PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 2, 1975

ON GROUPS WITH A SINGLE INVOLUTION

JERRY MALZAN

This paper is concerned with the "ordinary" (over the complex numbers) representation theory of finite groups and in particular with matrix groups of the first and second kinds (that is, matrix groups which are similar to real groups or, alternatively, have real character but are not similar to real groups. In the event that the character is non-real, we speak of the third kind.)

The purpose of this paper is associate groups with exactly one involution with representations of the second kind, and this we do in two ways: First, by showing that any group possessing an irreducible representation of the second kind involves a non-trivial group with only one involution. Second, by showing that a group with only one involution cannot have a faithful irreducible representation of the first kind.

It is well and long known that groups of odd order possess nontrivial irreducible representations of the third kind only, so that evenness of order is a necessity if matrix groups of the first or second kind are to be dealt with.

THEOREM 1. *If G is a finite group which admits a representation of the second kind then there is involved in G a group with exactly one involution which is neither cyclic nor a direct product. In fact G involves one of the groups mentioned at the conclusion of the proof.*

Proof. G possesses, by assumption, a representation ρ of the second kind. If ρ has as irreducible components only representations of the first or third kind then ρ would, in fact, be the first kind. Hence we may assume that G possesses an irreducible representation of the second kind. Using [1], we see that G has a subgroup H with this same property, and that furthermore $H = S_2 \cdot P$ where

(i) S_2 is a 2-group,

(ii) P is cyclic,

(iii) $(2, |P|) = 1$, and

(iv) $P \lhd H$.

It is enough to show Theorem 1 for H. To this end we first show that the irreducible representations of H are monomial. To do this, look at $C_H(P)$, the centralizer of P in H, and a representation $\gamma(C_H(P))$ of it in which P is presented faithfully. If P is trivial then H is a 2-group, and the monomial character of the representations of H is immediate. [2] Otherwise, we may assume that P is nontrivial, and that every element in H outside of $C_H(P)$ normalizes, but does not centralize, P. From this it follows directly [3] that $\gamma(C_H(P)) \uparrow H$ is irreducible. But $C_H(P)$ is a direct product of p-groups and so [2] $\gamma(C_H(P))$ is monomial. It follows that the irreducible representation $\gamma(C_H(P)) \uparrow H$ is monomial. Further, it is easy to see that each faithful irreducible representation of H arises in this manner. Finally, we note that since homomorphic images of H still have properties (i)–(iv), nothing is lost in assuming that the representation of H of the second kind in question is faithful. Hence we may assume that we have a faithful irreducible representation $\rho(H)$ of the second kind such that

$$\rho = \gamma(K) \uparrow H, \gamma(K) \quad \text{of degree 1, with} \quad P \subset K .$$

γ is necessarily complex. Also [5],

$$\sum_H \chi^\rho(g^2) = -|H| = \frac{|H|}{|K|} \sum{}' \gamma(g^2) ,$$

where the last sum is taken over those g in H satisfying $g^2 \in K$. Hence

$$\sum{}' \gamma(g^2) = -|K|, \text{ while } \sum_K \gamma(g^2) = 0 .$$

Consider the set $S = \{g \in H | g^2 \in K\}$. Split S into subsets $g(K \cap K^g)$. This is, in fact, a partitioning of S, for if g_1 and g_2 were in the same subset, there would exist elements k_1, k_2, k_3, k_4 of K such that

$$\begin{aligned}
&g_1k_1 = k_2g_1 = g_2k_3 = k_4g_2 \text{ whence} \\
&g_1 = g_2k_5 = k_6g_2 \text{ which implies} \\
&g_1 \in g_2(K \cap K^g2) \text{ and so} \\
&g_1(K \cap K^{g_1}) = g_2(K \cap K^{g_2}) .
\end{aligned}$$

Label a complete disjoint subset of the $g(K \cap K^g)$ as $g_i(K \cap K^{g_i})$. We define new groups $M_i = \langle g_i, K \cap K^{g_i} \rangle$. g_i normalizes $K \cap K^{g_i}$, and so $|M_i| = 2|K \cap K^{g_i}|$. Also,

$$\sum_S \gamma(g^2) = \sum_{i, M_i} \gamma(g^2) - \sum_{i, K \cap K^{g_i}} \gamma(g^2) = -|K| .$$

We would like to conclude that $\sum_{M_i} \gamma(g^2)$ is negative for at least one i. This follows unless it is the case that for some i

$$\sum_{K \cap K^{g_i}} \gamma(g^2) = |K \cap K^{g_i}| .$$

But consider $\omega_i(M_i) = \gamma(K \cap K^{g_i}) \uparrow M_i$, of degree 2. Since $\gamma(K \cap K^{g_i})$ is real so too is $\omega_i(M_i)$ and

$$\sum_{M_i} \chi^{\omega_i}(g^2) = |M_i|$$

(in which case the contribution of $\sum_{K \cap K^{g_i}} \gamma(g^2)$ is cancelled) or else $\sum_{M_i} \chi^{\omega_i}(g^2) = 0$. (The case of two complex components.) In order for this to be the case it would be necessary that $\gamma(K \cap K^{g_i})$ involve only ± 1, and that g_i be represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which generates all the matrices in $\omega_i(M_i)$. In this case, however, we have $\gamma(g_i k g_i^{-1}) = \gamma(k)$ whenever $k \in K \cap K^{g_i}$, and this contradicts the irreducibility of $\gamma \uparrow H$. [4] p. 329.

It follows that one of the ω_i, of degree 2, is of the second kind, and induced. The matrix group $\omega_i(M_i)$ has at least one involution, since it is of the second kind. But each such involution has an even number of -1 eigenvalues, as an immediate consequence of [5] p. 62, and so the *only* involution in this group is $-I$. It follows that M_i, hence G, involves a subgroup which has only one involution, which is neither cyclic nor a direct product.

While this completes the proof we can go further to give generating relations for a group that must be so involved. Such a group may be taken to be one of the following:

$$\{h, g \mid g^4 = h^n = 1, ghg^{-1} = h^{-1}\} \qquad (n \text{ odd}) \quad \text{or}$$
$$\{h, g \mid h^n = 1, g^2 = h^{n/2}, ghg^{-1} = h^{-1}\} \quad (n \text{ even}) .$$

THEOREM 2. *Let G be a group with a single involution which is neither trivial or C_2. Then G does not possess a faithful irreducible representation of the first kind.*

Proof. We begin by noting that every subgroup of G is, again, a group possessing no more than one involution so, since groups of odd order do not possess nontrivial irreducible representations of the first kind, we may proceed by induction on the subgroups of G. We note also that a Sylow 2-subgroup of G must [2] be cyclic or generalized quaternion.

Suppose first that G has a normal subgroup N of prime index p. Suppose, by way of contradiction, that G has an irreducible representation ρ of the first kind (which we also assume to be faithful), and consider the restriction $\rho \downarrow N$. By induction, none of the irreducible components of this restriction is the first kind. If $\rho \downarrow N$ is irreducible, then it is also real, a contradiction. Hence $\rho \downarrow N$ is reducible, with p irreducible components, no two of them similar. This last removes the possibility that the components of $\rho \downarrow N$ are of the second kind since $\rho \downarrow N$ is, like ρ, real. We see that these components must be of the third kind, and appear in complex conjugate pairs. It follows immediately that $p = 2$.

Let the irreducible components of $\rho \downarrow N$ be σ and $\bar{\sigma}$. We have

$$\sum_G \chi^\rho(g^2) = |G| \text{ and } \sum_N \chi^\sigma(g^2) = 0$$

and so

$$\sum_{G-N} \chi^\rho(g^2) = |G| .$$

Each element of $G - N$ is of the form hk, where h is a 2-element, k has odd order, and h and k commute. For each 2-element h in $G - N$ we define $C_N(h)$ to be the centralizer in N of h. Because a Sylow 2-subgroup of G is cyclic or generalized quaternion, $C_N(h)$ has cyclic Sylow 2-subgroup generated by h^2. Hence [2] the elements of odd order in $C_N(h)$ form a normal subgroup, Q. By definition of $C_N(h)$ we then have

$$C_N(h) = \langle h^2 \rangle \times Q .$$

We claim that the cosets $hC_N(h)$ partition $C - N$. That every element of $G - N$ arises in this fashion is obvious from the remarks above. If we suppose that $h_1C_N(h_1) \cap h_2C_N(h_2) \neq \emptyset$, then there are elements k_1 and k_2 of odd order in $C_N(h_1)$ and $C_N(h_2)$ respectively such that $h_1^ik_1 = h_2^jk_2$, with i and j odd. Suppose that that 2^q is the order of h_1. Then

$$(h_1^ik_1)^{2q} = k_1^{2q} = h_2^{j2q}k_2^{2q}$$

which implies that $h_2^{j2q} = 1$, and that $k_1 = k_2$. Hence $h_1^i = h_2^j$, and $h_1C_N(h_1) = h_2C_N(h_2)$.

It is enough, then, to show that for each h,

$$\sum_{hC_N(n)} \chi^\rho(g^2) \leqq 0 .$$

Let k be a fixed element, of odd order, of $C_N(h)$ and consider the sum

$$s_k = \sum_i \chi^\rho((h^ik)^2)$$

which is taken over $i = 1, 3, 5, \cdots, |h| - 1$. If $M(h)$ is the matrix for h in ρ, then

$$s_k \doteq \sum_i \chi(M(h^i)M(k))^2 = \chi(M^2(k) \sum_i (M^i(h))) .$$

The single central involution in G must be represented by the matrix $-I$ and so all the eigenvalues of $M(g)$ are primitive $|h|^{\text{th}}$

roots of unity. It follows that

$$\sum_i (M^i(h))^2 = 0 \quad \text{if} \quad |h| > 4 \quad \text{and}$$
$$\sum_{i,3} (M^i(h))^2 = -I \quad \text{if} \quad |h| = 4 \,.$$

In the first case we have

$$\sum_Q s_k = \sum_{hC_N(h)} \chi^\rho(g^2) = 0 \,.$$

In the second case we have

$$\sum_Q s_k = \sum_{hC_N(h)} \chi^\rho(g^2) = -\sum_{C_N(h)} \chi_\rho(g^2) \leqq 0 \,,$$

since $C_N(h)$ is a direct product of a group of odd order and a cyclic 2-group, so that $\rho \downarrow C_N(h)$ has no irreducible components of the second kind.

We conclude that $\sum_G \chi^\rho(g^2) \leqq 0$, and the case in which G admits a normal subgroup of prime index is disposed of.

Suppose now that a maximal proper normal subgroup N of G is larger than $Z(G)$. G/N is a noncyclic simple group having either dihedral or cyclic Sylow 2-subgroup. It could not be cyclic [2], and so N has cyclic Sylow 2-subgroup. It follows that N is a semi-direct product PQ, where Q contains all the elements of odd order in N, and is characteristic in N. Hence $Q \lhd G$. If $Q \neq 1$, we consider $\rho \downarrow Q$. The irreducible components of this representation occur with equal multiplicity and are permuted transistively among themselves by the action under conjugation of G. Further, since ρ is faithful, and is odd, these components are all of the third kind. Let σ be one of these components, and let H_σ be the subgroup of G which stabilizes σ, and let β be the irreducible representation of H_σ associated with σ in $\rho \downarrow H_\sigma$. In order that $\sum_G \chi^\rho(g^2)$ be positive it is necessary and sufficient (since $\rho = \beta \uparrow G$) that

$$\sum_S \chi^\beta(g^2) > 0$$

where $S = \{g \in G \mid g^2 \in H_\sigma\}$. As in the previous theorem, $S - H$ is partitioned by cosets $g_i(H_\sigma \cap H_\sigma^{g_i^2})$, and it is necessary that one of the representations in $\rho \downarrow \langle g_i, H_\sigma \cap H_\sigma^{g}i\rangle$ be of the first kind. This, however, is impossible since $\langle g_i, H_\sigma \cap H_\sigma^{g}i\rangle$ is a proper subgroup of G (otherwise, G would admit a subgroup of index 2) and all the irreducible components of $\rho \downarrow \langle g_i, H_\sigma \cap H_\sigma^{g}i\rangle$ take $Z(G)$ to $-I$, so that the homomorphic image of $\langle g_i, H_\sigma \cap H_\sigma^{g}i\rangle$ in any of these components is still a group with exactly one involution which is neither trivial nor C_2. We conclude that $Q = 1$.

The possibility remains that N is a cyclic 2-group greater than

$Z(G)$. But in this case, an element h of N of order 4 cannot be centralized by G for otherwise, by Schur's lemma, ρ could not possibly be real. Hence h is centralized by half the elements of G, and inverted by the rest. But now G has a subgroup of index 2, a possibility which has already been seen to.

We conclude that $G/Z(G)$ is simple, with dihedral Sylow 2-subgroup. Such simple groups have been classified [6] and are one of
(i) PSL $(2, q)$, q odd, $q > 3$,
(ii) A_7.

A_7 has no central extension of degree 2 [8] and so need not concern us here. The obvious central extension of degree 2 of PSL $(2, q)$ is to SL $(2, q)$, the group of 2×2 matrices wich determinant 1, and entries from the GF (q). But [7] this is the only such extension of PSL $(2, q)$. It is enough to show that none of the faithful irreducible representations of SL $(2, q)$ is real. To do this we first exhibit the classes, C, of SL $(2, q)$, showing a representative from each class, the number of such classes in each row, and the order, $|C|$, of each class, as well as the order of the restriction of each of these to the subgroup H of SL $(2, q)$ consisting of the matrices

$$\begin{pmatrix} x & 0 \\ y & x^{-1} \end{pmatrix}.$$

H has index $q + 1$ in SL $(2, q)$ which, in turn, has order $q(q^2 - 1)$.

$g \in C$	$\sharp C$	$\|C\|$	$\|C \cap H\|$
$\pm I$	2	1	1
$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$	$(q-3)/2$	$q(q+1)$	$q + q$ (splitting)
$\pm \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$	4	$(q^2-1)/2$	$(q-1)/2$
$\begin{pmatrix} \beta & 1 \\ -1 & 0 \end{pmatrix}$ $\beta \neq \alpha + \alpha^{-1}$.	$(q-1)/2$	$q(q-1)$	0

H is a semi-direct product of a cyclic group of order $q - 1$ and a group $C_p \times C_p \times \cdots \times C_p$, normal in H, where $q = p^n$, and there are n terms in this direct product. The irreducible characters of H can be directly calculated, as well as the characters of the induced representations. In these calculations, only representations of H which do not have $-I$ in the kernel will be considered since all the others yield representations of $\mathrm{PSL}(2, q)$ and are not faithful on $\mathrm{SL}(2, q)$. The table below lists representatives from the classes of G, the character of these representatives in an irreducible representation σ of H, the character of these representatives in the induced representation

$$\sigma \uparrow \mathrm{SL}(2, q)$$

and, finally, the number of such representations. In this table w will designate a generator of the multiplicative group of the $\mathrm{GF}(q)$ and z is a $(q-1)$th root of unity. λ_i and λ_i' are pth roots of unity (possibly 1) and the sums $\sum_i \lambda_i$ and $\sum_i \lambda_i'$ satisfy $\sum_i \lambda_i + \sum_i \lambda_i' = -1$. Each λ_i corresponds to a nontrivial irreducible character of $C_p \times C_p \times \cdots \times C_p$.

	$\pm I$	$\begin{pmatrix} w^s & 0 \\ 0 & w^{-s} \end{pmatrix}$	$\pm \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$	$\pm \begin{pmatrix} 1 & 0 \\ w^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \beta & 1 \\ -1 & 0 \end{pmatrix}$	$\#\sigma$
$\sigma_1(H)$	± 1	z^2	± 1	± 1	$\cdots$	$(q-1)/2$
$\sigma_2(H)$	$\pm (q-1)/2$	0	$\pm \sum_i \lambda_i$	$\pm \sum_i \lambda_i'$	$\cdots$	1
$\sigma_3(H)$	$\pm (q-1)/2$	0	$\pm \sum_i \lambda_i'$	$\pm \sum_i \lambda_i'$	$\cdots$	1
$\sigma_1 \uparrow \mathrm{SL}(2, q)$	$\pm (q+1)$	$(z^s + z^{-s})$	± 1	± 1	0	$\cdots$
$\sigma_2 \uparrow \mathrm{SL}(2, q)$	$\pm (q^2-1)/2$	0	$\pm \sum_i \lambda_i$	$\pm \sum_i \lambda_i'$	0	$\cdots$
$\sigma_3 \uparrow \mathrm{SL}(2, q)$	$\pm (q^2-1)/2$	0	$\pm \sum_i \lambda_i'$	$\pm \sum_i \lambda_i$	0	$\cdots$

Label the representations $\sigma_i \uparrow \mathrm{SL}(2, q)$ as $\rho_i (i = 1, 2, \cdots, (q+3)/2)$ and write $G = \mathrm{SL}(2, q)$. We compute

$$A_i = \frac{1}{|G|} \sum_G \chi^{\rho_i}(g^2) \quad \text{and} \quad (\rho_i, \rho_i) = \frac{1}{|G|} \sum_G |\chi^{\rho_i}(g)|^2$$

noting that if $\rho_i = \sum_j m_j^i \mu_j$, where μ_j is an irreducible representation of G, and a component of ρ_i with multiplicity m_j^i, then

$$A_i = \sum_j m_j^i c_j \quad \text{and} \quad (\rho_i, \rho_i) = \sum_j (m_j^i)^2$$

where $c_j = 1, -1$ or 0 according as m_j^i is, respectively, of the first, second, or third kind. Also, we note that since the centre of G is represented by $-I$ in each of the ρ_i, each of the μ_j is a faithful representation of G. Further, each faithful irreducible μ_j appears as a component in one of the ρ_i.

We note also that if

$$M = \begin{pmatrix} \beta & 1 \\ -1 & 1 \end{pmatrix}, \qquad \beta \neq \alpha + \alpha^{-1},$$

then M^2 has character 0 in each of the ρ_i unless $4 | q - 1$ and $\beta = 0$. This follows from the fact that M^2, as a matrix in $SL(2, q)$, has trace $\beta^2 - 2$ and, if it has nonzero character, this trace must satisfy $\beta^2 - 2 = \alpha + \alpha^{-1}$ for some α. But then $\beta^2 = \alpha^{-1}(\alpha + 1)^2$ whence either $\beta = 0$, and $4 | q - 1$, or else $\beta \neq 0$, α is a square, say $\alpha = \gamma^2$, and $\beta^2 = \gamma^{-2}(\gamma^2 + 1)^2$ and so $\beta = \pm\gamma^{-1}(\gamma^2 + 1) = \pm(\gamma + \gamma^{-1})$, a contradiction.

Using these observations one obtains, from a straightforward, although lengthy computation, the following results:

If $4 | q - 1$, then $A_1 = -1$ and $(\rho_1, \rho_1) = 1$, so that the first $(q - 1)/2$ irreducible induced representations of G in the last table are irreducible, and of the second kind. We find also that

$$A_2 = A_3 = -(q + 1)/2 , \quad (\rho_2 + \rho_3, \rho_2 + \rho_3) = 2q .$$

If $4 | q - 1$, then $A_1 = -1$ and $(\rho_1, \rho_1) = 1$ for all but one of the induced representations subsumed under ρ_1. This is induced from a real representation of degree 1 of H. For it we have $A_1 = 0$, $(\rho_1, \rho_1) = 2$, and we conclude that this representation has two irreducible components, both of which have complex character. We have also $A_2 = A_3 = -(q - 1)/2$, $(\rho_2 + \rho_3, \rho_2 + \rho_3) = 2q$.

We need concern ourselves only with the representations ρ_2, and ρ_3, for which we do not yet know enough to categorize their irreducible components by kind. Computation shows that $(\rho_1, \rho_2) = (\rho_1, \rho_3) = 1$ for each of the representations under ρ_1. Hence with the exception of the representation induced from a real character of H, ρ_2 and ρ_3 each contain each of the ρ_1 exactly once. In the exceptional case, ρ_2 and ρ_3 contain, separately, the two inequivalent complex irreducible components of ρ_1. Denote by ρ_4 and ρ_5 the characters obtained by subtracting from ρ_2 and ρ_3 respectively the various ρ_1. It is enough to deal with ρ_4 and ρ_5. For these we have:

If $4 | q - 1$, $A_4 = A_5 = -(q + 1)/2 + (q - 1)/2 = -1$ and

$$(\rho_4 + \rho_5, \rho_4 + \rho_5) = 2q - 4(q - 1)/2 = 2 .$$

Hence ρ_4 and ρ_5 are both irreducible, of the second kind and, in this case, we are done.

If $4 | q - 1$, $A_4 = A_5 = -(q - 1)/2 + (q - 3)/2 = -1$ while

$$(\rho_4 + \rho_5, \rho_4 + \rho_5) = 2q - 4(q - 3)/2 - 2 = 4 .$$

This can happen only in two ways: Either ρ_4 and ρ_5 are irreducible, and both of the second kind, or else ρ_4 and ρ_5 each have two irreducible components, no two of them similar, with at least two of the four of the second kind. But then, since $A_4 = A_5 = -1$, the remaining two components must be of the third kind, and the proof is complete.

We have, as an easy corollary, a result first proved by G. Vincent [9].

COROLLARY 1. *If G is a finite group of real matrices of degree >1, irreducible over the field of complex numbers, then one of the matrices of G, other than the identity, has a $+1$ eigenvalue.*

Proof. G must have even order and so has at least one involution. Theorem two excludes the possibility that G has only one involution. At least one of these lies outside the centre of G since, by Schur's lemma, each central involution in G is represented by $-I$. Now a noncentral involution M has eigenvalues ± 1 and, since $M \neq -I$, at least one of these is $+1$, as claimed.

We also have

COROLLARY 2. *If G is a real, irreducible group of matrices of degree greater than* 1, *then G has a dihedral subgroup.*

Proof. A dihedral subgroup is, by definition, a group generated by two distinct involutions. (Here we are admitting $C_2 \times C_2$ as dihedral.) Theorem 2 shows that G does, in fact, have two distinct involutions.

REFERENCES

1. R. Brauer, *On the algebraic structure of group rings*, J. Math. Soc. Japan, **3** (1951), 237-251.

2. W. Burnside, *Theory of groups of finite order*, Dover, 1955.

3. A. H. Clifford, *Representations induced in an invariant subgroup*, Ann. of Math., **38** (1937), 533-550.

4. C. Curtis and I. Reiner, *Representation Theory of Finite groups and Associative Algebras*, Interscience Publishers, New York, 1967.

5. W. Feit, *Characters of Finite Groups*, W. A. Benjamin Inc., 1967.

6. D. Gorenstein and J. H. Walter, *The characterization of finite groups with dihedral Sylow 2-subgroups, I, II, III*, J. Algebra, **2** (1965), 85-151, 218-270, 334-393.

7. I. Schur, *Utersuchen über die Darstellung der endlichen Gruppen durch gebrochemen linearen Substitutionen*, J. Reine. Angew. Math., **132** (1907), 85-137.

8. ———, *Über Darstellung der symmetrischen und der alternieren Gruppen durch gebrochenen linearen Substitutionen*, J. Reine. Angew. Math., **139** (1911).

9. G. Vihcent, *Les groupes lineaires finis sans points fixes*, Math. Helv., **20** (1947), 117-171.

Received July 10, 1974. The author was partially supported during the preparation of this paper by NRC Grant A-7263.

UNIVERSITY OF WATERLOO