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MONOTONE DECOMPOSITIONS INTO TREES OF HAUSDORFF CONTINUA IRREDUCIBLE ABOUT A FINITE SUBSET

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This paper deals with characterizing two types of monotone, upper semi-continuous decompositions of a Hausdorff continuum that is irreducible about a finite subset. One of the decompositions is minimal with respect to the property of having a quotient space which is a tree (a hereditarily unicoherent, locally connected continuum) and is characterized in terms of certain collections of subcontinua. The other decomposition is not only minimal but also unique with respect to the properties that the quotient space is a tree and the elements of the decomposition have void interiors. This decomposition is characterized quite simply by prohibiting the existence of indecomposable subcontinua with nonvoid interiors. The structure of the elements of the decompositions that have void interiors is very nice and is described by means of the aposyndetic set function *T*. In the case where elements exist with nonvoid interiors, the structure can be very complicated and a final result deals with this structure under some rather stringent conditions.

For a compact, metric continuum M that is irreducible about two points Thomas proved [7, p. 15] that M has a decomposition $\mathscr{D}$ such that

(1) $\mathscr{D}$ is upper semi-continuous,

(2) the elements of $\mathscr{D}$ are continua,

(3) each element of $\mathscr{D}$ has void interior, and

(4) the quotient space of $\mathscr{D}$ is an arc,

if and only if M contains no indecomposable subcontinuum with nonvoid interior. Gordh [3, p. 650] generalized this for a compact, Hausdorff continuum where the quotient space in his result is a generalized arc (a continuum in which every point except for two is a separating point). Theorem 1 in this paper generalizes the above decomposition theorem to a compact, Hausdorff continuum that is irreducible about a finite set of points where condition (4) is now that the quotient space is a tree (a locally connected, hereditarily unicoherent continuum). This result strengthens somewhat a theorem due to Russell [6, p. 260] which proves that a sufficient condition in order for a metric continuum M to have such a decomposition is that M be hereditarily decomposable.

If M is a compact, Hausdorff continuum that is irreducible about a finite set of points then in Theorem 2 necessary and sufficient conditions are given so that M has a nontrivial decomposition $\mathscr{D}$ for which

(1) $\mathscr{D}$ is upper semi-continuous,

(2) the elements of $\mathscr{D}$ are continua, and

(3) the quotient space is a tree.

These results generalize and strengthen a theorem of the author's [**8**] for a continuum that is irreducible about two points.

The difference in Theorems 1 and 2 is that the elements of the decomposition must have void interiors in Theorem 1. The elements in this decomposition have a very nice structure indeed and, in fact, the decomposition is unique. The decomposition in Theorem 2 is not unique with respect to properties (1), (2), (3), but is minimal in the sense of refinement and enjoying these properties. In this decomposition the element structure is quite complicated. A final theorem of this paper allows M to contain certain combinations of indecomposable subcontinua with nonvoid interiors and a decomposition results that is very similar to that of Theorem 1.

Some terminology and a few notions are necessary in order to prove the first theorem. Let M be a compact, Hausdorff continuum and A a subset of M. Then $T(A)$ is the set A together with all points $x \in M$ for which there does not exist an open set U and continuum H such that $x \in U \subset H \subset M - A$. If $n = 1$ let $T^n(A) = T(A)$ and if $n \geq 2$ let $T^n(A) = T(T^{n-1}(A))$. If $T(A) = A$ let us say that A is *T-closed* and if $T(x) = \{x\}$ for each $x \in M$ we say that M is *semi-locally connected.* The set function T is the aposyndetic set function first defined by Jones [**5**]. For more information on T and T^n see [**1**]. If M is irreducible about the n points $x_1, x_2, \ldots, x_n$ but is not irreducible about any fewer number of points, then M is *minimally irreducible* about $x_1, x_2, \ldots, x_n$ and each x_i, $1 \leq i \leq n$, is an *endpoint* of M. A decomposition $\mathscr{D}$ of M that has a specific set of properties is a *core* decomposition with respect to these properties if $\mathscr{D}$ refines every other decomposition of M having these properties. FitzGerald and Swingle have proved that if M is a compact, Hausdorff continuum, then M has a core decomposition with respect to the properties of being monotone and having a semi-locally connected quotient space [**2**, p. 37]. We will use this result.

THEOREM 1. *Let M be a compact, Hausdorff continuum that is minimally irreducible about n points, $n \geq 2$. If M contains no indecomposable subcontinuum with nonvoid interior then there exists a decomposition $\mathscr{D}$ of M such that*

(1) *$\mathscr{D}$ is upper semi-continuous,*

(2) *the elements of* $\mathscr{D}$ *are continua,*
(3) *the quotient space of* $\mathscr{D}$ *is locally connected, and*
(4) *each element of* $\mathscr{D}$ *has void interior.*

Furthermore, the quotient space is a tree minimally irreducible about n points and $\mathscr{D}$ *is the only decomposition satisfying* (1), (2), (3), (4). *Conversely, if* $\mathscr{D}$ *is a decomposition of M satisfying* (1), (2), (3), (4) *then M contains no indecomposable subcontinuum with nonvoid interior.*

Proof. Let each indecomposable subcontinuum of M have void interior. We will first show that $T^n(x) = T^{n+1}(x)$ for all $x \in M$ and that $\{T^n(x) \mid x \in M\}$ is a decomposition of M. Then by [2, p. 39], $\{T^n(x), x \in M\}$ will be the core decomposition of M with respect to being monotone, upper semi-continuous and having a semi-locally connected quotient space. Given $x \in M$ clearly $T^n(x) \subset T^{n+1}(x)$. To show $T^{n+1}(x) \subset T^n(x)$ suppose $y \notin T^n(x)$. There exists a continuum H_1 such that $y \in H_1^o \subset H_1 \subset M - T^{n-1}(x)$. The components of $M - H_1$ are open sets since there can be only a finite number due to the fact that M is irreducible about a finite set. So H_1 can be chosen so that $M - H_1$ is connected. Let $K_1 = \overline{M - H_1}$. For each $z \in H_1$ there is a continuum H_z such that $z \in H_z^o \subset H_z \subset M - T^{n-2}(x)$ and hence, by the compactness of the continuum H_1, there exists a continuum H_2 such that $H_1 \subset H_2^o \subset H_2 \subset M - T^{n-2}(x)$. Again we may assume that $M - H_2$ is connected and we set $K_2 = \overline{M - H_2}$. Continuing this process we obtain n continua $H_1, H_2, \dots, H_n$ for which the following sequence of inclusions holds:

$$y \in H_1^o \subset H_1 \subset H_2^o \subset H_2 \subset \cdots \subset H_n^o \subset H_n \subset M - \{x\}.$$

Let $K_i = \overline{M - H_i}$, $i = 1, \dots, n$, where $M - H_i$ is connected. Consider the pairwise disjoint open sets

$$H_1^o, H_2^o \cap K_1^o, H_3^o \cap K_2^o, \dots, H^o_{nm-m+1} \cap K^o_{nm-m}, K^o_{nm-m+1}.$$

Since K_1 and H_n are continua and M is irreducible about $x_1, x_2, \dots, x_n$, H_1^o and K_n^o must each contain an end point of M. Because M has but n end points, one of the above $n+1$ open sets, say $H_i^o \cap K_{i-1}^o$, contains no end point of M. Let L be an irreducible continuum from H_{i-1} to K_i. Then $H_i^o \cap K_{i-1}^o \subset L$ or else $H_{i-1} \cup L \cup K_i$ is a proper subcontinuum of M containing all of the end points. Therefore $L^o \neq \Phi$ and $L = L_1 \cup L_2$ where L_1 and L_2 are proper subcontinua of L. Due to L being irreducible we have $L_1 \cap H_{i-1} \neq \Phi \neq L_2 \cap K_i$ and $L_1 \cap K_i = \Phi = L_2 \cap H_{i-1}$. We now have the inclusions $T(x) \subset K_n,\ T^2(x) \subset K_{n-1}, \dots, T^{n-i+1}(x) \subset K_i,$

$$T^{n-i+2}(x) \subset K_i \cup L_2,\ T^{n-i+3}(x) \subset K_{i-1}, \ldots,$$
$$T^n(x) \subset K_2,\ T^{n+1}(x) \subset K_1.$$

Therefore $y \notin T^{n+1}(x)$ and it follows that $T^n(x) = T^{n+1}(x)$.

To show $\{T^n(x) \mid x \in M\}$ is a decomposition, suppose $z \in T^n(x)$. By the above argument it is clear that $x \in T^n(z)$. We then have

$$T^n(z) \subset T^n(T^n(x)) = T^{2n}(x) = T^n(x) \quad \text{and} \quad T^n(x) \subset T^{2n}(z) = T^n(z).$$

Then $T^n(x) = T^n(z)$. So if $z \in T^n(x) \cap T^n(y)$ we must have $T^n(x) = T^n(z) = T^n(y)$ and, consequently, $\{T^n(x) \mid x \in M\}$ is a decomposition. As mentioned, the decomposition is monotone, upper semi-continuous and the quotient space M' is semi-locally connected. Since M is minimally irreducible about n points so is M', and it follows easily from this and the semi-local connectedness that M' is locally connected.

The continuum M' is a tree if and only if given any two distinct points x and y of M' there is a point $z \in M'$ such that z separates x from y [9]. Take $x, y \in M'$ and let H be an irreducible subcontinuum of M' from x to y. Choose $z \in H$ such that $z \notin \{x, y, f(x_1), f(x_2), \ldots, f(x_n)\}$ where f is the quotient map from M onto M'. The components of $M' - \{z\}$ are open sets since M' is locally connected. Suppose x and y are in the same component C of $M' - \{z\}$. Let U be an open set such that $z \in U$, $x, y \notin U$, and $f(x_i) \notin U$, $i = 1, \ldots, n$. By the local connectedness of M' and because z does not separate $\bar{C} = C \cup \{\}$, there exists a continuum K such that $C - U \subset K^o \subset K \subset C$. The open set $C - K \subset H$ for otherwise $(M' - C) \cup K \cup H$ is a proper subcontinuum of M' containing $f(x_i)$, $i = 1, \ldots, n$. Choose $w \in C - K$. If w separates z from K in $\bar{C}$ then w separates $\{x, y\}$ from z in H which means that H is not irreducible from x to y. So w does not separate z from K in $\bar{C}$ and now, because M' is locally connected, there exists a subcontinuum L of $\bar{C} - \{w\}$ such that $z \in L$ and $L \cap K \neq \Phi$. But $(M' - C) \cup L \cup K$ is a proper subcontinuum of M' containing $f(x_i)$, $i = 1, \ldots, n$, a contradiction. So the assumption that x and y are in the same component of $M' - \{z\}$ is contradictory. Hence x and y are in different components of $M' - \{z\}$ which means that z separates x from y in M'.

To show (4) suppose for some $x \in M$ that $T^n(x)$ has nonempty interior. First suppose $T^n(x)$ contains none of the end points. Let $K_1, K_2, \ldots, K_m$ be the closures of the finite number of components of $M - T^n(x)$ and let Q be a subcontinuum of $T^n(x)$ irreducible about the closure of $(T^n(x))^\circ$. Now Q is irreducible about a finite set of points, is not indecomposable and contains no indecomposable subcontinuum with nonvoid interior. Therefore there exist two points x', y' such that $x' \in K_i \cap Q, y' \in K_j \cap Q$ for some pair of integers i, j; $1 \leq i, j \leq m$; and for which the following inclusions hold:

$$y' \in Q_1^o \subset Q_1 \subset Q_2^o \subset Q_2 \subset \cdots \subset Q_n^o \subset Q_n \subset M - \{x'\}$$

where each Q_i is a continuum. But then $y' \notin T^n(x') = T^n(x)$, a contradiction. If $T^n(x)$ contains one of the end points x_i from $\{x_1, x_2, \dots, x_n\}$, then let L be a subcontinuum of M irreducible about $\{x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$ and let L' be an irreducible subcontinuum of M from x_i to L. Since L' contains no indecomposable subcontinuum with nonvoid interior it has a monotone decomposition $\mathscr{D}'$, the elements of which have void interiors, and for which the quotient space is a generalized arc [3, p. 650]. Let D_{x_i} be the element of $\mathscr{D}'$ that contains x_i. Clearly $T^n(x) \subset D_{x_i}$ since $x_i \in T^n(x)$ and this is a contradiction to $(T^n(x))^\circ \neq \Phi$.

To complete the proof of sufficiency we need to show that $\{T^n(x) \mid x \in M\}$ is the only decomposition satisfying (1), (2), (3), (4). Let $\mathscr{H}$ be any other decomposition satisfying these properties. Since $\{T^n(x) \mid x \in M\}$ refines $\mathscr{H}$, there exists $h \in \mathscr{H}$ and $T^n(x)$ for some $x \in M$ such that $T^n(x) \quad h$. Let $y \in h - T^n(x)$. Borrowing the construction from the first part of the proof we have the continua $H_1, H_2, \dots, H_n$; $K_1, K_2, \dots, K_n$ where $y \in H_1$ and $x \in K_n$. For some i, $H_i^o \cap K_{i-1}^o \subset L$ where L is the irreducible subcontinuum of M from H_{i-1} to K_i. Clearly L separates M into the open sets H_{i-1}^o and K_i^o with $y \in H_{i-1}^o$ and $x \in K_i^o$. But h contains both x and y so h must contain L. Because L contains an open set and h cannot contain such a set we have arrived at a contradiction.

Suppose $\mathscr{D}$ is any decomposition of M satisfying (1), (2), (3), (4) and let I be an indecomposable subcontinuum of M such that $I^\circ \neq \Phi$. First suppose I contains an end point x_i. Let L be a subcontinuum of M irreducible about $\{x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$ and let L' be an irreducible subcontinuum of M from x_i to L. If D_{x_i} is the element of $\mathscr{D}$ containing x_i then it is clear that $D_{x_i} \cap L = \Phi$. Let us denote by f the quotient map of M onto the locally connected quotient space M' of the decomposition $\mathscr{D}$. Because of the local connectedness of $M', f(L') \cap f(L)$ is a single element $x \in M'$ and $f(L')$ is a generalized arc. So L' contains no indecomposable subcontinuum with nonvoid interior. Thus $L' \subsetneqq I$. But it is clear then that for some element $D \in \mathscr{D}$ that lies in L', the continuum I is separated by D, a contradiction.

Next suppose I contains no end point. Let $K_1, K_2, \dots, K_m$ be the closures of the components of $M - I$. Consider K_1. There do not exist $D_2, D_3, \dots, D_m \in \mathscr{D}$ each intersecting K_1 such that $D_i \cap K_i \neq \Phi, i = 2, \dots, m$;

for otherwise $\cup_{i=1}^{m} K_i \cup \cup_{i=2}^{m} D_i$ would be a proper subcontinuum of M containing all the end points. A similar statement is true for each of the m K_i's. Using this fact plus the local connectedness of the quotient space M' it follows easily that there exists an element $D \in \mathscr{D}$ such that $D \subset I°$. Because $\mathscr{D}$ is upper semi-continuous there exists an open set U such that $D \subset U \subset I°$ and U is the union of members of $\mathscr{D}$. Now $f(D) \in f(U)$ and since M' is locally connected there is a connected open set $W \in M'$ such that $f(D) \in W \subset \overline{W} \subset f(U)$. Then $D \subset f^{-1}(W) \subset f^{-1}(\overline{W}) \subset U$ which implies that $f^{-1}(\overline{W})$ is a proper subcontinuum of the indecomposable continuum I with a nonvoid interior. This is impossible and the proof is complete.

Next we need to borrow a definition from Whyburn. A collection $\mathscr{G}$ of subsets (not necessarily disjoint) of M is a *saturated* collection if whenever $G \in \mathscr{G}$ and $p \notin G$ there exists $G' \in \mathscr{G}$ such that G' separates p from G in M, i.e., $M - G' = A \cup B$ where $p \in A$, $G \subset B$, and $\bar{A} \cap B = \Phi = A \cap \bar{B}$ [**10**, p. 45]. Also we call a subset G of M a *separator* if $M - G$ is not connected.

THEOREM 2. *Let M be a compact, Hausdorff continuum that is minimally irreducible about the n points $\{x_1, x_2, \ldots, x_n\}$. Suppose there exists a nonvoid saturated collection of separators each element of which is a continuum. Then M has a nontrivial decomposition $\mathscr{D}$ such that*

(1) *$\mathscr{D}$ is upper semi-continuous*

(2) *the elements of $\mathscr{D}$ are continua*

(3) *the quotient space of $\mathscr{D}$ is locally connected.*

Moreover, $\mathscr{D}$ is the core decomposition with respect to these three properties and the quotient space is a tree minimally irreducible about m points, $m \leq n$. Conversely, if $\mathscr{D}$ is a nontrivial decomposition satisfying (1), (2), (3), *there exists a nonvoid saturated collection of separators each element of which is a continuum.*

Proof. Consider the union of all saturated collections of separators where the separators are continua and let us denote this collection by $\mathscr{G}$. Clearly $\mathscr{G}$ is itself a saturated collection of separators each element of which is a continuum and is the unique maximal such collection. Let S_x be the set of all points y such that there does not exist $G \in \mathscr{G}$ which separates x from y. Now take $G \in \mathscr{G}$ and let $M - G = A \cup B$, a separation of M. Choose $a \in A$, $b \in B$, $c \in G$. There exists $G' \in \mathscr{G}$ such that G' separates a from G and clearly $G' \subset A$. So G' separates a from c in M and hence by [**2**, p. 49] $\mathscr{S} = \{S_x \mid x \in M\}$ is an upper semi-continuous decomposition of M into closed sets. Now let $\mathscr{C}$ denote the core decomposition with respect to the properties of being monotone and having a semi-locally connected quotient space $(M,\mathscr{C})$ and let f be the quotient map from M onto $(M,\mathscr{C})$. Since $(M,\mathscr{C})$ is semi-locally connected and clearly irreducible about m points, $m \leq n$, it follows that $(M,\mathscr{C})$ is locally connected. As in the proof of Theorem 1, $(M,\mathscr{C})$ can be shown to be a tree. So what remains to be shown is that $\mathscr{C} = \mathscr{S}$. Let $K = \{f^{-1}(k) \mid k$ is a point of $(M,\mathscr{C})$ and k separates $(M, \mathscr{C})\}$. Since $(M,\mathscr{C})$ is a tree obviously $K \neq \Phi$ so let $f^{-1}(k) \in K$, $x \in M - f^{-1}(k)$ and consider $f(x)$, $k \in (M,\mathscr{C})$. Due to the fact that $(M,\mathscr{C})$ is hereditarily unicoherent there exists a unique subcontinuum of $(M,\mathscr{C})$, H, irreducible from $f(x)$ to k. Choose $k' \in H - \{f(x), k\}$. Because $(M,\mathscr{C})$ is locally connected it follows that k' separates $f(x)$ from k in $(M,\mathscr{C})$. Then $f^{-1}(k')$ separates x from $f^{-1}(k)$ in M and K is a saturated collection of separators of M. By the monotonicity of f each element of K is a continuum. The elements of $\mathscr{C}$ are of the form $f^{-1}(k)$ where $k \in (M,\mathscr{C})$ so take $f^{-1}(k) \in \mathscr{C}$ and $x \in M - f^{-1}(k)$. Exactly as above there exists $k' \in (M,\mathscr{C})$ such that $f^{-1}(k')$ separates x from $f^{-1}(k)$. Since $f^{-1}(k') \in K$ then $f^{-1}(k') \in \mathscr{G}$ (because $\mathscr{G}$ is maximal) and it follows that $\mathscr{S} \leq \mathscr{C}$.

To prove that $\mathscr{C} \leq \mathscr{S}$ let us note that FitzGerald and Swingle have proved that $\mathscr{C}$ can alternately be expressed as the core decomposition with respect to the properties of being upper semi-continuous and having T-closed elements [2, p. 37]. We have already established that $\mathscr{S}$ is upper semi-continuous and if we show that the elements of $\mathscr{S}$ are T-closed it will follow that $\mathscr{C} \leq \mathscr{S}$. For this purpose take $S_x \in \mathscr{S}$ and $y \in M - S_x$. There exists $G \in \mathscr{G}$ such that $M - G = A \cup B$, a separation, with $x \in A$ and $y \in B$. Because $\mathscr{G}$ is a saturated collection it follows easily that G can be chosen so that $S_x \subset A$ and $y \in B$. But then $G \cup B$ is a continuum containing y in its interior and not intersecting S_x. Hence $y \notin T(S_x)$ so S_x is T-closed and $\mathscr{S}$ is the core decomposition $\mathscr{D}$ of the theorem.

Conversely, suppose that $\mathscr{D}$ is a nontrivial decomposition satisfying conditions (1), (2), (3) in Theorem 2. Let the quotient space of $\mathscr{D}$ be denoted by $(M,\mathscr{D})$ and let f be the quotient map of M onto $(M,\mathscr{D})$. We know from the proof of Theorem 1 that $(M,\mathscr{D})$ is a tree. Let $K = \{f^{-1}(k) \mid k \in (M,\mathscr{D})$ and k separates $(M,\mathscr{D})\}$ and let $f^{-1}(k) \in K$, $y \in M - f^{-1}(k)$. Because $(M,\mathscr{D})$ is hereditarily unicoherent there exists a unique continuum H irreducible between $f(y)$ and k. As previously proved there exists $k' \in H - \{f(y), k\}$ such that k' separates $f(y)$ from k in $(M,\mathscr{D})$. Then $f^{-1}(k')$ separates y from $f^{-1}(k)$ in M. So K is a nonvoid saturated collection of separators of M and each separator is a continuum since $\mathscr{D}$ is monotone.

The next theorem is a generalization of Theorem 1 where M is allowed to contain simple chains of indecomposable subcontinua with nonvoid interiors. A collection of subsets of M, $\{C_i \mid i = 1, \dots, m\}$, is a *simple chain* if

$C_i \cap C_j \neq \Phi$ if and only if $|i - j| \leq 1$. Each subset C_i is called a *link* of the chain.

THEOREM 3. *Let M be a compact, Hausdorff continuum minimally irreducible about n points, $n \geq 2$. If M contains no simple chain of m links, $m \geq 1$, in which each link is an indecomposable continuum with nonvoid interior then $\mathscr{D} = \{T^{nm-m+1}(x) \mid x \in M\}$ is the core decomposition of M with respect to being monotone, upper semi-continuous with a locally connected quotient space. Moreover, the quotient space is a tree minimally irreducible about p points where $p \leq n$.*

Proof. It is only necessary to show that $T^{nm-m+2}(x) = T^{nm-m+1}(x)$ for each $x \in M$ and that $\{T^{nm-m+1}(x) \mid x \in M\}$ is a decomposition of M. The conclusion will then follow by exactly the same reasoning as in Theorem 1. Given $x \in M$ it is clear that $T^{nm-m+1}(x) \subset T^{nm-m+2}(x)$. Suppose $y \notin T^{nm-m+1}(x)$. As in the construction of Theorem 1, a sequence of continua $H_1, \ldots, H_{nm-m+1}$ can be constructed such that

$$y \in H_1^o \subset H_1 \subset H_2^o \subset H_2 \subset \cdots \subset H_{nm-m+1}^o \subset H_{nm-m+1} \subset M - \{x\}.$$

As before let K_i be the continuum $\overline{M - H_i}$, $i = 1, \ldots, nm - m + 1$, and consider the pairwise disjoint open sets

$$H_1^o,\ H_2^o \cap K_1^o,\ H_3^o \cap K_2^o, \ldots, H_n^o \cap K_{n-1}^o,\ K_n^o.$$

Let L_i be an irreducible subcontinuum of M from H_i to K_{i+1}, $i = 1, \ldots, nm - m$. Because M is irreducible about n points H_1^o and K_{nm-m+1}^o must each contain an end point and at least $nm - m - (n - 2)$ of the L_i's do not contain an end point. It follows that each of these L_i's must contain $H_{i+1}^o \cap K_i^o$ because of the irreducibility of M. Also because M contains no simple chain of m links where each link is an indecomposable continuum with nonvoid interior, at most $(m - 1)(n - 1)$ of the L_i's that do not contain an end point are indecomposable. Therefore at least $mn - m - (n - 2) - (m - 1)(n - 1) = 1$ of the L_i's does not contain an end point and is decomposable. For one such L_i let $L_i = L_a \cup L_b$ where L_a and L_b are proper subcontinua of L_i. Since L_i is irreducible from H_i to K_{i+1} we have $L_a \cap H_i \neq \Phi \neq L_b \cap K_{i+1}$ and $L_a \cap K_{i+1} = \Phi = L_b \cap H_i$. As in the proof of Theorem 1 this leads immediately to the conclusion that $T^{nm-m+2}(x) \subset K_1$. Hence $y \notin T^{nm-m+2}(x)$ and it follows that $T^{nm-m+1}(x) = T^{nm-m+2}(x)$.

To show that $\{T^{nm-m+1}(x) \mid x \in M\}$ is a decomposition of M, the reasoning in Theorem 1 may be employed with T^n replaced by T^{nm-m+1}.

In Theorems 1 and 3 it is established that the structure of the elements in the decompositions are precisely $T^n(x)$ and $T^{nm-m+1}(x)$, respectively. It is not hard to show by means of simple examples that the exponents of T cannot be reduced. Also it can be shown easily that the elements D of the decomposition $\mathscr{D}$ in Theorem 2 that have void interiors are of the form $T^n(x)$ for each $x \in D$. But the structure of the elements with nonvoid interiors is considerably more complicated.

It might be pointed out too that, while Theorem 1 assures a decomposition space which preserves the number of "ends," the decomposition spaces of Theorems 2 and 3 may greatly reduce this number; in the case of Theorem 3, the decomposition space may be degenerate.

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