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GENERALIZED LERCH ZETA FUNCTION

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The purpose of this paper is to establish certain properties of the generalized Lerch zeta function $\theta(z, \nu, a, b) = \sum_{n=0}^{\infty} (n + a)^{-\nu} z^{(n+a)^b}$. The main result yields another infinite series representation for θ. A generalization of Hardy's relation follows as an immediate corollary.

1. Introduction. The function $\Phi(z, \nu, a)$ defined by the power series

$$\Phi(z, \nu, a) = \sum_{n=0}^{\infty} (n + a)^{-\nu} z^n , \tag{1}$$

for $|z| < 1$, $0 < a \leqq 1$ and arbitrary ν, is called Lerch's zeta function. For $z = 1$, this function becomes Hurwitz' zeta function

$$\Phi(1, \nu, a) = \zeta(\nu, a) = \sum_{n=0}^{\infty} (n + a)^{-\nu}, \ \mathrm{Re}\,\nu > 1 \text{ and } 0 < a \leqq 1 . \tag{2}$$

Lerch's function has been extensively investigated in [1], [2], [3], [5, v. 1, p. 27], [7], [8], and [12]. One important result

$$\Phi(z, \nu, a) = \Gamma(1 - \nu) z^{-a} (\log 1/z)^{\nu-1} + z^{-a} \sum_{r=0}^{\infty} \zeta(\nu - r, a) \frac{(\log z)^r}{r!} , \tag{3}$$

for $|\log z| < 2\pi$, $0 < a \leqq 1$, $\nu \neq 1, 2, 3, \cdots$, which transforms Lerch's series into another series, is derived in Erdélyi [5, v. 1, p. 29] by using Lerch's transformation formula and Hurwitz' series for the Hurwitz zeta function. Hardy's relation (see Hardy [7] and Mellin [10]) follows immediately from (3):

$$\lim_{z \to 1} \{\Phi(z, \nu, a) - \Gamma(1 - \nu)(\log 1/z)^{\nu-1} z^{-a}\} = \zeta(\nu, a) . \tag{4}$$

The purpose of this paper is to establish certain properties of the function $\theta(z, \nu, a, b)$ where

$$\theta(z, \nu, a, b) = \sum_{n=0}^{\infty} (n + a)^{-\nu} z^{(n+a)^b}, \text{ for } |z| < 1, 0 < a \leqq 1, 0 < b . \tag{5}$$

It is appropriate to call θ the generalized Lerch zeta function because

$$\theta(z, \nu, a, 1) = z^a \Phi(z, \nu, a) .$$

Using an approach which is more direct than the above mentioned derivation of equation (3), we will establish

$$\theta(z, \nu, a, b) \tag{6}$$
$$= b^{-1}\Gamma\Big(\frac{1-\nu}{b}\Big)(\log 1/z)^{(\nu-1)/b} + \sum_{r=0}^{\infty} \zeta(\nu - br, a)\frac{(\log z)^r}{r!} ,$$

where $\nu \neq 1,\ 1+b,\ 1+2b, \cdots,\ 0 < a,\ b \leqq 1$. Formula (6) is valid for unrestricted z if $0 < b < 1$, and for $|\log z| < 2\pi$ if $b = 1$. In the latter case equation (6) becomes equation (3). Furthermore, from (6) we immediately obtain the following generalization of Hardy's relation:

$$\lim_{z\to 1}\Big\{\theta(z, \nu, a, b) - b^{-1}\Gamma\Big(\frac{1-\nu}{b}\Big)(\log 1/z)^{(\nu-1)/b}\Big\} = \zeta(\nu, a) , \tag{7}$$

for $0 < b \leqq 1$.

2. **Derivation of formula (6).** Consider the function

$$f(x) = x^{-\nu}e^{-\beta x^b} , \qquad \operatorname{Re}\beta > 0,\ b > 0.$$

The Mellin transform of $f(x)$ with respect to the parameter s is

$$g(s) = b^{-1}\beta^{(\nu-s)/b}\Gamma\Big(\frac{s-\nu}{b}\Big), \qquad \operatorname{Re} s > \operatorname{Re}\nu,\ \operatorname{Re}\beta > 0,\ b > 0 .$$

For Mellin transform theory see [4, v. 1, Ch. 2], [10], [11], and [13, p. 46]; for tables see [6, v. 1, p. 303]. Writing $f(x)$ in its Mellin inversion integral form with $x = n + a$, we obtain by summing on n and interchanging the order of summation and integration

$$\sum_{n=0}^{\infty}(n+a)^{-\nu}e^{-\beta(n+a)^b} = \frac{\beta^{\nu/b}}{2\pi i b}\int_{\sigma_0 - i\infty}^{\sigma_0+i\infty}\beta^{-s/b}\zeta(s, a)\Gamma\Big(\frac{s-\nu}{b}\Big)ds , \tag{8}$$

where $\sigma_0 > \max\{1, \operatorname{Re}\nu\}$. The left hand side of (8) is $\theta(e^{-\beta}, \nu, a, b)$. To evaluate the right hand side integral we will use the residue theorem.

If we denote

$$h(s) = \beta^{-s/b}\zeta(s, a)\Gamma\Big(\frac{s-\nu}{b}\Big) , \tag{9}$$

then $h(s)$ has a first order pole at $s = 1$ with residue $\beta^{-1/b}\Gamma([1 - \nu/b])$, and an infinite set of first order poles at $s = \nu - br$ with residues

$$\frac{(-1)^r}{r!}b\beta^{r-\nu/b}\zeta(\nu - br, a) , \qquad r = 0, 1, 2, \cdots .$$

Consider the contour integral

$$\int_C h(s)ds , \tag{10}$$

where the path of integration C is indicated in Figure 1 below, such that the half-circle C' of radius d separates the poles $s = \nu - Nb$ and $s = \nu - (N+1)b$. Then $h(s)$ is one-valued and analytic inside and on C except at the points $s = 1$, $s = \nu - rb$ $(r = 0, 1, 2, \cdots, N)$.

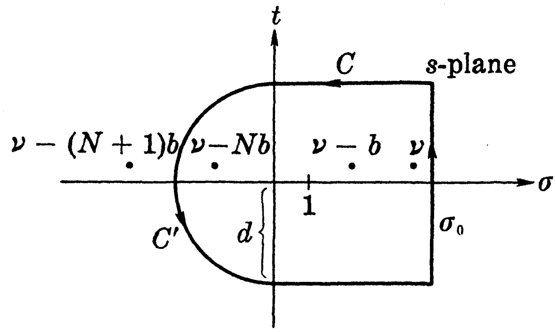

FIG. 1

Now we let N tend to infinity through positive integers.

To investigate the contributions along individual segments of the contour C we will need the following well-known properties of the gamma function and Hurwitz' zeta function, which can be found in Erdélyi [5] and/or Whittaker and Watson [14]:

$$\Gamma(s) = (2\pi/s)^{1/2}e^{-s}e^{s\log s}\left(1 + O\left(\frac{1}{s}\right)\right) \text{ as } |s| \longrightarrow \infty,\ |\arg s| < \pi\,. \tag{11}$$

$$\frac{\Gamma(s+\alpha)}{\Gamma(s+\beta)} = s^{\alpha-\beta}\left(1 + O\left(\frac{1}{s}\right)\right) \text{ as } |s| \longrightarrow \infty,\ |\arg s| < \pi\,. \tag{12}$$

$$|\Gamma(\sigma + it)| = O(|t|^{\sigma-1/2}e^{-\pi|t|/2}) \text{ as } |t| \longrightarrow \infty \text{ with } \sigma \text{ fixed } (\sigma, t \text{ real})\,. \tag{13}$$

$$\zeta(s, a) = 2(2\pi)^{s-1}\Gamma(1-s)\sum_{n=1}^{\infty} n^{s-1}\sin\left(2\pi na + \frac{\pi s}{2}\right), \quad \operatorname{Re} s < 0,\ 0 < a \leqq 1\,. \tag{14}$$

$$\zeta(\sigma + it, a) = o(|t|) \text{ as } |t| \longrightarrow \infty \text{ with } 0 \leqq \sigma \text{ fixed } (\sigma, t \text{ real})\,. \tag{15}$$

It is clear that the contributions to the integral (10) along the horizontal lines of length σ_0 (see Figure 1) vanish as $d \to \infty$ because of (13) and (15). To find the contribution along the half-circle C', it is sufficient to investigate the behavior of $h(s)$ on the quarter-circle of radius d for $\pi/2 < \arg s = \phi < \pi$, since the modulus of $h(s)$ on the quarter-circle for $-\pi < \phi < -\pi/2$ is the same by Schwarz's reflection principle. From (11), (12), and (14) we obtain

$$
\begin{aligned}
h(s) = O\{|s|^{-\nu/b} e^{|s|\cos\phi(\log 2\pi e - b^{-1}\log\beta e)} \\
\times e^{|s|\sin\phi(\pi/2-\phi/b)} e^{|s|\cos\phi(b^{-1}\log(|s|/b)-\log|s|)}\} \\
\text{as } |s| \longrightarrow \infty \text{ in } \frac{\pi}{2} < \phi < \pi,\ s = |s|e^{i\phi} .
\end{aligned} \tag{16}
$$

In formula (16) we have assumed β to be real and positive. The analytic continuation to complex β will be obtained later. The following three cases are possible:

(i) $b > 1$: Then (16) is dominated by the last exponential function and $h(s)$ tends to infinity when $d \to \infty$. Thus, the contribution over the semi-circle tends to infinity as $d \to \infty$ and formula (8) is not applicable, although the series in (8) converges for $\operatorname{Re}\beta > 0$.

(ii) $b = 1$: It is clear from (16) that the integral over the semi-circle C' vanishes as $d \to \infty$, provided $\beta < 2\pi$.

(iii) $0 < b < 1$: The integral over the semi-circle C' vanishes as $d \to \infty$, regardless of β.

Hence, in cases (ii) and (iii) we obtain by (8) and the residue theorem

$$
\theta(e^{-\beta}, \nu, a, b) - \frac{1}{b}\Gamma\left(\frac{1-\nu}{b}\right)\beta^{(\nu-1)/b} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!}\zeta(\nu - br, a)\beta^r . \tag{17}
$$

The *r.h.s.* series in (17) is a Taylor series around the origin and is therefore an analytic function of β in its circle of convergence, while the *l.h.s.* expression is valid only when $\operatorname{Re}\beta > 0$ and ν arbitrary; or $\operatorname{Re}\beta = 0$, $\operatorname{Im}\beta \neq 0$, and $\operatorname{Re}\nu > 0$; or $\beta = 0$ and $\operatorname{Re}\nu > 1$. Therefore, (17) represents the analytic continuation with respect to β of the *l.h.s.* of (17) valid for $\operatorname{Re}\beta > 0$ into the *r.h.s.* which is valid for unrestricted β when $0 < b < 1$, or for $|\beta| < 2\pi$ when $b = 1$. If $b > 1$, (17) is not valid.

Formula (6) is obtained from (17) by setting

$$
z = e^{-\beta} , \qquad \beta = \log 1/z .
$$

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