that $m \geqq q$.*

(b) *If* $Q \subseteqq P$, $p \in P$ *and* $0 < \Pi(Q) \leqq p$, *then* $q \leqq p$, *for some* $q \in Q$.

(c) *There exists an* $m \in M$ *such that* $m \geqq p$ *for every* $p \in P - M$.

Proof of this follows from Theorems 4.1 and 4.2.

THEOREM 4.4. *If* P *is a finite partially ordered set and* M *is the set of maximal elements of* P. *Suppose*

(a) *For every* $p \in P$, *there exists an* $m \in M$ *such that* $p \nless m$.

(b) *There exists a* $m \in M$ *such that* $m \geqq p$, *for every* $p \in P - M$.

Then the semi-lattice with 0 *which is freely generated by* P *with the defining relation* $\Pi(M) = 0$, *is a projective* PCSL, *and every finite projective* PCSL *can be so obtained.* Proof of this follows from Lemma 4.1 and Theorem 4.2.

REMARK. To the conditions of Theorem 4.2 and 4.3, we could add the following, though redundent condition: If $Q \subseteqq M$, then $\Pi(Q) = 0$ iff $Q = M$.

REFERENCES

1. Raymond Balbes and Alfred Horn, *Stone lattices*, Duke Math. J., **38** (1970).
2. O. Frink, *Pseudo complements in semi-lattices*, Duke Math. J., **29** (1962).
3. P. R. Halmos, *Lectures on Boolean Algebras*, Van Nostrand, 1963.
4. G. Grätzer, *Universal Algebra*, Van Nostrand, 1968.
5. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloquium Publications, 1967.
6. A. Horn, *A property of free Boolean algebras*, Proc. Amer. Math. Soc., **19** (1968).
7. P. H. Dwinger, *Introduction to Boolean Algebras*, Physica-Verlag, Wurzberg, 1961.

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