PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 2, 1975

SEMIPRIME RINGS WITH THE SINGULAR SPLITTING PROPERTY

MARK L. TEPLY

A (right nonsingular) ring R is called a splitting ring if, for every right R-module M, the singular submodule $Z(M)$ is a direct summand of M. If R is a semiprime splitting ring with zero right socle, then R contains no infinite direct sum of two-sided ideals. As applications of this result, the center of a semiprime splitting ring with zero socle is analyzed, and the study of splitting ring is completely reduced to the case where R is a prime ring. The center of a semiprime splitting ring is a von Neumann regular ring.

1. Introduction. In this paper R denotes an associative ring with identity element. Unless otherwise noted, all modules will be unital right modules.

Considerable work has recently been done on the problem of characterizing the rings for which the singular submodule $Z(M)$ of every module M is a direct summand. Such rings will be called *splitting rings* in this paper. Every splitting ring is a right nonsingular ring. Rotman [6] showed that a commutative integral domain is a splitting ring if and only if it is a field. Cateforis and Sandomierski [1] characterized the commutative splitting rings as the von Neumann regular rings R with the property that, for every essential ideal I of R, R/I is a direct sum of fields. In a series of papers [2, 3, 4], Goodearl (a) reduced the study of splitting rings to the study of rings with essential right socle or zero right socle, (b) characterized the splitting rings with essential right socle, and (c) showed via a triangular matrix ring structure theorem that, in order to complete the characterization of splitting rings, it is sufficient to study the semiprime splitting rings with zero right socle.

In Theorem 7 of this paper, we show that a semiprime splitting ring with zero right socle is an essential product of *finitely* many prime splitting rings with zero right socle. (A ring R is an essential product of the rings $R_1, R_2, \cdots, R_n$, if R is a subdirect product of $R_1, R_2, \cdots R_n$ which contains an essential right ideal of $\prod_{i=1}^{n} R_i$.) Each prime ring used for the essential product in Theorem 7 is a homomorphic image of R which is determined in a natural way; so the product of prime rings is constructable from R. Moreover, *Theorem 7 can be used in the following way to reduce the study of splitting rings to the case where R is a prime ring with zero right socle.* By the discussion above, we only need to construct the semiprime splitting rings with zero right socle from the prime ones. By an iterative use of [3, Theorem 12], all (necessarily semiprime) splitting rings which are essential products of prime splitting rings with zero right socle can be constructed. But Theorem 7 says that every semiprime splitting ring with zero right socle is such an essential product. Thus all semiprime splitting rings with zero right socle can be constructed from the prime splitting rings with zero right socle.

In Theorem 4 and Corollaries 5 and 6, we find useful necessary conditions for semiprime and prime rings to be a splitting ring.

2. The results. The proof of the following lemma is contained in the proof of [2, Theorem 5.3].

LEMMA 1. *If R is a splitting ring and I is a two-sided ideal of R, then R/I is a right perfect ring whenever I is essential as a right ideal of R.*

THEOREM 2. *If R is a semiprime splitting ring with zero right socle, then R contains no infinite direct sum of two-sided ideals.*

Proof. Let A be an index set such that $I = \bigoplus_{\alpha \in A} I_\alpha$ is a maximal direct sum of two-sided ideals of R. If L is a nonzero right ideal of R such that $L \cap I = 0$, then RL is a nonzero two-sided ideal of R. By our choice of I, $I \cap RL \neq 0$. But $(I \cap RL)^2 \subseteq RL \cdot I \subseteq 0$, which contradicts R semiprime. Thus I is an essential right ideal of R.

If $|A|$ is infinite, partition A into a countable number of disjoint infinite sets $\{A_i\}_{i=1}^\infty$. If $Z(R/\bigoplus_{\alpha \in A_i} I_\alpha) = 0$, then choose $M_\alpha (\alpha \in A_i)$ to be an essential submodule of I_α such that $M_\alpha \neq I_\alpha$. (We can do this since R has zero right socle.) Thus

$$Z\Big(R\Big/\bigoplus_{\alpha \in A_i} M_\alpha\Big) = \Big(\bigoplus_{\alpha \in A_i} I_\alpha\Big)\Big/\Big(\bigoplus_{\alpha \in A_i} M_\alpha\Big) \cong \bigoplus_{\alpha \in A_i} I_\alpha/M_\alpha$$

is not finitely generated. Hence $Z(R/\bigoplus_{\alpha \in A_i} M_\alpha)$ cannot be a direct summand of $R/\bigoplus_{\alpha \in A_i} M_\alpha$, which contradicts the splitting hypothesis. Hence $Z(R/\bigoplus_{\alpha \in A_i} I_\alpha) \neq 0$.

By the splitting hypothesis, there exists $e_i \in R - (\bigoplus_{\alpha \in A_i} I_\alpha)$ for each $i = 1, 2, 3, \cdots$ such that

(a) $e_i^2 = e_i \pmod{\bigoplus_{\alpha \in A_i} I_\alpha}$,

(b) $e_i R + (\bigoplus_{\alpha \in A_i} I_\alpha)$ is an essential extension of $\bigoplus_{\alpha \in A_i} I_\alpha$, and

(c) $e_i R + (\bigoplus_{\alpha \in A_i} I_\alpha)$ is a two-sided ideal of R.

Hence $e_i e_j \in (e_i R + (\bigoplus_{\alpha \in A_i} I_\alpha)) \cap (e_j R + (\bigoplus_{\alpha \in A_j} I_\alpha)) = 0$ whenever $i \neq j$.

Thus, $\{e_i + I\}_{i=1}^\infty$ is a set of orthogonal idempotents in R/I.

If $e_i \in I$, then $e_i = \sum_{j=1}^n x_j$, where $x_j \in I_{\alpha_i}$ and at least one $\alpha_j \notin A_i$. By (b) there exists an essential right ideal K such that $e_i K \subseteq \bigoplus_{\alpha \in A_i} I_\alpha$;

so by properties of the direct sum $x_j K = 0$ for $\alpha_j \notin A_i$. But this contradicts $Z(R_R) = 0$; hence $e_i \notin I$ for any i. Thus $\{e_i + I\}_{i=1}^{\infty}$ forms an infinite set of distinct orthogonal idempotents in R/I, and hence R/I cannot be right perfect. This contradicts Lemma 1, and therefore A must be finite.

COROLLARY 3. *The center C of a semiprime splitting ring R with zero right socle is a semiprime Goldie ring.*

Proof. Let A be an index set, and let $\bigoplus_{\alpha \in A} C_\alpha$ be a direct sum of ideals of C. Now $C_\beta R \cap (\bigoplus_{\alpha \in A - \{\beta\}} C_\alpha R) = 0$; for

$$\Big(C_\beta R \cap \Big(\bigoplus_{\alpha \in A - \{\beta\}} C_\alpha R\Big)\Big)^2 = 0$$

and R is semiprime. Hence $\bigoplus_{\alpha \in A} C_\alpha R$ is a direct sum of two-sided ideals of R. So A must be a finite set by Theorem 2.

If $N \subseteqq C$ and $N^2 = 0$, then $(RN)^2 = 0$; so, since R is semiprime, $N = 0$. Therefore, C is semiprime and hence nonsingular. By [7, Lemma 3] C must have acc on annihilators.

THEOREM 4. *The center C of a semiprime splitting ring R is von Neumann regular.*

Proof. Let $0 \neq d \in C$, and let $K = \{r \in R \mid dr = 0\}$. Since $d \in C$, K is a two-sided ideal of R. Since $Z(R_R) = 0$, then $Z(R/K) = 0$. Hence R/K is a splitting ring (see [2, Proposition 1.11]). From the definition of K and the semiprimeness of R, it follows that $\bar{d} = d + K$ is not a zero-divisor in R/K.

Let $R' = R/K$. Define $M = \prod_{n=1}^{\infty} R'/\bar{d}^n R'$. Assume that $\bar{d}^{-1} \notin R'$, so that $\bar{d}R' \supsetneqq \bar{d}^2 R' \supsetneqq \bar{d}^3 R' \supsetneqq \cdots$. Note that $\bigoplus_{n=1}^{\infty} R'/\bar{d}^n R' \subseteqq Z(M)$. A nonzero element x of M is said to have infinite height in M if there exists $y_n \in M$ for each positive integer n such that $y_n \bar{d}^n = x$. Since $\bar{d}^n$ is an element of the center of R', $\bar{d}^n$ annihilates the first n coordinates of M; so M has no elements of infinite height. Since $M/Z(M)$ is a direct summand of M by the splitting hypothesis, then in order to get $\bar{d}^{-1} \in R'$ it is sufficient to show that $M/Z(M)$ has elements of infinite height.

Let

$$x = (1 + \bar{d}R', 1 + \bar{d}^2 R', \bar{d} + \bar{d}^3 R', \bar{d} + \bar{d}^4 R', \cdots, \bar{d}^n + \bar{d}^{2n+1} R', \\ \bar{d}^n + \bar{d}^{2n+2} R', \cdots) .$$

If $x \in Z(M)$, then there exists an essential right ideal J of R' such that $xJ = 0$. Thus $\bar{d}^n J \subseteqq \bar{d}^{2n+1} R'$ for all n. But $\bar{d}$ is not a zero-divisor in R'; so $J \subseteqq \bar{d}^{n+1}R'$ for each n. Hence $I = \bigcap_{n=1}^{\infty} \bar{d}^{n+1}R' \supseteqq J$ is a two-sided ideal of R', and I is essential as a right ideal of R'. Thus R'/I is a right perfect ring by Lemma 1. Thus R'/I has dcc on principal left ideals; but this contracts the assumption that

$$\bar{d}R' \supsetneqq \bar{d}^2R' \supsetneqq \bar{d}^3R' \supsetneqq \cdots \supsetneqq I .$$

Therefore, $x \notin Z(M)$. Let

$$y_n = (0, 0, \cdots, 1 + \bar{d}^{2n+1}R', 1 + \bar{d}^{2n+2}R', \bar{d} + \bar{d}^{2n+3}R', \bar{d} + \bar{d}^{2n+4}R', \cdots) .$$

Then

$$\begin{aligned} x + Z(M) = (1 + \bar{d}R', 1 + \bar{d}^2R', \cdots, \bar{d}^{n-1} + \bar{d}^{2n}R', 0, 0, \cdots) \\ + y_n\bar{d}^n + Z(M) = y_n\bar{d}^n + Z(M) . \end{aligned}$$

So $x + Z(M)$ has infinite height in $M/Z(M)$.

Therefore $\bar{d}^{-1} \in R' = R/K$. Then $R = dR + K$. Since R is semiprime, $dR \cap K = 0$. So $R = dR + K$ is a ring direct sum, and hence there exists $e = e^2 \in C$ such that $eR = K$. Consequently, $(d + e)^{-1} \in C$, and $d = d(d + e)^{-1}d$.

COROLLARY 5. *The center of a semiprime splitting ring with zero right socle is a direct sum of finitely many fields.*

Proof. By Corollary 3 and Theorem 4, C is a commutative, von Neumann regular, Goldie ring. Such a ring must be a direct sum of finitely many fields.

As the example in the second remark on page 161 of [1] shows, the "zero right socle" hypothesis cannot be dropped from Corollary 5. However, if R is a prime ring, then the "zero right socle" hypothesis can be dropped.

COROLLARY 6. *The center of a prime splitting ring is a field.*

Proof. Since R is prime, then each nonzero element in the center C of R is not a zero-divisor in C; so it follows from Theorem 4 that C is a field.

Any field can be the center of a prime splitting ring with essential right socle. For example, given a field F, let R be the algebra over F whose basis consists of the identity of F and the set $\{e_{ij}\}_{i,j=1}^{\infty}$, where multiplication is defined by

$$e_{hi}e_{jk} = \begin{cases} e_{hk} & \text{if } i = j \\ 0 & \text{otherwise} . \end{cases}$$

Then the center of R is F; the right socle of R is $\sum_{i,j} e_{ij}R$; the singular submodule of any R-module is injective and isomorphic to a direct sum of copies of $R/\sum_{i,j} e_{ij}R$.

If F is a field of characteristic 0 or if F has finite cardinality, then examples of prime splitting rings with zero right socle and center F are known.

THEOREM 7. *A semiprime splitting ring with zero right socle is an essential product of finitely many prime splitting rings with zero right socle.*

Proof. By Theorem 2, R has no infinite direct sum of two-sided ideals. Hence R is an essential product of finitely many prime rings by [5, Proposition 9]. Each of these prime rings is a splitting ring, as each is nonsingular and a homomorphic image of R. Moreover, since any simple submodule of one of these prime rings would also be simple submodule of R, then each of the prime rings must have zero right socle.

REFERENCES

1. V. Cateforis and F. Sandomierski, *The singular submodule splits off*, J. Algebra, **10** (1968), 149-165.

2. K. R. Goodearl, *Singular torsion and the splitting properties*, Mem. Amer. Math. Soc., **124** (1972).

3. ———, *Essential products of nonsingular rings*, Pacific J. Math., **45** (1973), 493-505.

4. ———, *Triangular representations of splitting rings*, Trans. Amer. Math. Soc., **185** (1973), 271-285.

5. D. Handelman, *Strongly semiprime rings*, to appear in Pacific J. Math.

6. J. Rotman, *A characterization of fields among integral domains*, Anales Acad. Brasil Ciênc., **32** (1960), 193-194.

7. J. Zelmanowitz, *A shorter proof of Goldie's theorem*, Canad. Bull., **12** (1969), 597-602.

Received November 18, 1974.

UNIVERSITY OF FLORIDA