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SECOND ORDER DIFFERENTIAL OPERATORS WITH SELF-ADJOINT EXTENSIONS

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Let $\mathscr{H}$ denote the Hilbert space of square summable analytic functions on the unit disk, and consider those formal differential operators

$$L = p_2 \frac{d^2}{dz^2} + p_1 \frac{d}{dz} + p_0$$

which give rise to symmetric operators in $\mathscr{H}$. Examples have been given where the symmetric operators associated with these formal operators have defect indices (0, 0) and (2, 2) and hence are either self-adjoint or have self-adjoint extensions in $\mathscr{H}$. In this note a class of symmetric operators with defect indices (1, 1) is given.

Let $\mathscr{A}$ denote the space of functions analytic on the unit disk and $\mathscr{H}$ the subspace of square summable functions in $\mathscr{A}$ with inner product

$$(f, g) = \iint_{|z|<1} f(z)\overline{g(z)}\, dxdy.$$

A complete orthonormal set for $\mathscr{H}$ is obtained by normalizing the powers of z. From this it follows that $\mathscr{H}$ is identical with the space of power series $\sum_{n=0}^{\infty} a_n z^n$ which satisfy

$$\sum_{n=0}^{\infty} |a_n|^2/(n+1) < \infty. \tag{1.1}$$

Let L be such that it maps polynomials into $\mathscr{H}$ and has the property $(Lz^n, z^m) = (z^n, Lz^m)$, $n, m = 0, 1, 2, \cdots$. Let $\mathscr{D}_0$ be the subspace of polynomials and set $T_0 f = Lf$ for f in $\mathscr{D}_0$. Then T_0 is symmetric and the defect indices m^+ and m^- of its closure, S, are just the number of linearly independent solutions of $Lu = iu$ and $Lu = -iu$ respectively which are in $\mathscr{H}$. See [2]. In [2] and [3] examples of such symmetric operators S with defect indices (0, 0) and (2, 2) are provided. We now give a class of operators with defect indices (1, 1).

2. Consider the operator L,

$$L = (c_1z^3 + \bar{c}_1z)\frac{d^2}{dz^2} + ((c_2 + 3c_1)z^2 + \bar{c}_2)\frac{d}{dz} + 2c_2z. \tag{2.1}$$

In **[3]** it is shown that L gives rise to symmetric T_0. Concerning the defect indices of its closure S, we have the following.

THEOREM 2.1. *Let L be the operator of* (2.1) *then S has defect indices $m^+ = m^- = 1$.*

Proof. The idea of the proof is to show that the equation $L\phi = \pm i\phi$ has precisely one power series solution $\phi(z) = \sum_{j=0}^{\infty} a_j z^j$ and that there exists a $K > 0$ and a positive integer p such that $|a_j| \leqq Kj^{-1/p}$ for j sufficiently large. Consequently the series $\sum_{j=0}^{\infty} |a_j|^2/(j+1)$ converges and ϕ belongs to $\mathcal{H}$, and $m^+ = m^- = 1$.

Dividing $L\phi = \pm i\phi$ by c_1 we have the differential equation

$$(z^3 + \omega z)\phi'' + [(3+\alpha)z^2 + \beta]\phi' + 2\alpha z\phi = \lambda\phi, \tag{2.2}$$

where $\omega = \bar{c}_1/c_1$, $\alpha = c_2/c_1$, $\beta = \bar{c}_2/c_1$, and $\lambda = \pm i/c_1$.

Substituting $\sum_{j=0}^{\infty} a_j z^j$ into (2.2) we obtain

$$\beta a_1 + \sum_{j=1}^{\infty} [(j+1)(\omega j + \beta)a_{j+1} + (j^2 + j\alpha + \alpha - 1)a_{j-1}]z^j \tag{2.3}$$

$$= \lambda a_0 + \sum_{j=1}^{\infty} \lambda a_j z^j \qquad \lambda \neq 0.$$

If $\beta = 0$ we have $a_0 = 0$ and (2.3) can be solved recursively for $a_2, a_3, \cdots$, in terms of a_1 since $\omega j + \beta$ never vanishes. Thus we have but one analytic solution

$$\phi(z) = z(1 + a_2z^2 + \cdots).$$

If $\beta \neq 0$, we have $a_1 = \lambda a_0/\beta$ and (2.3) can be solved recursively for a_2, a_3, etc., provided that $(\omega j + \beta)$ never vanishes for $j = 1, 2, \cdots$. Thus we are able to obtain the single formal power series solution $\phi(z) = 1 + a_1z + a_2z^2 + \cdots$. The case when $(\omega j + \beta)$ vanishes for some positive integer j presents some complications and will be considered later in the proof. Solving (2.3) for a_{j+1} we have

$$a_{j+1} = \frac{1}{\omega}\left\{\frac{-[j^2 + j\alpha + (\alpha - 1)]a_{j-1} + \lambda a_j}{j^2 + \left(1 + \frac{\beta}{\omega}\right)j + \frac{\beta}{\omega}}\right\}. \tag{2.4}$$

But $\beta/\omega = \bar{c}_2/\bar{c}_1 = \bar{\alpha}$, hence (2.4) becomes

$$a_{j+1} = \frac{1}{\omega}\left\{\frac{-[j^2 + j\alpha + (\alpha - 1)]a_{j-1} + \lambda a_j}{j^2 + (1+\bar{\alpha})j + \bar{\alpha}}\right\}. \tag{2.4}$$

Thus we obtain the estimate

$$|a_{j+1}| \leqq \frac{1}{|\omega|}\left|\frac{j^2 + j\alpha + (\alpha - 1)}{j^2 + (1+\bar{\alpha})j + \bar{\alpha}}\right| |a_{j-1}| + \frac{|\lambda|}{|\omega|}\frac{1}{|j^2 + (1+\bar{\alpha})j + \bar{\alpha}|}|a_j|. \tag{2.5}$$

Since $|\omega| = 1$ we have

$$|a_{j+1}| \leqq |u_1(j)|\,|a_{j-1}| + |u_2(j)|\,|a_j|, \tag{2.6}$$

where

$$u_1(j) = \frac{j^2 + j\alpha + (\alpha - 1)}{j^2 + (1+\bar{\alpha})j + \bar{\alpha}},$$

and

$$u_2(j) = \frac{\lambda}{j^2 + (1+\bar{\alpha})j + \bar{\alpha}}.$$

We now estimate $|u_1(j)|$ and $|u_2(j)|$ for large j. Since $|u_2(j)|$ tends to zero as j^{-2} it follows that there exists an $M > 0$ such that

$$|u_2(j)| \leqq \frac{M}{j^2}, \qquad \text{for } j \text{ sufficiently large.} \tag{2.7}$$

Concerning $|u_1(j)|$ we obtain, upon dividing,

$$u_1(j) = \left(1 - \frac{1}{j}\right) + \frac{2}{j}\,\mathrm{Im}(\alpha)i + O(j^{-2}).$$

Thus $|u_1(j)|^2 = 1 - 2/j + O(j^{-2})$, and hence by a direct calculation,

$$|u_1(j)| = 1 - \frac{1}{j} + O(j^{-2}).$$

For $\xi > 0$, we note that $|u_1(j)| \leqq 1 - \xi j^{-1}$ for j sufficiently large if and only if $-1 < -\xi$, or $\xi < 1$. Hence we have

$$|u_1(j)| \leqq 1 - \frac{\xi}{j}, \quad \text{for } j \text{ sufficiently large} \tag{2.8}$$

and $0 < \xi < 1$.

Using (2.6), (2.7), and (2.8) we obtain, for j sufficiently large,

$$\begin{aligned} |a_{j+1}| &\leqq (1 - \xi j^{-1}) |a_{j-1}| + M j^{-2} \| a_j| \\ &\leqq (1 - \xi j^{-1} + M j^{-2}) M(j), \qquad 0 < \xi < 1, \end{aligned}$$

where $M(j) = \max\{|a_{j-1}|, |a_j|\}$.

Thus, for sufficiently large j, we have

$$|a_{j+1}| \leqq (1 - \gamma j^{-1}) M(j), \tag{2.9}$$

where $0 < \gamma = \xi/2 < \frac{1}{2}$.

Now consider the expression $(1 - \gamma j^{-1})(j-1)^{-1/p}$, where p is a positive integer. This is dominated by $(j+1)^{-1/p}$ for j sufficiently large if and only if

$$j^{p+1} + (-p\gamma + 1) j^p + \cdots \leqq j^{p+1} - j^p.$$

Hence, if and only if $-p\gamma + 1 < -1$ or $-p\gamma < -2$. Since $\gamma > 0$, $p > 2/\gamma$. Thus we have

$$(1 - \gamma j^{-1})(j-1)^{-1/p} \leqq (j+1)^{-1/p}, \quad p > \frac{2}{\gamma}. \tag{2.10}$$

We now show that there exists a positive constant K for which $|a_j| \leqq K j^{-1/p}$ for $j \geqq 1$. Let j_1 be such that (2.9) and (2.10) hold for $j > j_1$. Let $K = \max_{j \leqq j_1} |a_j| j^{1/p}$ so that $|a_j| \leqq K j^{-1/p}$ for $j \leqq j_1$. Using (2.9) it follows that

$$|a_{j_1+1}| \leqq (1 - \gamma j_1^{-1}) M(j_1),$$

where

$$\begin{aligned} M(j_1) &= \operatorname{Max}(K j_1^{-1/p}, K(j_1 - 1)^{-1/p}) \\ &= K(j_1 - 1)^{-1/p}. \end{aligned}$$

Hence,

$$|a_{j_1+1}| \leqq (1 - \gamma j_1^{-1}) K (j_1 - 1)^{-1/p},$$

and using (2.10) we have

$$|a_{j_1+1}| \leqq K(j_1+1)^{-1/p}. \tag{2.11}$$

We now proceed inductively to establish

$$|a_{j_1+k}| \leqq K(j_1+k)^{-1/p}, \qquad k = 2, 3, \cdots. \tag{2.12}$$

Let

$$\begin{aligned} K_1 &= \max_{j \leqq j_1+1} |a_j| j^{1/p} \\ &= \max\{K, K(j_1+1)^{-1/p}\} \leqq K, \end{aligned}$$

making use of (2.11). Using (2.9) we have

$$|a_{j_1+2}| \leqq (1 - \gamma(j_1+1)^{-1} M(j_1+1),$$

where,

$$\begin{aligned} M(j_1+1) &= \operatorname{Max}(|a_{j_1+1}|, |a_{j_1}|) \\ &= \operatorname{Max}(K(j_1+1)^{-1/p},\ K(j_1)^{-1/p}) \\ &= K(j_1)^{-1/p}. \end{aligned}$$

It follows from (2.10) that

$$\begin{aligned} |a_{j_1+2}| &\leqq (1 - \gamma(j_1+1)^{-1}) K(j_1)^{-1/p} \\ &\leqq K(j_1+2)^{-1/p}. \end{aligned}$$

Continuing on in this manner we establish (2.12). Hence any solution $\Sigma_{j=0}^{\infty} a_j z^j$ whose coefficients satisfy (2.4) is in $\mathscr{H}$. To complete the proof we have only to deal with the case where $j\omega + \beta$ vanishes for some positive integer j.

We now consider the case when $j\omega + \beta$ vanishes for some positive integer n. The analytic solution obtained from (2.3) by taking $a_0 = a_1 = \cdots = a_n = 0$, and solving recursively for $a_{n+2}, a_{n+3}, \cdots$, in terms of a_{n+1} is, as we have seen, in $\mathscr{H}$. If there were a second analytic solution corresponding to $a_0 \neq 0$ it would be in $\mathscr{H}$ as well, and $m^+ (m^-)$ would be 2. We now show that this is not the case, i.e., $m^+ = m^- = 1$. To do this we make use of the following result.

Let μ be such that $\operatorname{Im}(\mu) > 0$ and let $\mathscr{D}_\mu^+$ be the nullspace of the operator $S^* - \mu$. Then the dimension of $\mathscr{D}_\mu^+$ is equal to m^+. Similarly, let $\mathrm{Im}(\mu) < 0$ and let $\mathscr{D}_\mu^-$ be the nullspace of the operator $S^* - \mu$, then the dimension of $\mathscr{D}_\mu^-$ is equal to m^-, [**1**, p. 1232].

Using this we see that m^+ is just the number of linearly independent solutions of $L\phi = \mu\phi$ in $\mathscr{H}$ for any μ such that $\mathrm{Im}(\mu) > 0$. Similarly, m^- is the number of linearly independent solutions of $L\phi = \mu\phi$ in $\mathscr{H}$ for any μ such that $\mathrm{Im}(\mu) < 0$. Hence, if we can show that there exist μ such that $\mathrm{Im}\,\mu > 0$ ($\mathrm{Im}\,\mu < 0$) for which there is no analytic solution corresponding to $a_0 \neq 0$ we will have shown that $m^+ = m^- = 1$.

Consider (2.3), where λ is now μ/c_2, and suppose that $\beta = -n\omega$. Taking $j = 1, 2, \cdots, n$ we obtain the following set of $n+1$ linear equations in a_0 thru a_n:

$$\begin{aligned} -n\omega a_1 &= \lambda a_0 \\ (j+1)(j-n)\omega a_{j+1} + (j^2 + j\alpha + \alpha - 1)a_{j-1} &= \lambda a_j, \\ &\qquad j = 1, 2, \cdots, n-1 \\ (n^2 + n\alpha + \alpha - 1)a_{n-1} &= \lambda a_n. \end{aligned}$$

Thus we are led to consider the homogeneous system

$$\begin{aligned} -\lambda a_0 - n\omega a_1 &= 0 \\ 2\alpha a_0 - \lambda a_1 + 2(2-n)\omega a_2 &= 0 \\ (n^2 + n\alpha - 2n)a_{n-2} - \lambda a_{n-1} - n\omega a_n &= 0 \\ (n^2 + n\alpha + \alpha - 1)a_{n-1} - \lambda a_n &= 0 \end{aligned}$$

Since the parameter $\lambda = \mu/c_2$ appears only on the diagonal the system determinant $D_n(\lambda)$ is a polynomial in λ of degree $n+1$,

$$D_n(\lambda) = (-1)^{n+1}\lambda^{n+1} + \cdots.$$

Thus $D_n(\lambda)$ vanishes at most $n+1$ points in the complex plane, and we can find μ in the upper half-plane and lower half-plane for which $D_n(\mu/c_2) \neq 0$. Thus $a_0 = a_1 = \cdots = a_n = 0$ and there is only one analytic solution of $L\phi = \mu\phi$.

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