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ALGEBRAIC MAXIMAL SEMILATTICES

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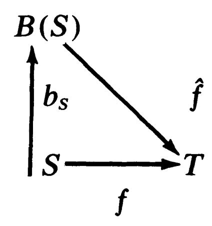
A topological semigroup S is maximal if it is closed in each topological semigroup that contains it. The semigroup S is called absolutely maximal if each continuous image is maximal. In this paper we are concerned with those discrete semilattices that are absolutely maximal. Thus we are concerned with those algebraic conditions on a semilattice which force it to be topologically closed.

In [9] Stralka studies those semigroups which have the congruence extension property. The semilattices we are concerned with and all their homomorphic images have this property. In fact, every congruence on such a semilattice S is closed. Thus S admits a compact Hausdorff topology $\mathcal{F}(S)$ under which multiplication is continuous. By [5] S admits a unique such topology. Also, since S has the congruence extension property for finite subsemilattices, the topology $\mathcal{F}(S)$ has a base which consists of subsemilattices [3].

In §II we give definitions, and we give necessary and sufficient conditions for a sublattice of a compact lattice to be closed. In §III we characterize those discrete semilattices and lattices which are absolutely maximal. Also, we show $(S, \mathcal{F}(S))$ is stable and 0-dimensional. In §IV we indicate how absolutely maximal discrete semilattices are constructed from a class of simple examples.

II. Definitions. Let S denote a topological semilattice. The Bohr compactification of S is a pair $(B(S), b_s)$ where $B(S)$ is a compact semilattice, $b_s\colon S \rightarrow B(S)$ is a continuous homomorphism and if $f\colon S \rightarrow T$ is a continuous homomorphism with T a compact semilattice, then there is a unique continuous homomorphism which makes the following diagram commute:



For the existence of the Bohr Compactification see either [1] or [2].

For each $U \subseteq S$ with $U \neq \emptyset$ let $M(U) = \{y \in S \mid$ there is an $x \in U$ with $xy = x\}$, $L(U) = U \cdot S$ and $CL(U)$ denotes the closure of U. Define $\leqq$ on S by $x \leqq y$ if and only if $xy = x$. Let $\{x_a\}_{a \in A}$ be a net in S. To say $x_\alpha \uparrow x$ means the net converges to x and $x_\alpha \leqq x_\beta$ whenever $\alpha \leqq \beta$. We define $x_\alpha \downarrow x$ is a similar manner. For a topological semilattice T Hom(S, T) denotes the collection of continuous homomorphisms from S to T. Let I denote the unit interval with $xy = \min\{x, y\}$ and let $I_1 = \{0, 1\} \subseteq I$.

PROPOSITION 1. *Let L be a compact topological semilattice with identity element and let A be a sublattice of S. Then A is closed if and only if A is complete.*

Proof. Assume A is complete and let $x \in \mathrm{CL}(A)$. Let $\mathcal{U}$ be the collection of sequences of open sets about x having the following property; $\{U_n\}_{n=1}^\infty \in \mathcal{U}$ if and only if $\mathrm{CL}(U_{n+1}) \wedge \mathrm{CL}(U_{n+1}) \subseteq U_n$ for $n = 1, 2, \cdots$. Partially order $\mathcal{U}$ by $\{U_n\}_{n=1}^\infty \leqq \{V_n\}_{n=1}^\infty$ if $V_n \subseteq U_n$ for all n. Then $\mathcal{U}$ with this partial order is a directed set. Now fix $\alpha = \{U_n\}_{n=1}^\infty \in \mathcal{U}$. Note that $\bigcap_{i=1}^\infty U_n = \bigcap_{i=1}^\infty \mathrm{CL}(U_n)$ is a sublattice of L and if $(\bigcap_{i=1}^\infty U_n) \cap A \neq \emptyset$, then $(\bigcap_{i=1}^\infty U_n) \cap A$ is closed under taking inf s and thus has a zero which will be denoted by $z(\alpha)$. Thus we show this intersection exists.

For each n let $b_n \in U_n \cap A$ and let $\{a_p^n\}_{p=1}^\infty$ be the sequence given by $a_p^n = \wedge_{j=1}^p b_{n+j}$. Then $\{a_p^n\}_{p=1}^\infty \subseteq U_n$ and is a decreasing sequence and thus has a limit point t in $\mathrm{CL}(U_n) \cap A$. Clearly, $t \in \mathrm{CL}(U_m)$ for all $m > n$ and thus $(\bigcap_{i=1}^\infty U_n) \cap A \neq \varnothing$. It is clear that if $\alpha, \beta \in \mathcal{U}$ with $\alpha < \beta$, then $z(\alpha) \leqq z(\beta)$. Thus $\{z(\alpha)\}_{\alpha \in \mathcal{U}}$ is an increasing net in A which converges to x. Since A is complete, and l compact, $x \in \mathrm{CL}(A)$.

In [5] Lawson defines B^+ for an ideal in a semilattice S to be $\{x \mid$ there is a net $\{x_\alpha\}_{\alpha \in \Gamma} \subseteq B$ with $x_\alpha \uparrow x\}$. He shows for an ideal B in a compact semilattice S is closed if and only if $B^+ = B$. Thus one has

PROPOSITION 2. *Let B be a subsemilattice of a compact semilattice S. Then B is closed if and only if B contains arbitrary infs and* $B = B^+$.

We also need the following from [5].

PROPOSITION 3. *Let S and T be compact semilattices and lef f be a homomorphism from S to T. Then f is continuous if and only if f has the property that* $f(x_\alpha) \uparrow f(x)$ *whenever* $x_\alpha \uparrow x$ *and* $f(y_\alpha) \downarrow f(y)$ *whenever* $y_\alpha \downarrow y$.

Comment 4. It is not the case that a complete lattice necessarily admits a compact Hausdorff topology for which both operations are continuous. For consider the lattice on the integers with 0 the smallest element, 1 the largest element and each maximal chain having three elements. However, $(Z, \wedge)$ does admit a compact Hausdorff topology with $\wedge$ continuous.

III. Maximal semilattices and lattices. Throughout this section S will denote a discrete semilattice which is absolutely maximal. Since S is a locally compact semilattice with a base for the topology which consist of subsemilattices, $\mathrm{Hom}(S, I)$ separates points **[4]**. Thus there exists a continuous injection α from S into a compact semilattice. Since $\alpha(S)$ is closed it is compact and S therefore admits a compact topology $\mathscr{F}(S)$ with multiplication continuous. By **[5]**, $\mathscr{F}(S)$ is unique, and therefore $(\alpha(S), \alpha)$ is the Bohr compactification of S. Note that $\alpha(S)$ is the Bohr compactification of $\alpha(S)$ with the discrete topology. Therefore, we first characterize those compact semilattices T which are the Bohr compactification of T with the discrete topology.

For a semilattice T we let T_d denote T with the discrete topology.

PROPOSITION 5. *Let T be a compact semilattice with $T = B(T_d)$. Then*

(a) $\mathrm{Hom}(T, I_1)$ *separates points.*

(b) *If U is a subsemilattice of T, then $M(U)$ is both open and closed.*

(b′) *Each prime ideal of T is both open closed*

(c) $\dim S = 0$.

Proof. (a) Let $x, y \in T$ and assume $x \not\in M(y)$. Let $\phi: T \to I_1$ be given by $\phi(s) = 1$ if $s \in M(y)$ and 0 otherwise. Since $T = B(T_d)$, $\phi \in \mathrm{Hom}(T, I_1)$, and $\phi(y) = 1 \neq 0 = \phi(x)$. It now follows that Hom (T, I_1) separates points.

(b), (b′) Same as (a).

(c) Since $\mathrm{Hom}(S, I_1)$ separates points and S is compact, S can be embedded in a 0-dimensional semilattice and is therefore 0-dimensional.

LEMMA 6. *Let 1 be a compac semilattice with $M(U)$ both open and closed for each subsemilattice U of S. If C is a chain in T, then C is finite.*

Proof. Assume T has an infinite chain C. Then $\mathrm{CL}(C)$ is a chain and must have a limit point z. Since $M(z)$ is open, there is net $\{x_\alpha\}_{\alpha \in \Gamma}$ in C with $x_\alpha \downarrow z$ and $x_\alpha \neq z$ for each $\alpha \in \Gamma$. Let $N = \bigcap_{\alpha \in \Gamma} M(x\alpha)$; then

$M(U)$ is closed with $z \notin M(U)$ with is a contradiction. Thus C must be finite.

PROPOSITION 7. *Let T be a compact semilattice. Then the following are equivalent*:

(a) $T = B(T_d)$.

(b) *$M(U)$ is both open and closed for each subsemilattice U of S.*

(c) *Each chain in T is finite.*

(d) $\mathrm{Hom}(T, I_1) = \mathrm{Hom}(T_d, I_1)$.

(e) *There is a compact semilattice R with $|R| > 1$ with* $\mathrm{Hom}(T, R) = \mathrm{Hom}(T_d, R)$.

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (e) trivial, (e) $\Rightarrow$ (b) by proof Proposition 5, (b) $\Rightarrow$ (c) by Lemma 6. Thus we show (c) $\Rightarrow$ (a).

Let $f \in \mathrm{Hom}(T_d, R)$ where R is a compact semilattice and each chain in T is finite. Let $\{x_\alpha\}_{\alpha \in \Gamma}$ be a net in T with $x_\alpha \downarrow x$. Since each chain is finite, eventually $x_\alpha = x$ and $f(x_\alpha) \downarrow f(x)$. Similarly, if $y_\alpha \uparrow y$ then $f(y_\alpha) \uparrow f(y)$. By Proposition 4, $f \in \mathrm{Hom}(T, R)$ and thus $T = B(T_d)$.

LEMMA 8. *Let T be a topological semilattice and let R be a subsemilattice of T with each chain finite. Then R is closed.*

Proof. Let $x \in \mathrm{CL}(R)$. Let $\mathcal{U}$ be the collection of sequences of open sets about x satisfying; $\{U_n\}_{n=1}^\infty \in \mathcal{U}$ if and only if $U_{n+i} U_{n+1} \leqq U_n$ for all n. Partially order $\mathcal{U}$ by $\{U_n\}_{n=1}^\infty \leqq \{V_n\}_{n=1}^\infty$ if and only if $V_n \subseteq U_n$ for all n. Then $\mathcal{U}$ with this partial order is a directed set. Fix $\{U_n\}_{n=1}^\infty = \alpha \in \mathcal{U}$. Then $\bigcap_{i=1}^\infty U_n$ is a subsemilattice of T and if $(\bigcap_{i=1}^\infty U_n) \cap R \neq \emptyset$, then $(\bigcap_{i=1}^\infty U_n) \cap R$ has a zero. For each positive integer n let $b_n \in U_n \cap R$, and for each positive integer p let $a_p^n = b_{n+1} b_{n+2} \cdots b_{n+p}$. As before, $a_p^n \in U_n$ for all p. Since each chain in R is finite, there is a q such that if $p > q$ then $a_q^n = a_q^n$. Thus $\{a_p^n\}_{p=1}^\infty$ converges to a^n in U_n. Clearly, if $m > n$ then $a^m \geqq a^n$. Thus there is a m_0 such that if $n \geqq m_0$ then $a^n = a^{m_0}$. It now follows that $a^{m_0} \in (\bigcap_{i=1}^\infty U_n) \cap R$. Let $z(\alpha)$ be the zero of $(\bigcap_{i=1}^\infty U_n) \cap R$. Thus $\{z(\alpha)\}_{\alpha \in \mathcal{U}}$ converges to x. Thus $r = x \in R$ and R is closed.

We now summarize our results in the form of a theorem.

THEOREM 9. *Let T be a discrete semilattice. Then T is absolutely maximal if and only if each maximal chain in T is finite.*

It is clear that we also have

COROLLARY 10. *Let L be a discrete lattice with each chain finite. Then each lattice homomorphic image of L is closed.*

We close this section with some additional properties a semilattice T with $T = B(T_d)$ must have. The proofs are all straightforward and will be omitted.

PROPOSITION 11. *Let T be a compact semilattice with* $T = B(T_d)$. *Then*

(a) *Each semilattice of T is closed.*
(b) *If R is a sublattice of T, then* $R = B(R_d)$.
(c) *If R is a homomorphic image of T, then* $R = B(R_d)$.
(d) *T is stable (that is, there are no dimension raising homomorphisms on T).*

IV. Examples. Throughout this section S_d is assumed to be a discrete absolutely maximal semilattice and S will denote $B(S_d)$. For each $x \in S_d$ let $A(x) = \{y \in S_d \mid x < y$ and $M(x) \cap L(y) = \{x, y\}$.

LEMMA 12. *For each* $x \in S_d$ $A(x)$ *is infinite if and only if x is a limit point of S. Further, if* $A(x)$ *is infinite, then* $\mathrm{CL}_S(A(x)) = A(x) \cup \{x\}$.

Proof. Assume $A(x)$ is infinite and let $\{y_\alpha\}_{\alpha \in \Gamma}$ be a net in $A(x)$ which converges (in S) to y. Assume each $y_\alpha \neq y$. Let $z \in A(x)$; then $zy_\alpha = x$ if $y_\alpha \neq z$. Thus $zy = x$. It now follows that $y = y^2 = \lim yy_\alpha = \lim x = x$. Thus $\mathrm{CL}_S(A(x)) = A(x) \cup \{x\}$ and x is a limit point of S.

Now assume x is a limit point of S and let $\{z_\alpha\}_{\alpha \in \Gamma}$ be a net in S which converges to x and $z_\alpha \neq x$. For each $\alpha \in \Gamma$ let $x_\alpha \in A(x) \cap L(Z_\alpha)$. Such x_α's exist since each chain in T is finite. Thus $\{x_\alpha\}_{\alpha \in \Gamma}$ is a net in $A(x)$ which converges to x. It now follows that $A(x)$ is infinite.

EXAMPLE 13. Let X be a compact well-ordered space and let B be the set of limit points of X. Let ρ be defined on X by $x\rho y$ if and only if $x = y$ or $x, y \in B$. Then X/ρ is a compact Hausdorff space. Define multiplication on X/ρ by $[x][y] = [x]$ if $[x] = [y]$ and B otherwise. Then X/ρ with this multiplication is a compact semilattice with each chain finite. Thus $(X/\rho)_d$ is an absolutely maximal semilattice.

EXAMPLE 14. *Let* $T=\{((1/n),(1/p))\mid n,p$ positive integers, $n \leqq p \leqq 2n\} \cup \{(0,0)\}$ with multiplication defined by

$$\left(\frac{1}{n},\frac{1}{p}\right)\left(\frac{1}{m},\frac{1}{q}\right)=\begin{cases}(0,0) & \text{if} \quad n \neq m \\ \left(\frac{1}{n},\min\left\{\frac{1}{p},\frac{1}{q}\right\}\right) & \text{if} \quad n = m.\end{cases}$$

Then each chain in T_d is finite and thus T_d is absolutely maximal. Note that although chain in T is finite there is no upper bound on the length of chains.

Observation 15. Let x be a limit point of S. Then $\mathrm{CL}_S(A(x))$ is isomorphic to X/ρ for a suitable compact well-ordered space X (see example 13).

Observation 16. There is a discrete semilattice T_d which is absolutely maximal and the set of limit points of T is S.

Question 17. If S is a maximal semilattice is it absolutely maximal?

Question 18. Are these reasonable conditions one can impose on a locally compact semilattice to guarantee that it be maximal?

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