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BIHOLOMORPHIC APPROXIMATION OF PLANAR DOMAINS

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This paper establishes the existence of a domain (open connected subset) B of the complex plane C such that for every domain $\Omega \subset C$ and every compact set $K \subset \Omega$, there is a biholomorphic embedding $e: B \rightarrow \Omega$, such that $K \subset e(B) \subset \mathrm{cl}\,[e(B)] \subset \Omega$.

1. Introduction. Let Ω_1 and Ω_2 be domains (i.e., open connected sets) in the complex plane C such that $\mathrm{cl}\,\Omega_1 \subset \Omega_2$ (cl = closure). A domain Ω is a biholomorphic approximation of Ω_1 with respect to Ω_2 provided that there exists an invertible holomorphic function e defined on Ω such that

$$\mathrm{cl}\,\Omega_1 \subset e(\Omega) \subset \mathrm{cl}\,[e(\Omega)] \subset \Omega_2 .$$

The mapping e is a biholomorphic embedding (bh-embedding) of Ω into Ω_2. (Ω may also be considered a biholomorphic approximation of Ω_2 with respect to Ω_1.)

Homeomorphic domains may, of course, be biholomorphically inequivalent, and, moreover, may not even be close biholomorphic approximations of each other. For example, let $A(r, s) = \{z \in C: r < |z| < s\}$ when $0 < r < s < \infty$. Suppose that $0 < \varepsilon < 1 < t < \infty$ and that e is a bh-embedding of $A = A(r, s)$ such that

$$\mathrm{cl}\,A(1, t) \subset e(A) \subset \mathrm{cl}\,[e(A)] \subset A(1 - \varepsilon, t + \varepsilon) .$$

By taking the modules of these ring domains (cf. [1]) we obtain the inequality $t < s/r < (t + \varepsilon)/(1 - \varepsilon)$ which is precisely the condition r and s must satisfy for such an embedding e to exist.

Our main result establishes the existence of a domain $B \subset C$ which is a biholomorphic approximation of every bounded domain Ω_1 with respect to every domain Ω_2 containing $\mathrm{cl}\,\Omega_1$.

2. The main theorem. Let $\hat{C}$ denote the Riemann sphere.

THEOREM 2.1. *There exists a domain $B \subset C$ such that for every domain $\Omega \subset \hat{C}$ and for every compact set $K \subset \Omega$ other than $\hat{C}$ there exists a biholomorphic embedding $e: B \rightarrow \Omega$ such that $K \subset e(B) \subset \mathrm{cl}\,[e(B)] \subset \Omega$.*

REMARK. Actually such an embedding will exist if Ω is any connected Riemann surface (without boundary) and $K \subset \Omega$ is any planar compact surface other than $\hat{C}$. ("Planar" means homeomorphic to a subset of $\hat{C}$.) Indeed, by the trianguability of Ω there must exist a planar domain Ω_0 such that $K \subset \Omega_0 \subset \Omega$, and so it suffices to consider the planar case.

The following theorems are corollaries of Theorem 2.1.

COROLLARY 2.2. *Let* $K \neq \hat{C}$ *be a compact connected subset of a domain* $\Omega \subset \hat{C}$. *Then* $K = \bigcap_{i=1}^{\infty} B_i$ *where each* B_i *is bh-equivalent to* B *and* $\mathrm{cl}\, B_{i+1} \subset B_i$ *for* $i = 1, 2, \cdots$.

COROLLARY 2.3. *Let* $\Omega \neq \phi$ *be a domain in* C. *Then* $\Omega = \bigcup_{i=1}^{\infty} B_i$ *where each* B_i *is bh-equivalent to* B *and* $\mathrm{cl}\, B_i \subset B_{i+1}$ *for* $i = 1, 2, \cdots$.

3. *Proofs*. For each $a \in C$ and $r > 0$ set $D(a, r) = \{z: |z - a| < r\}$ and let $\bar{D}(a, r)$ denote $\mathrm{cl}\, D(a, r)$. Set $D = D(0, 1)$. A circle $\{z: |z - a| = r\}$ will be called "rational" provided that $\mathrm{Re}\, a, \mathrm{Im}\, a$, and $r > 0$ are rational numbers. The topological boundary of a domain Ω will be denoted $\partial\Omega$.

To construct B consider the domains Ω satisfying: (1) $\partial\Omega$ has finitely many components, (2) each component of $\partial\Omega$ is a rational circle, (3) $\mathrm{cl}\, \Omega \subset D$ and its outer boundary is centered at the origin. Let $E_1, E_2, \cdots$ be an enumeration of these domains. Let s_j be the radius of the outer boundary of E_j and let ϕ_j be the linear fractional transformation of D onto $H = \{z: \mathrm{Re}\, z > 0\}$ which carries -1 to 0, $+1$ to ∞, and $-s_j$ to 1 if $j = 1$ and to $\phi_{j-1}(s_{j-1})$ if $j > 1$. Let $B = H \backslash \bigcup_{j=1}^{\infty} \phi_j[D(0, s_j) \backslash E_j]$.

To show that B has the desired properties, we prove the following lemma using the "small mesh grid" technique (often employed in texts on function theory), rather than the theory of trianguability. A bounded domain $\Omega \subset C$ will be called a Jordan domain if $\partial\Omega$ consists of finitely many disjoint Jordan curves.

LEMMA 3.1. *Let* K *be a compact subset of a domain* $\Omega \subset C$. *Then there exists a Jordan domain* Ω_0 *such that* $K \subset \Omega_0 \subset \mathrm{cl}\, \Omega_0 \subset \Omega$.

Sketch of proof. Since Ω is connected, there exists a connected compact set K_0 such that $K \subset K_0 \subset \Omega$. Thus we may assume that K is connected. With r picked so small that $[K + \bar{D}(0, \sqrt{2}r)] \subset \Omega$ let L be the union of those squares of a grid of squares with edge length r which intersect K. If $a \in L$ is a vertex of precisely two squares of L select the positive number $s_a < r/2$ to be so small that $\bar{D}(a, s_a) \subset \Omega$. Let L_0 denote the union of all the $\bar{D}(a, s_a)$'s. Then straightforward arguments show that $\Omega_0 = \mathrm{int}\,(L \cup L_0)$ is the desired Jordan domain.

Now let Ω and K be as described in Theorem 2.1. Lemma 3.1

provides a Jordan domain Ω_0 such that $K \subset \Omega_0 \subset \text{cl}\,\Omega_0 \subset \Omega$. According to Theorem 2 page 237 of [2] there is a bh-embedding h of Ω_0 into D such that (1) the outer boundary of $h(\Omega_0)$ is ∂D and (2) $\partial[h(\Omega_0)]$ has finitely many components and each is a circle. Each of the circles bounding $h(\Omega_0)$ can be "approximated" arbitrarily closely by a rational circle which lies in $h(\Omega_0)$. We require that the approximation to the unit circle be centered at 0. Since $h(K)$ is a compact subset of $h(\Omega_0)$, when the approximations are close enough, the approximating circles will bound a domain which contains $h(K)$. This region, by its definition, is one of the E_j's, say E_k. Then

$$h(K) \subset E_k \subset \phi_k^{-1}(B) \subset h(\Omega_0)$$

and so applying h^{-1} will establish Theorem 2.1.

To prove Corollary 2.2 we let $B_1 = e_1(B)$ where e_1 is the bh-embedding of B such that $K \subset B_1 \subset \text{cl}\,B_1 \subset \Omega$. For $i > 1$ we let G_i be the component of $[K + D(0, 1/(i-1))] \cap B_{i-1}$ which contains K, and we set $B_i = e_i(B)$ where e_i is the bh-embedding of B, given by Theorem 2.1, such that $K \subset B_i \subset \text{cl}\,B_i \subset G_i$.

To prove Corollary 2.3 we pick $a \in \Omega$ and for large n we can let G_n be the component of $\{z\colon \text{dist}\,(z, \boldsymbol{C}\backslash\Omega) > 1/n \text{ and } |z| < n\}$ which contains a. Since $\text{cl}\,G_n$ is a compact subset of G_{n+1} there exists a bh-embedding $e_n\colon B \to G_{n+1}$ such that $B_n = e_n(B) \supset \text{cl}\,G_n$. That $\Omega = \bigcup G_n$ (and hence $\Omega = \bigcup B_n$) follows from the arc connectedness of Ω. These B_n's are the required domains (except for re-indexing).

4. Some applications to holomorphic extension problems. Let $K \subset \boldsymbol{C}$ be compact and let $f\colon K \to \boldsymbol{C}$. It is easy to extend f to a holomorphic function F defined on a domain containing K (caution: domains are connected) if there exist: (1) a domain Ω, (2) a biholomorphic function e on Ω such that $K \subset e(\Omega)$, and (3) a holomorphic extension G of $g = f \circ e\,|_{e^{-1}(K)}$ to all of Ω. Indeed $F = G \circ e^{-1}$ is the required extension. Conversely if f has such an extension F the existence of Ω, e, and G is trivial. For let the domain Ω be the domain of F, set $e(z) = z$, and take $G = F$. Thus we have an equivalent formulation of the problem of holomorphically extending a function $f\colon K \to \boldsymbol{C}$ to a domain containing K. Theorem 4.2 shows that another equivalent formulation is obtained when in the discussion above the variable domain Ω is replaced by the fixed domain B. We first show that for a more restricted class of sets K this extension question is very naturally formulated with D in the role of Ω.

THEOREM 4.1. *Let $K \subset \boldsymbol{C}$ be compact and let $f\colon K \to \boldsymbol{C}$. Suppose* *that K and $C\backslash K$ are connected. Then there exists a holomorphic extension F of f to a domain containing K if and only if there exist* (a) *a bh-embedding e of D such that $K \subset e(D)$ and* (b) *a holomorphic extension G of $g = f \circ e|_{e^{-1}(K)}$ to all of D.*

Proof. Since the "if" part of this theorem is treated in the discussion above we confine our remarks to the "only if" part. Assume that the extension F exists, and let $\Omega \supset K$ be its domain. It suffices to find a bh-mapping e of D such that $K \subset e(D) \subset \Omega$. This is trivial if K is a singleton: so we assume K is not a singleton. Then the Riemann Mapping theorem shows that $\hat{C}\backslash K$ is bh-equivalent to D (it is simply connected because K is connected). Let $h: \hat{C}\backslash K \to D$ be the Riemann mapping. Since $h^{-1}(\bar{D}(0, r))$ is simply connected for $0 < r < 1$ we know that $V_r = \hat{C}\backslash h^{-1}(\bar{D}(0, r))$ is nonempty, open, and simply connected for $0 < r < 1$. Thus each V_r with $0 < r < 1$ is bh-equivalent to D. Since $h(\hat{C}\backslash \Omega)$ is a compact subset of D it lies in $D(0, s)$ for some $s < 1$, and the Riemann mapping e of D onto V_s is the required map.

If in Theorem 4.1 D is replaced by B the assumption that K and $C\backslash K$ are connected may be dropped.

THEOREM 4.2. *Let $K \subset \boldsymbol{C}$ be compact and let $f: K \to \boldsymbol{C}$. There exists a holomorphic extension F of f to a domain containing K if and only if there exist* (a) *a bh-mapping e of B such that $K \subset e(B)$ and* (b) *a holomorphic extension G of $g = f \circ e|_{e^{-1}(K)}$ to all of B.*

Proof. As in the proof of Theorem 4.1 the "if" part has already been settled and we begin the "only if" part by letting $\Omega \supset K$ be the domain of F. An application of Theorem 2.1 gives a bh-embedding e of B such that $K \subset e(B) \subset \Omega$. This is the required mapping.

REMARK. Comparing Theorems 4.1 and 4.2 tempts one to conjecture the existence of a sequence of domains $D = \Omega_1, \Omega_2, \cdots, \Omega_\infty = B$ such that $\hat{C}\backslash \Omega_n$ has n components and for which Theorem 4.1 will remain true when it is modified by: (1) Replacing its second sentence with "Suppose K is connected and $C\backslash K$ has n components", and (2) Replacing D with Ω_n. The discussion in the introduction shows that this conjecture fails, since for $n = 2$, Ω_2 must be bh-equivalent to $A(r, s)$ for some r, s with $0 \leqq r < s \leqq \infty$ and so Ω_2 cannot be embedded between $A(1, t)$ (the domain of f) and $A(1 - \varepsilon, t + \varepsilon)$ (the domain of the extension F) unless $t < s/r < (t + \varepsilon)/(1 + \varepsilon)$.

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